INTERNATIONAL MATHEMATICAL OLYMPIAD 2022
PROBLEM SET

OSLO2022

English (eng), day 1

Monday, 11. July 2022

Problem 1. The Bank of Oslo issues two types of coin: aluminium (denoted A) and bronze (denoted B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A *chain* is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leqslant 2n$, Marianne repeatedly performs the following operation: she identifies the longest chain containing the k^{th} coin from the left, and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be

$$AAB\underline{B}BABA \to BBB\underline{A}AABA \to AAA\underline{B}BBBA \to BBB\underline{B}AAAA \to BBB\underline{B}AAAA \to \cdots.$$

Find all pairs (n, k) with $1 \leqslant k \leqslant 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 2. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leqslant 2.$$

Problem 3. Let k be a positive integer and let S be a finite set of odd prime numbers. Prove that there is at most one way (up to rotation and reflection) to place the elements of S around a circle such that the product of any two neighbours is of the form $x^2 + x + k$ for some positive integer x.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Tuesday, 12. July 2022

Problem 4. Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Problem 5. Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Problem 6. Let n be a positive integer. A *Nordic square* is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. Two different cells are considered adjacent if they share a common side. Every cell that is adjacent only to cells containing larger numbers is called a *valley*. An *uphill path* is a sequence of one or more cells such that:

(i) the first cell in the sequence is a valley,

(ii) each subsequent cell in the sequence is adjacent to the previous cell, and

(iii) the numbers written in the cells in the sequence are in increasing order.

Find, as a function of n, the smallest possible total number of uphill paths in a Nordic square.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Solution

1. *Answer.* All pairs (n, k) such that $n \leq k \leq \frac{3n+1}{2}$.

   Solution. Define a *block* to be a maximal subsequence of consecutive coins made out of the same metal, and let M^b denote a block of b coins of metal M. The property that there is at most one aluminium coin adjacent to a bronze coin is clearly equivalent to the configuration having two blocks, one consisting of all A's and one consisting of all B's.

   First, notice that if $k < n$, the sequence $A^{n-1}B^{n-1}AB$ remains fixed under the operation and will therefore always have 4 blocks. If $k > \frac{3n+1}{2}$, let $a = k-n-1, b = 2n-k+1$. Then $k > 2a+b, k > 2b+a$, so the configuration $A^aB^bA^bB^a$ will always have 4 blocks:

$$A^aB^bA^b\underline{B^a} \longrightarrow B^aA^aB^b\underline{A^b} \longrightarrow A^bB^aA^a\underline{B^b} \longrightarrow B^bA^bB^a\underline{A^a} \longrightarrow A^aB^bA^b\underline{B^a} \longrightarrow \cdots$$

   Therefore a pair (n, k) can have the desired property only if $n \leq k \leq \frac{3n+1}{2}$. We claim that all such pairs in fact do have the desired property. Clearly, the number of blocks in a configuration cannot increase, so whenever the operation is applied, it either decreases or remains constant. We show that unless there are only two blocks, after a finite amount of steps the number of blocks will decrease.

   Consider an arbitrary configuration with $c \geq 3$ blocks. We note that as $k \geq n$, the leftmost block cannot be moved, because in this case all n coins of one type are in the leftmost block, meaning there are only two blocks. If a block which is not the leftmost or rightmost block is moved, its neighbor blocks will be merged, causing the number of blocks to decrease.

   Hence the only case in which the number of blocks does not decrease in the next step is if the rightmost block is moved. If c is odd, the leftmost and the rightmost blocks are made of the same metal, so this would merge two blocks. Hence $c \geq 4$ must be even. Suppose there is a configuration of c blocks with the i-th block having size a_i so that the operation always moves the rightmost block:

$$A^{a_1}\ldots A^{a_{c-1}}\underline{B^{a_c}} \longrightarrow B^{a_c}A^{a_1}\ldots \underline{A^{a_{c-1}}} \longrightarrow A^{a_{c-1}}B^{a_c}A^{a_1}\ldots \underline{B^{a_{c-2}}} \longrightarrow \cdots$$

   Because the rightmost block is always moved, $k \geq 2n+1-a_i$, for all $i = 1, \ldots, c$. Thus summing over i, we have $ck \geq 2nc+c-\sum_{i=1}^{c} a_i = 2nc+c-n$ so that $k \geq 2n+1-\frac{2n}{c} \geq 2n+1-\frac{2n}{4} = \frac{3n}{2}+1 > \frac{3n+1}{2}$. This contradicts $k \leq \frac{3n+1}{2}$. Hence at some point the operation will not move the rightmost block, meaning that the number of blocks will decrease, as desired.

2. (*Gabriel Goh Hao Xiang*) The only function satisfying the condition of the problem is $f(x) = \frac{1}{x}$, for $x > 0$.

   Let's check that $f(x) = \frac{1}{x}$ satisfies the condition of the problem. Note that $xf(y) + yf(x) = \frac{x}{y} + \frac{y}{x} \geq 2\sqrt{\frac{x}{y} \cdot \frac{y}{x}} = 2$ and equality holds if and only if $\frac{x}{y} = \frac{y}{x}$, which is equivalent to $x = y$. This means that, for every $x > 0$, there is a unique $y > 0$ such that $\frac{x}{y} + \frac{y}{x} \leq 2$, namely $y = x$.

   Now suppose f is a function satisfying the condition of the problem. We shall prove $f(x) = \frac{1}{x}$.

   Claim 1. f is strictly decreasing.

   Proof. Let $a, b > 0$. Suppose $f(a+b) \geq f(b)$. Let t be the unique number such that $(a+b)f(t)+tf(a+b) \leq 2$. Then $bf(t) + tf(b) \leq (a+b)f(t) + tf(a+b) \leq 2$. This implies that there is more than one y (namely $a+b$ and b) satisfying $yf(t)+tf(y) \leq 2$, which is a contradiction. Therefore, $f(a+b) < f(b)$. This proves that f is strictly decreasing.

   Claim 2. x is the unique y satisfying $xf(y) + yf(x) \leq 2$. Thus $f(x) \leq \frac{1}{x}$, for $x > 0$.

   Proof. Suppose $x \neq y$ satisfy $xf(y) + yf(x) \leq 2$. Then $xf(x) + xf(x) > 2$ and $yf(y) + yf(y) > 2$. Adding the two inequalities, we have $xf(x)+yf(y) > 2$. Now $xf(x)+yf(y) > 2 \geq xf(y)+yf(x)$ implies $(x-y)(f(x)-f(y)) > 0$, which contradicts the fact that f is strictly decreasing. Thus $2xf(x) \leq 2$ so that $f(x) \leq \frac{1}{x}$, for $x > 0$.

Claim 3. $xf(x) \geq 1$ for all $x > 0$.

Proof. Suppose there exists $a > 0$ such that $af(a) = 1-k < 1$ for some $0 < k < 1$. By claim 2, for any $y \neq a$, we have $af(y) + af(y) > 2$. Let $\epsilon > 0$ and take $y = a+\epsilon$. Then $af(a+\epsilon) + (a+\epsilon)f(y) > 2 \Leftrightarrow af(a+\epsilon) > 2-(a+\epsilon)f(a)$. By claim 1, f is decreasing, so that $af(a) > af(a+\epsilon) > 2-(a+\epsilon)f(a) \Leftrightarrow 2af(a) + \epsilon f(a) > 2 \Leftrightarrow 2(1-k) + \epsilon f(a) > 2 \Leftrightarrow \epsilon f(a) > 2k$. Taking $\epsilon = \frac{2k}{f(a)}$ gives a contradiction. Therefore, $xf(x) \geq 1$ for all $x > 0$.

Then claim 1 and 2 imply that $f(x) = \frac{1}{x}$ for $x > 0$.

3. Let us allow the value x to be 'nonnegative' integer; we prove the same statement under this more general constraint. Obviously that implies the statement with the original conditions.

   Call a pair $\{p, q\}$ of primes with $p \neq q$ *special* if $pq = x^2 + x + k$ for some nonnegative integer x.

   Claim. (a) For every prime r, there are at most two primes less than r forming a special pair with r.

   (b) If two such primes p and q exist, then $\{p, q\}$ is itself special.

   Proof. (a) The equation $x^2 + x + k \equiv 0 \pmod{r}$ is quadratic and thus has at most two solutions in x for $0 \leq x < r$. Note that for each solution x, we have $x^2 + x + k = ur$ for some positive integer u; and if this u is a prime less than r, then $\{u, r\}$ is a special pair.

   (b) Now suppose there are primes p, q with $p < q < r$ and nonnegative integers x, y such that

   $$x^2 + x + k = pr \text{ and } y^2 + y + k = qr.$$

   From $p < q < r$, we have $0 \leq x < y \leq r-1$. As x and y are solutions of the congruence equation: $z^2 + z + k \equiv 0 \pmod{r}$, we have by Vieta's formulas $x + y \equiv -1 \pmod{r}$, so $x + y = r - 1$.

   We also have $(y^2 + y + k) - (x^2 + x + k) = qr - pr \Leftrightarrow (y-x)(y+x+1) = r(q-p) \Leftrightarrow (y-x)r = r(q-p) \Leftrightarrow y - x = q - p$.

   Then $y = r-1-x, q = p+(r-1-2x), pq = p(p+r-1-2x) = p^2+pr-p-2px = p^2+(x^2+x+k)-p-2px = (x-p)^2 + (x-p) + k$.

   If $t = x-p$ is nonnegative, claim is proved. Otherwise, replace it by $(-t-1)$, as $(-t-1)^2+(-t-1)+k = t^2 + t + k$.

   Now we prove the statement using induction on $|S|$. The statement is clear true for $|S| \leq 3$. Suppose the statement is true for $|S| = n \geq 3$, and consider $|S| = n+1$. Let r be the largest prime in S. The claim tells that in any valid cycle of primes:

   - the neighbors of r are uniquely determined, and
   - removing r from the cycle results in a smaller valid cycle.

   It follows that there is at most one valid cycle, completing the inductive step.

4. (*Drew Michael Terren Ramirez*) From $TB = TD, TC = TE, BC = DE$, we have $\triangle TBC$ is congruent to $\triangle TDE$ by SAS. Therefore, $\angle BTC = \angle DTE$ and so $\angle BTY = \angle XTE$, where X is the intersection of DT and AB, and Y is the intersection of CT and AE.

   Claim 1. S, X, Y, Q are concyclic.

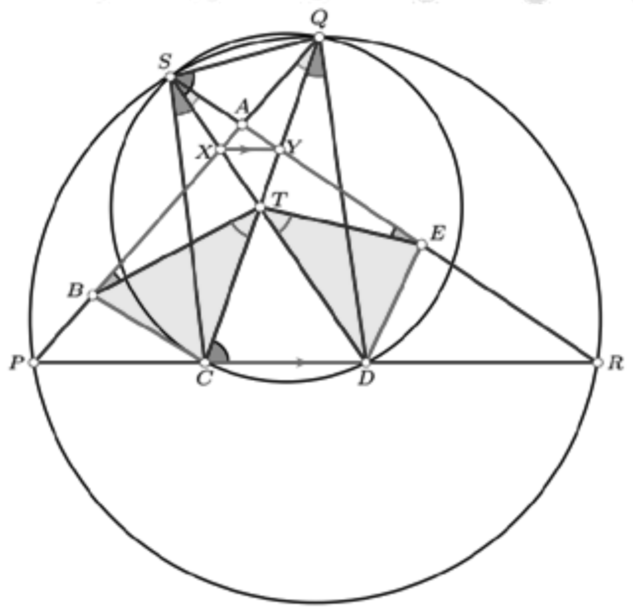
   Proof. $\angle SXQ = 180° - \angle AXT = 180° - (360° - \angle XTE - \angle TEA - \angle EAX) = 180° - (360° - \angle BTY - \angle ABT - \angle YAB) = 180° - \angle AYT = \angle SYQ$.

   Claim 2. $XY \parallel PR$.

   Proof. Since $\angle XBT = \angle YET$ and $\angle XTB = \angle YTB - \angle YTX = \angle ETX - \angle YTX = \angle YTE$, we have $\triangle XTB$ is similar to $\triangle YTE$.

   So $\frac{XT}{TB} = \frac{YT}{TE} \Rightarrow \frac{XT}{YT} = \frac{TB}{TE} = \frac{TD}{TC}$. Together with the fact that $\angle XTY = \angle DTC$, this gives $\triangle XTY$ is similar to $\triangle DTC$. Therefore, $\angle XYT = \angle DCT \Rightarrow XY \parallel PR$. As P, C, D, R are collinear, $XY \parallel DR$.

   By claim 1 and claim 2, we have $\angle QPR = \angle QXY = \angle QSY = \angle QSR$, and P, S, Q, R are concyclic.



5. The solutions are $(2,2,2)$ and $(3,4,3)$.

   Clearly, $a > 1$. We consider three cases.

   Case 1: $a < p$.

   Then we either have $a \leq b$ which implies a divides $a^p - b! = p$ leading to a contradiction as p is a prime, or $a > b$ which is also impossible since in this case we have $b! \leq a! < a^p - p$, where the last inequality is true for any $p > a > 1$.

   Case 2: $a > p$.

   In this case, $b! = a^p - p > p^p - p \geq p!$ so $b > p$, which means that $a^p = b! + p$ is divisible by p. Hence a is divisible by p, and $b! = a^p - p$ is not divisible by p^2. This means $b < 2p$. If $a < p^2$, then $a/p < p$ divides both a^p and $b!$ and hence it also divides $p = a^p - b!$ which is impossible. On the other hand, the case $a \geq p^2$ is also impossible since then $a^p \geq (p^2)^p > (2p-1)! + p \geq b! + p$.

   Case 3: $a = p$.

   In this case, $b! = p^p - p$. One can check that the values $p = 2, 3$ lead to the claimed solutions and $p = 5$ does not lead to a solution. So we now assume that $p \geq 7$. Clearly, $p < b$. We have

   $$v_2((p+1)!) \leq v_2(b!) = v_2(p^{p-1} - 1) \stackrel{LTE}{=} 2v_2(p-1) + v_2(v+1) - 1 = v_2\left(\frac{p-1}{2} \cdot (p-1)(p+1)\right),$$

   where in the middle we used lifting-the-exponent lemma. On the RHS we have at most three even factors of $(p+1)!$. But due to $p + 1 \geq 8$, there are at least 4 even numbers among $1, 2, \ldots, p+1$, so this case is not possible.

6. *Answer.* $2n^2 - 2n + 1$.

   Solution. We will call any cell that is only adjacent to cells with larger numbers a valley. Other cells will be called non-valleys. Let us make a second $n \times n$ board B where in each cell we will write the number of uphill paths which end on the corresponding cell in the original board A. We will thus look for the minimal possible value of the sum of all entries in B.

   First note that there is at least 1 valley, namely the cell with the number 1 in it. Next, for each pair of adjacent cells, we can trace back an uphill path by following an decreasing sequence of numbers until a valley is reached. In other words, the number of uphill paths is as many as the number of adjacent cells of the board. Including the uphill path with just the cell labeled 1, we see that the number of uphill paths is at least $2n(n-1) + 1 = 2n^2 - 2n + 1$.

We will now prove that the lower limit of $2n^2 - 2n + 1$ entries can be achieved. This amounts to finding a way of marking a certain set of cells, those that have a value of 1 in B, such that no two unmarked cells are adjacent and that the marked cells form a connected tree with respect to adjacency.

Once we have marked cells, the construction of number comes natural.

1. Take a marked cell, call it the root. It gets the number 1.

2. We fill with numbers in increasing order the marked cell in such a way that we only fill cell which has an already filled neighbour. From the tree property we get that it can have at most one filled neighbour.

3. We fill with the remaining "large" numbers the unmarked cell in any way.

This way each edge can be the last edge of at most one uphill path.

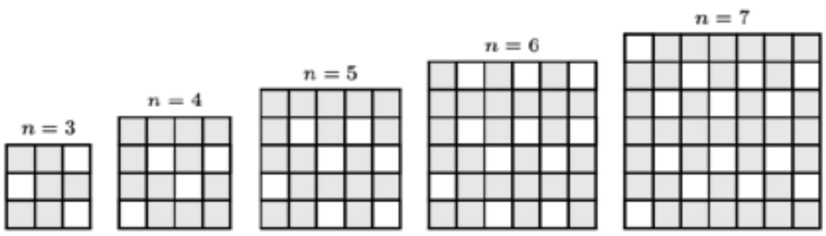
For $n = 1$ and $n = 2$ the markings are respectively the single cell and the L-trimino.

Now, for $n > 2$, let $s = 2$ for $n \equiv 0$ or $2 \pmod 3$; and $s = 1$ for $n \equiv 1 \pmod 3$. We will take indices k and ℓ to be arbitrary non-negative integers.

For $n \geq 3$ we will construct a path of marked cells in the first two columns consisting of all cells of the form $(1, i)$ where i is not of the form $6k + s$, and $(2, j)$ where j is of the form $6k + s - 1$, $6k + s$, $6k + s + 1$, or $6k + s + 3$.

Obviously, this path is connected. Now, let us consider the cells $(2, 6k + s)$ and $(1, 6k + s + 3)$. For each considered cell (i, j) we will mark all cells of the form (ℓ, j) for $\ell > i$ and $(i + 2k, j \pm 1)$.

One can easily see that no set of marked cells will produce a cycle. Also the only unmarked cells are of the form $(1, 6k + s)$, $(2 + 2\ell, 6k + s + 3 \pm 1)$ and $(2 + 2\ell + 1, 6k + s \pm 1)$ and that no two such unmarked cells are adjacent, since the consecutive considered cells are in columns of opposite parity. Examples of markings are given for $n = 3, 4, 5, 6, 7$, and the corresponding constructions for A and B are given for $n = 5$. Here marked cells are colored yellow.



$n = 5$

Board A

12	13	14	15	16
11	24	17	25	18
10	9	22	8	23
21	3	4	5	6
1	2	19	7	20

$n = 5$

Board B

1	1	1	1	1
1	4	1	4	1
1	1	4	1	3
3	1	1	1	1
1	1	3	1	2