

SIMO 2022 NATIONAL SELECTION TEST

Singapore International Mathematical Olympiad 2022
National Team Selection Test
Day 0

1. Let n be an integer, and let A be a subset of $\{0, 1, 2, 3, \dots, 5^n\}$ consisting of $4n+2$ numbers. Prove that there exist $a, b, c \in A$ such that $a < b < c$ and $c + 2a > 3b$.
2. Let \mathcal{S} be an infinite set of positive integers, such that there exist four distinct $a, b, c, d \in \mathcal{S}$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three distinct $x, y, z \in \mathcal{S}$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.
3. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B . The circumcircle of the triangle ACD meets the segment PD again at Q , and the circumcircle of the triangle APQ meets the segment PC again at R . Prove that the lines CD, AQ and BR are concurrent.
4. Determine all integers $n \geq 1$ for which there exists a pair of positive integers (a, b) such that no cube of a prime divides $a^2 + b + 3$ and

$$\frac{ab + 3b + 8}{a^2 + b + 3} = n.$$

Time allowed: 4 hours

Singapore International Mathematical Olympiad 2022

National Team Selection Test

Day 1

5. Let $n \geq 3$ be an integer. An integer $m \geq n + 1$ is called *n-colourful* if, given infinitely many marbles in each of n colours C_1, C_2, \dots, C_n , it is possible to place m of them around a circle so that in any group of $n + 1$ consecutive marbles there is at least one marble of colour C_i for each $i = 1, \dots, n$. Prove that there are only finitely many positive integers which are not *n-colourful*. Find the largest among them.
6. Determine all positive integers n whose positive divisors can be arranged as (d_1, d_2, \dots, d_k) such that for every $i = 1, 2, \dots, k$, the number $d_1 + \dots + d_i$ is a perfect square.
7. Let $ABCD$ be a cyclic quadrilateral whose side lengths are distinct. Let O be the circumcentre of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1 , respectively. Let O_B be the centre of the circle which passes through B and is tangent to AC at D_1 . Similarly, let O_D be the centre of the circle which passes through D and is tangent to AC at B_1 . Assume that $BD_1 \parallel DB_1$. Prove that O lies on the line $O_B O_D$.

Time allowed: 4.5 hours

Singapore International Mathematical Olympiad 2022
National Team Selection Test
Day 2

8. Let $ABCD$ be a quadrilateral inscribed in a circle Ω . Let the tangent to Ω at D intersect the rays BA and BC at points E and F , respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.
9. Given a positive integer n , find the smallest value of $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor$ over all permutations (a_1, a_2, \dots, a_n) of $(1, 2, \dots, n)$.
10. Let n and k be two integers with $n > k \geq 1$. There are $2n + 1$ students standing in a circle. Each student S has $2k$ neighbours - namely, the k students closest to S on the right, and the k students closest to S on the left.
- Suppose that $n + 1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.

Time allowed: 4.5 hours

Solutions to National Team Selection Test 2022

1. Let n be an integer, and let A be a subset of $\{0, 1, 2, 3, \dots, 5^n\}$ consisting of $4n+2$ numbers. Prove that there exist $a, b, c \in A$ such that $a < b < c$ and $c + 2a > 3b$.

Solution. We prove by way of contradiction. Suppose that there exist $4n+2$ non-negative integers $x_0 < x_1 < \dots < x_{4n+1}$ that violate the problem statement. Then in particular $x_{4n+1} + 2x_i \leq 3x_{i+1}$ for all $i = 0, \dots, 4n-1$, which gives

$$x_{4n+1} - x_i \geq \frac{3}{2}(x_{4n+1} - x_{i+1}).$$

By a trivial induction we then get

$$x_{4n+1} - x_i \geq \left(\frac{3}{2}\right)^{4n-i} (x_{4n+1} - x_{4n}),$$

which for $i = 0$ yields the contradiction

$$x_{4n+1} - x_0 \geq \left(\frac{3}{2}\right)^{4n} (x_{4n+1} - x_{4n}) = \left(\frac{81}{16}\right)^n (x_{4n+1} - x_{4n}) > 5^n \cdot 1.$$

2. Let \mathcal{S} be an infinite set of positive integers, such that there exist four distinct $a, b, c, d \in \mathcal{S}$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three distinct $x, y, z \in \mathcal{S}$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

Solution. There exists $\alpha \in \mathcal{S}$ so that $\{\gcd(\alpha, s) \mid s \in \mathcal{S}, s \neq \alpha\}$ contains at least two elements. Since α has only finitely many divisors, there is a $d \mid \alpha$ such that the set $B = \{\beta \in \mathcal{S} \mid \gcd(\alpha, \beta) = d\}$ is infinite. Pick $\gamma \in \mathcal{S}$ so that $\gcd(\alpha, \gamma) \neq d$. Pick $\beta_1, \beta_2 \in B$ so that $\gcd(\beta_1, \gamma) = \gcd(\beta_2, \gamma) =: d'$. If $d = d'$, then $\gcd(\alpha, \beta_1) = \gcd(\gamma, \beta_1) \neq \gcd(\alpha, \gamma)$. If $d \neq d'$, then either $\gcd(\alpha, \beta_1) = \gcd(\alpha, \beta_2) = d$ and $\gcd(\beta_1, \beta_2) \neq d$ or $\gcd(\gamma, \beta_1) = \gcd(\gamma, \beta_2) = d'$ and $\gcd(\beta_1, \beta_2) \neq d'$.

3. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B . The circumcircle of the triangle ACD meets the segment PD again at Q , and the circumcircle of the triangle APQ meets the segment PC again at R . Prove that the lines CD, AQ and BR are concurrent.

Solution. Let the circle of $APQR$ be α . First $\angle CRA = 180^\circ - \angle ARP = 180^\circ - \angle AQP = \angle DQA = \angle DCA = \angle CBA$, so that the points A, B, C, R lies on some circle γ . Let X be the intersection of AQ and CD . We wish to prove that B, R, X are collinear. By means of the circle α , we have $\angle RQX = 180^\circ - \angle AQR = \angle RPA = \angle RCX$, which means that the points C, Q, R, X lie on some circle δ . Using the circles δ and γ , we obtain $\angle XRC = \angle XQC = 180^\circ - \angle CQA = \angle ADC = \angle BAC = 180^\circ - \angle CRB$, that proves the desired collinearity.

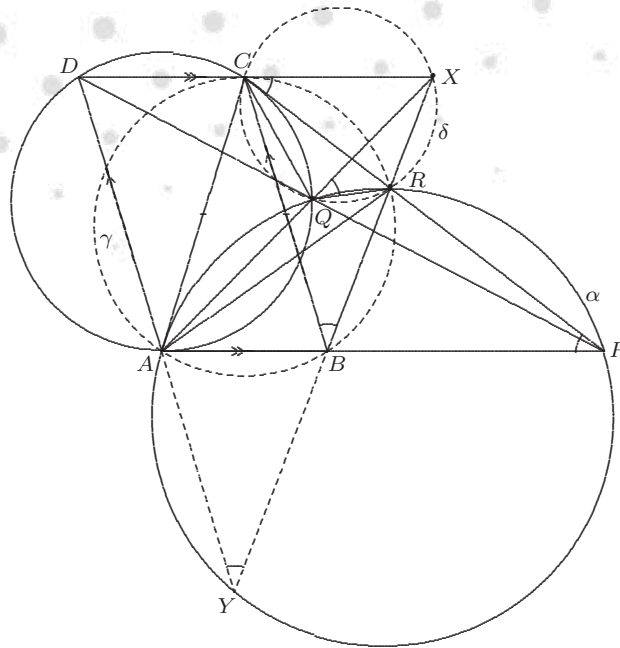


Figure 1: CD , AQ and BR are concurrent.

4. Determine all integers $n \geq 1$ for which there exists a pair of positive integers (a, b) such that no cube of a prime divides $a^2 + b + 3$ and

$$\frac{ab + 3b + 8}{a^2 + b + 3} = n.$$

Answer: The only integer with that property is $n = 2$.

Solution. As $b \equiv -a^2 - 3 \pmod{a^2 + b + 3}$, the numerator of the given fraction satisfies

$$ab + 3b + 8 \equiv a(-a^2 - 3) + 3(-a^2 - 3) + 8 \equiv -(a + 1)^3 \pmod{a^2 + b + 3}.$$

As $a^2 + b + 3$ is not divisible by p^3 for any prime p , if $a^2 + b + 3$ divides $(a + 1)^3$ then it does also divide $(a + 1)^2$. Since $0 < (a + 1)^2 < 2(a^2 + b + 3)$, we conclude that $(a + 1)^2 = a^2 + b + 3$. This yields $b = 2(a - 1)$ and $n = 2$. The choice $(a, b) = (2, 2)$ with $a^2 + b + 3 = 9$ shows that $n = 2$ indeed is a solution.

5. Let $n \geq 3$ be an integer. An integer $m \geq n + 1$ is called n -colourful if, given infinitely many marbles in each of n colours C_1, C_2, \dots, C_n , it is possible to place m of them around a circle so that in any group of $n + 1$ consecutive marbles there is at least one marble of colour C_i for each $i = 1, \dots, n$. Prove that there are only finitely many positive integers which are not n -colourful. Find the largest among them.

Answer: $m_{max} = n^2 - n - 1$.

Solution. First suppose that there are $n(n-1) - 1$ marbles. Then for one of the colours, say blue, there are at most $n-2$ marbles, which partition the non-blue marbles into at most $n-2$ groups with at least $(n-1)^2 > n(n-2)$ marbles in total. Thus one of these groups contains at least $n+1$ marbles and this group does not contain any blue marble.

Now suppose that the total number of marbles is at least $n(n-1)$. Then we may write this total number as $nk+j$ with some $k \geq n-1$ and with $0 \leq j \leq n-1$. We place around a circle $k-j$ copies of the colour sequence $[1, 2, 3, \dots, n]$ followed by j copies of the colour sequence $[1, 1, 2, 3, \dots, n]$.

6. Determine all positive integers n whose positive divisors can be arranged as (d_1, d_2, \dots, d_k) such that for every $i = 1, 2, \dots, k$, the number $d_1 + \dots + d_i$ is a perfect square.

Answer: $n = 1$ and $n = 3$.

Solution. For $i = 1, 2, \dots, k$, let $d_1 + \dots + d_i = s_i^2$, and define $s_0 = 0$ as well. Obviously $0 = s_0 < s_1 < s_2 < \dots < s_k$, so

$$s_i \geq i \text{ and } d_i = s_i^2 - s_{i-1}^2 = (s_i + s_{i-1})(s_i - s_{i-1}) \geq s_i + s_{i-1} \geq 2i - 1. \quad (6.1)$$

The number 1 is one of the divisors of d_1, \dots, d_k but, due to $d_i \geq 2i - 1$, the only possibility is $d_1 = 1$.

Now consider d_2 and $s_2 \geq 2$. By definition, $d_2 = s_2^2 - 1 = (s_2 - 1)(s_2 + 1)$, so the numbers $s_2 - 1$ and $s_2 + 1$ are divisors of n . In particular, there is some index j such that $d_j = s_2 + 1$.

Notice that

$$s_2 + s_1 = s_2 + 1 = d_j \geq s_j + s_{j-1}; \quad (6.2)$$

since the sequence $s_0 < s_1 < \dots < s_k$ increases, the index j cannot be greater than 2. Hence, the divisors $s_2 - 1$ and $s_2 + 1$ are listed among d_1 and d_2 . That means $s_2 - 1 = d_1 = 1$ and $s_2 + 1 = d_2$; therefore $s_2 = 2$ and $d_2 = 3$.

We can repeat the above process in general.

Claim. $d_i = 2i - 1$ and $s_i = i$ for $i = 1, 2, \dots, k$.

Proof. Apply induction on i . The claim has been proved for $i = 1, 2$. Suppose that we have already proved $d_1 = 1, d_2 = 3, \dots, d_i = 2i - 1$, and consider the next divisor d_{i+1} :

$$d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2 = (s_{i+1} + i)(s_{i+1} - i).$$

The number $s_{i+1} + i$ is a divisor of n , so there is some index j such that $d_j = s_{i+1} + i$.

Similarly to (6.2), by (6.1) we have

$$s_{i+1} + s_i = s_{i+1} + i = d_j \geq s_j + s_{j-1}; \quad (6.3)$$

since the sequence $s_0 < s_1 < \dots < s_k$ increases, (6.3) forces $j \leq i + 1$. On the other hand, $d_j = s_{i+1} + i > 2i > d_i > d_{i-1} > \dots > d_1$, so $j \leq i$ is not possible. The only possibility is $j = i + 1$.

Hence, $s_{i+1} + i = d_j = d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2$, so that $s_{i+1}^2 - s_{i+1} = i(i + 1)$.

By solving this equation we have $s_{i+1} = i + 1$ and $d_{i+1} = 2i + 1$, that finishes the proof.

Now we know that the positive divisors of the number n are $1, 3, 5, \dots, n - 2, n$. The greatest divisor is $d_k = 2k - 1 = n$ itself, so n must be odd. The second greatest divisor is $d_{k-1} = n - 2$; then $n - 2$ divides $n = (n - 2) + 2$, so $n - 2$ divides 2. Therefore, n must be 1 or 3.

The numbers $n = 1$ and $n = 3$ obviously satisfy the requirements: for $n = 1$ we have $k = 1$ and $d_1 = 1^2$; for $n = 3$ we have $k = 2$, $d_1 = 1^2$ and $d_1 + d_2 = 1 + 3 = 2^2$.

7. Let $ABCD$ be a cyclic quadrilateral whose side lengths are distinct. Let O be the circumcentre of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1 , respectively. Let O_B be the centre of the circle which passes through B and is tangent to AC at D_1 . Similarly, let O_D be the centre of the circle which passes through D and is tangent to AC at B_1 . Assume that $BD_1 \parallel DB_1$. Prove that O lies on the line $O_B O_D$.

Solution. Let Ω be the circle circumscribing the cyclic quadrilateral $ABCD$, and let γ_B and γ_D denote the two circles from the problem statement with centres O_B and O_D respectively. Clearly all three centres O_B, O_D , and O are distinct.

Assume, without loss of generality, that $AB > BC$. Suppose that $AD > DC$, and let H be the intersection of AC and BD . Then the rays BB_1 and DD_1 lie on one side of BD , as they contain the midpoints of the arcs ADC and ABC , respectively. However, if $BD_1 \parallel DB_1$, then B_1 and D_1 should be separated by H . This contradiction shows that $AD < CD$.

Let γ_B and γ_D meet Ω again at T_B and T_D , respectively. The common chord BT_B of Ω and γ_B is perpendicular to their line of centres $O_B O$; likewise, $DT_D \perp O_D O$. Therefore, O lies on $O_B O_D \Leftrightarrow O_B O \parallel O_D O \Leftrightarrow BT_B \parallel DT_D$, and the problem reduces to showing that

$$BT_B \parallel DT_D \quad (7.1)$$

Let H be the intersection of AC and BD . Consider the homothety h centred at H and mapping B to D . Since $BD_1 \parallel DB_1$, we have $h(D_1) = B_1$.

Let the tangents to Ω at B and D meet AC at L_B and L_D , respectively. We have

$$\angle L_B B B_1 = \angle L_B B C + \angle C B B_1 = \angle B A L_B + \angle B_1 B A = \angle B B_1 L_B,$$

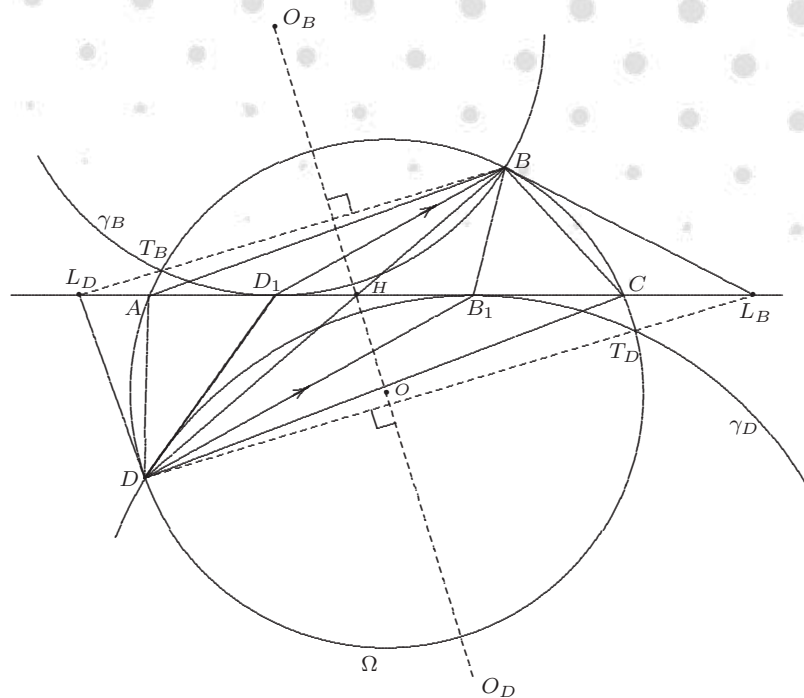


Figure 2: O, O_B, O_D are collinear.

which means that the triangle $L_B B B_1$ is isosceles, $L_B B = L_B B_1$. The powers of L_B with respect to Ω and γ_D are $L_B B^2$ and $L_B B_1^2$, respectively; so they are equal, whence L_B lies on the radical axis $T_D D$ of those two circles. Similarly, L_D lies on the radical axis $T_B B$ of Ω and γ_B .

By sine rule in the triangle $B H L_B$, we obtain

$$\frac{H L_B}{\sin \angle H B L_B} = \frac{B L_B}{\sin \angle B H L_B} = \frac{B_1 L_B}{\sin \angle B H L_B}; \quad (7.2)$$

similarly,

$$\frac{H L_D}{\sin \angle H D L_D} = \frac{D L_D}{\sin \angle D H L_D} = \frac{D_1 L_D}{\sin \angle D H L_D}. \quad (7.3)$$

Clearly, $\angle B H L_B = \angle D H L_D$. In the circle Ω , tangent lines $B L_B$ and $D L_D$ form equal angles with the chord $B D$, so $\sin \angle H B L_B = \sin \angle H D L_D$. (This equality does not depend on the picture). Thus dividing (7.2) by (7.3), we get $\frac{H L_B}{H L_D} = \frac{B_1 L_B}{D_1 L_D}$. Hence, $\frac{H L_B}{H L_D} = \frac{H L_B - B_1 L_B}{H L_D - D_1 L_D} = \frac{H B_1}{H D_1}$. Since $h(D_1) = B_1$, the obtained relation yields $h(L_D) = L_B$, so h maps the line $L_D B$ to $L_B D$, and these lines are parallel as desired.

8. Let $ABCD$ be a quadrilateral inscribed in a circle Ω . Let the tangent to Ω at D intersect the rays BA and BC at points E and F , respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

Solution. Let the segments TE and TF cross AC at P and Q , respectively. Since $PE \parallel CD$ and ED is tangent to the circumcircle Ω of $ABCD$, we have

$$\angle EPA = \angle DCA = \angle EDA,$$

and so the points A, P, D and E lie on some circle α . Similarly, the points C, Q, D and F lie on some circle γ .

We now want to prove the line DT is tangent to both α and γ at D . Indeed, since $\angle FCD + \angle EAD = 180^\circ$, the circles α and γ are tangent to each other at D . To prove that T lies on their common tangent line at D (i.e. on their radical axis), it suffices to check that $TP \cdot TE = TQ \cdot TF$, or that the quadrilateral $PEFQ$ is cyclic. This fact follows from

$$\angle QFE = \angle ADE = \angle APE.$$

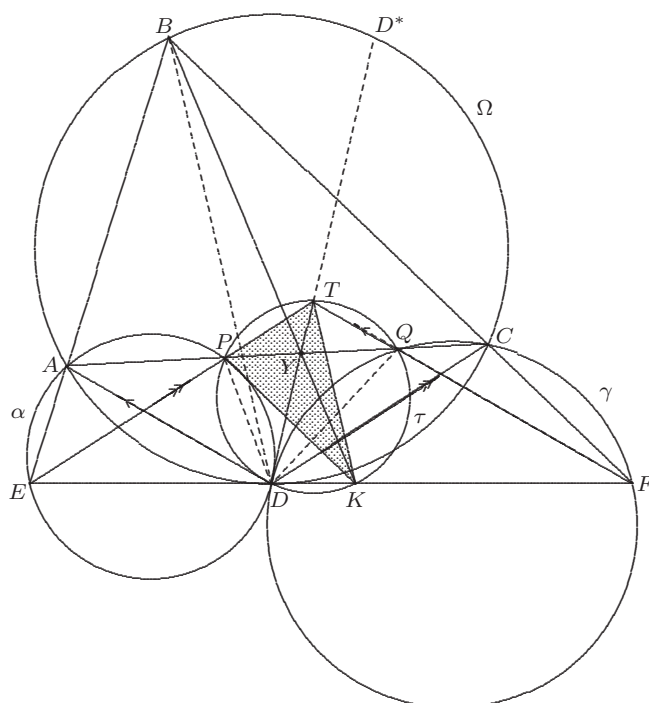


Figure 3: AC, DT and BK are concurrent.

Since $TD = TK$, we have $\angle TKD = \angle TDK$. Next, as TD and DE are tangent to α and Ω , respectively, we obtain

$$\angle TKD = \angle TDK = \angle EAD = \angle BDE,$$

which implies $TK \parallel BD$.

Next we prove that the five points T, P, Q, D and K lie on some circle τ . Indeed, since TD is tangent to the circle α we have

$$\angle EPD = \angle TDF = \angle TKD,$$

which means that the point P lies on the circle (TDK) . Similarly, Q lies on the circle (TDK) .

Finally, we prove that $PK \parallel BC$. Indeed, using the circles τ and γ we conclude that

$$\angle PKD = \angle PQD = \angle DFC,$$

which means $PK \parallel BC$.

Triangles TPK and DCB have pairwise parallel sides, which by the converse of Desargues's theorem implies that TD, PC and KB are concurrent, as desired.

9. Given a positive integer n , find the smallest value of $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor$ over all permutations (a_1, a_2, \dots, a_n) of $(1, 2, \dots, n)$.

Answer: The minimum of such sums is $\lfloor \log_2 n \rfloor + 1$; so if $2^k \leq n < 2^{k+1}$, the minimum is $k + 1$.

Solution. Suppose $2^k \leq n < 2^{k+1}$ with some nonnegative integer k . First we show a permutation (a_1, a_2, \dots, a_n) such that $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor = k + 1$; then we will prove that $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor \geq k + 1$ for every permutation. Hence the minimal possible value will be $k + 1$.

Consider the permutation

$$(a_1) = (1), \quad (a_2, a_3) = (3, 2), \quad (a_4, a_5, a_6, a_7) = (7, 4, 5, 6), \quad \dots$$

$$(a_{2^k-1}, \dots, a_{2^k-1}) = (2^k - 1, 2^{k-1}, 2^{k-1} + 1, \dots, 2^k - 2),$$

$$(a_{2^k}, \dots, a_n) = (n, 2^k, 2^k + 1, \dots, n - 1).$$

This permutation consists of $k + 1$ cycles. In every cycle $(a_p, \dots, a_q) = (q, p, p + 1, \dots, q - 1)$, we have $q < 2p$, so

$$\sum_{i=p}^q \left\lfloor \frac{a_i}{i} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor + \sum_{i=p+1}^q \left\lfloor \frac{i-1}{i} \right\rfloor = 1.$$

The total sum over all cycles is precisely $k + 1$.

Now we prove the lower bound.

Lemma. $\left\lfloor \frac{a}{b} \right\rfloor \geq \log_2 \frac{a+1}{b}$, for every pair a, b of positive integers.

Proof. Let $t = \left\lfloor \frac{a}{b} \right\rfloor$, so $t \leq \frac{a}{b}$ and $\frac{a+1}{b} \leq t+1$. By applying the inequality $2^t \geq t+1$, we obtain

$$\left\lfloor \frac{a}{b} \right\rfloor = t \geq \log_2(t+1) \geq \log_2 \frac{a+1}{b}.$$

By applying the lemma to each term, we get

$$\sum_{i=1}^n \left\lfloor \frac{a_i}{i} \right\rfloor \geq \sum_{i=1}^n \log_2 \frac{a_i+1}{i} = \sum_{i=1}^n \log_2(a_i+1) - \sum_{i=1}^n \log_2 i.$$

Notice that the numbers $a_1+1, a_2+1, \dots, a_n+1$ form a permutation of $2, 3, \dots, n+1$. Hence in the last two sums all terms cancel out, except for $\log_2(n+1)$ in the first sum and $\log_2 1 = 0$ in the second sum. Therefore,

$$\sum_{i=1}^n \left\lfloor \frac{a_i}{i} \right\rfloor \geq \log_2(n+1) > k.$$

As the left-hand side is an integer, it must be at least $k+1$.

10. Let n and k be two integers with $n > k \geq 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ neighbours - namely, the k students closest to S on the right, and the k students closest to S on the left.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.

Solution. We replace the girls by 1's and the boys by 0's, getting the numbers $(a_1, a_2, \dots, a_{2n+1})$ arranged in a circle. We extend this sequence periodically by letting $a_{2n+1+k} = a_k$ for all $k \in \mathbb{Z}$. We get an infinite periodic sequence

$$\dots, a_1, a_2, \dots, a_{2n+1}, a_1, a_2, \dots, a_{2n+1}, \dots$$

Consider the numbers $b_i = a_i + a_{i-k-1} - 1 \in \{-1, 0, 1\}$ for all $i \in \mathbb{Z}$. We know that

$$b_{m+1} + b_{m+2} + \dots + b_{m+2n+1} = 1, \quad m \in \mathbb{Z}. \quad (10.1)$$

In particular, this yields that there exists some i_0 with $b_{i_0} = 1$. Now we want to find an index i such that

$$b_i = 1 \quad \text{and} \quad b_{i+1} + b_{i+2} + \dots + b_{i+k} \geq 0. \quad (10.2)$$

This will imply $a_i = 1$ and

$$(a_{i-k} + a_{i-k+1} + \dots + a_{i-1}) + (a_{i+1} + a_{i+2} + \dots + a_{i+k}) \geq k;$$

as desired.

Suppose to the contrary, that for every index i with $b_i = 1$ the sum $b_{i+1} + b_{i+2} + \dots + b_{i+k}$ is negative. We start from some index i_0 with $b_{i_0} = 1$ and construct a sequence i_0, i_1, i_2, \dots , where i_j ($j > 0$) is the smallest possible index such that $i_j > i_{j-1} + k$ and $b_{i_j} = 1$. We can choose two numbers among $i_0, i_1, \dots, i_{2n+1}$ which are congruent modulo $2n+1$ (without loss of generality, we may assume that these numbers are i_0 and i_T).

On the one hand, for every j with $0 \leq j \leq T-1$ we have

$$S_j := b_{i_j} + b_{i_j+1} + b_{i_j+2} + \dots + b_{i_{j+1}-1} \leq b_{i_j} + b_{i_j+1} + b_{i_j+2} + \dots + b_{i_j+k} \leq 0$$

since $b_{i_j+k+1}, \dots, b_{i_{j+1}-1} \leq 0$. On the other hand, since $(2n+1) \mid (i_T - i_0)$, from (10.1) we deduce that

$$S_0 + \dots + S_{T-1} = \sum_{i=i_0}^{i_T-1} b_i = \frac{i_T - i_0}{2n+1} > 0.$$

This contradiction finishes the solution.