International Mathematical Olympiad

English (eng), day 1

Monday, 19. July 2021

Problem 1. Let $n \geqslant 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Problem 2. Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots, x_n$.

Problem 3. Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

English (eng), day 2

Tuesday, 20. July 2021

Problem 4. Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Problem 5. Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Problem 6. Let $m \geqslant 2$ be an integer, A be a finite set of (not necessarily positive) integers, and $B_1, B_2, B_3, \ldots, B_m$ be subsets of A. Assume that for each $k = 1, 2, \ldots, m$ the sum of the elements of B_k is m^k. Prove that A contains at least $m/2$ elements.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Solutions

1. Let $n > 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution. The following lemma can be proved using induction on n.

Lemma. Given an integer $n \geq 100$, there exists an integer $x \geq 9$ such that

$$x^2 + 2x \leq n \leq 2x^2 - 4x.$$

By the lemma, there exists an integer $x \geq 9$ such that

$$n \leq 2x^2 - 4x < 2x^2 + 1 < 2x^2 + 4x \leq 2n.$$

So the cards $3x^2 - 4x, 2x^2 + 1, 2x^2 + 4x$ exist. By pigeonhole principle, two of them are in the same pile. One can check that any two of them add up to a perfect square: $(2x^2 - 4x) + (2x^2 + 1) = (2x - 1)^2, (2x^2 + 1) + (2x^2 + 4x) = (2x + 1)^2, (2x^2 + 4x) + (2x^2 - 4x) = (2x)^2$.

2. Show that the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots, x_n$.

Solution. If we add t to all the variables then the left-hand side remains constant and the right-hand side becomes

$$H(t) := \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|2t + x_i + x_j|}.$$

Let T be large enough such that both $H(-T)$ and $H(T)$ are larger than the value L of the left-hand side of the inequality we want to prove. Not necessarily distinct points $p_{i,j} := -(x_i + x_j)/2$ together with T and $-T$ split the real line into segments and two rays such that on each of these segments and rays the function $H(t)$ is concave since $f(t) := \sqrt{|\ell + 2t|}$ is concave on both intervals $(-\infty, -\ell/2]$ and $[-\ell/2, \infty)$. Let $[a, b]$ be the segment containing zero. Then concavity implies $H(0) \geq \min H(a), H(b)$ and, since $H(\pm T) > L$, it suffices to prove the inequalities $H(-(x_i + x_j)/2) \geq L$, that is to prove the original inequality in the case when all numbers are shifted in such a way that two variables x_i and x_j add up to zero. In the following we denote the shifted variables still by x_i.

If $i = j$, i.e. $x_i = 0$ for some index i, then we can remove x_i which will decrease both sides by $2\sum_k \sqrt{|x_k|}$. Similarly, if $x_i + x_j = 0$ for distinct i and j we can remove both x_i and x_j which decreases both sides by

$$2\sqrt{2|x_i|} + 2 \cdot \sum_{k \neq i,j} \left(\sqrt{|x_k + x_i|} + \sqrt{|x_k + x_j|} \right).$$

In either case we reduce our inequality to the case of smaller n. It remains to note that for $n = 0$ and $n = 1$ the inequality is trivial.

3. Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

   Solution. Let Q be the isogonal conjugate of D with respect to the triangle ABC. Since $\angle BAD = \angle DAC$, the point Q lies on AD. Then $\angle QBA = \angle DBC = \angle FDA$, so thepoi nts Q, D, F, and B are concyclic. Analogously, the points Q, D, E, and C are concyclic. Thus $AF \cdot AB = AD \cdot AQ = AE \cdot AC$ and so the points B, F, E, and C are also concyclic.

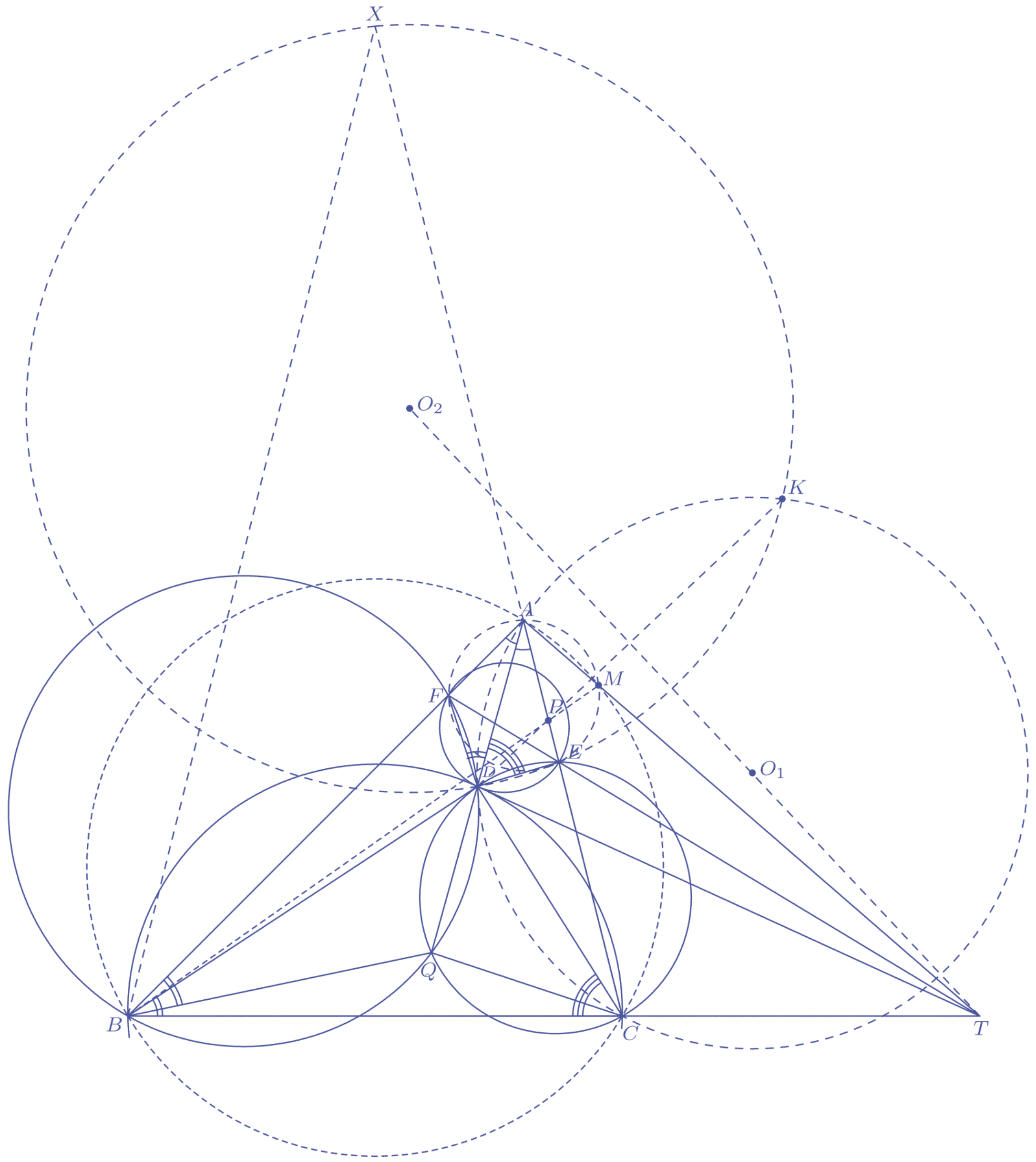

Figure 1: BC, EF, and O_1O_2 are concurrent.

Let T be the intersection of BC and FE.

Claim. $TD^2 = TB \cdot TC = TF \cdot TE$.

Proof. We will prove that the circles (DEF) and $(BDC)$are tangent to each other. Indeed, using the above arguments, we get

$$\begin{aligned}&\angle BDF = \angle AFD - \angle ABD = (180^\circ - \angle FAD - \angle FDA) - (\angle ABC - \angle DBC)\\&= 180^\circ - \angle FAD - \angle ABC = 180^\circ - \angle DAE - \angle FEA = \angle FED + \angle ADE\\&= \angle FED + \angle DCB,\end{aligned}$$

which implies the desired tangency.

Since the points B, C, E, and F are concyclic, the powers of the point T with respect to the circles (BDC) and (EDF) are equal. So their radical axis, which coincides with the common tangent at D, passes through T, and hence $TD^2 = TE \cdot TF = TB \cdot TC$. □

Let TA intersect the circle (ABC) again at M. Due to the circles $(BCEF)$ and $(AMCB)$, and using the above Claim, we get $TM \cdot TA = TF \cdot TE = TB \cdot TC = TD^2$; in particular, the points A, M, E, and F are concyclic.

Under the inversion with centre T and radius TD, the point M maps to A, and B maps to C, which implies that the circle (MBD) maps to the circle (ADC). Their common point D lies on the circle of the inversion, so the second intersection point K also lies on that circle, which means $TK = TD$. It follows that the point T and the centres of the circles (KDE) and (ADC) lie on the perpendicular bisector of KD.

Since the center of (ADC) is O_1, it suffices to show now that the points D, K, E, and X are concyclic (the center of the corresponding circle will be O_2).

The lines BM, DK, and AC are the pairwise radical axes of the circles $(ABCM), (ACDK)$ and $(BMDK)$, so they are concurrent at some point P. Also, M lies on the circle (AEF), thus

$$\begin{aligned}&\measuredangle(EX, XB) = \measuredangle(CX, XB) = \measuredangle(XC, BC) + \measuredangle(BC, BX) = 2\measuredangle(AC, CB)\\&= \measuredangle(AC, CB) + \measuredangle(EF, FA) = \measuredangle(AM, BM) + \measuredangle(EM, MA) = \measuredangle(EM, BM),\end{aligned}$$

so the points M, E, X, and B are concyclic. Therefore, $PE \cdot PX = PM \cdot PB = PK \cdot PD$, so the points E, K, D, and X are concyclic, as desired.

4. Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Solution. We refer to the configuration in the figure.

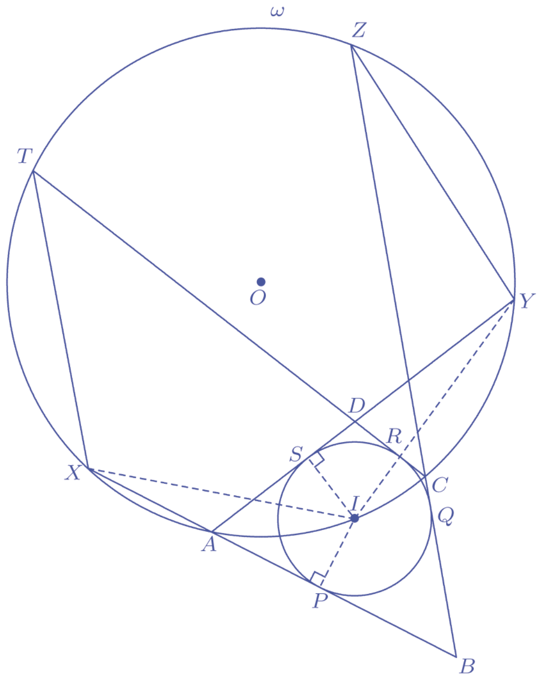

Figure 2: $AD + DT + TX + XA = CD + DY + YZ + ZC$.

The point I is the intersection of the external bisector of $\angle TCZ$ with the circumcircle ω of the $\triangle TCZ$, so I is the midpoint of $T\widehat{C}Z$ and $IT = IZ$. Similarly, I is the midpoint of $Y\widehat{A}X$ and $IX = IY$. Let O be the centre of ω. Then X and T are the reflections of Y and Z in IO, respectively. So $XT = YZ$.

Let the incircle of $ABCD$ touch AB, BC, CD, and DA at points P, Q, R, and S, respectively.

The right-angled triangles IXP and IYS are congruent, since $IP = IS$ and $IX = IY$. Similarly, the right-angled triangles IRT and IQZ are congruent. Therefore, $XP = YS$ and $RT = QZ$.

Since $AS = AP$, $CQ = RC$, and $SD = DR$, we have

$XT + XA + (AS + SD) + DT = XT + XP + RT = YZ + YS + QZ = YZ + YD + (DR + RC) + CZ$ as required.

5. Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k. Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Solution. We shall use the following notation. Before the k-th move each walnut a with $a < k$ is called *used*, and every walnut b with $b \geq k$ is called *active*. Thus, at the beginning of the process, all walnuts are active, and on each move exactly one walnut changes its state from being active to being used. We argue indirectly. That is, on the

k-th move, the squirrel swaps either two walnuts $a, b < k$, or two walnuts $a, b > k$. In the former case we say that walnut k (and the number k) are *large*, otherwise they are *small*. Clearly, 1 is small, while 2021 is large.

At any moment in the process, the used walnuts are split into several groups consisting of one or more contiguous used walnuts each; different groups are separated by active walnuts. We prove by induction on $1 \le k \le 2020$ that, after the k-th move, all groups of used walnuts have odd sizes. The base case $k = 1$ is obvious. To prove the step, consider the current, k-th, move. Two cases are possible:

Case 1: k is small.

In this case, both neighbours of walnut k remain active, so k forms a separate group.

Case 2: k is large.

In this case, the neighbours of k are both used, so they belong to two groups containing, say, p and q walnuts, respectively (both p and q are odd). Now, when k becomes used, those two groups merge into a single group consisting of an odd number of walnuts-namely, $p + q + 1$.

Now, after the 2020-th move, the 2020 used walnuts should form several groups of odd sizes. However, they in fact form just one group of size 2020. This is a contradiction.

6. Let $m > 2$ be an integer, A be a finite set of (not necessarily positive) integers, and $B_1, B_2, B_3, \ldots, B_m$ be subsets of A. Assume that for each $k = 1, 2, \ldots, m$ the sum of the elements of B_k is mk. Prove that A contains at least $m/2$ elements.

Solution. Let $A = \{a_1, \ldots, a_k\}$. Assume that, on the contrary, $k = |A| < m/2$. Let

$$s_i := \sum_{j : a_j \in B_i} a_j$$

be the sum of elements of B_i. We are given that $s_i = m^i$ for $i = 1, \ldots, m$.

Now consider all m^m expressions of the form

$$f(c_1, \ldots, c_m) := c_1 s_2 + c_2 s_2 + \cdots + c_m s_m, \; c_i \in \{0, 1, \ldots, m-1\} \text{ for all } i = 1, 2, \ldots m.$$

Note that every number $f(c_1, \ldots, c_m)$ has the form

$$\alpha_1 a_1 + \cdots + \alpha_k a_k, \quad \alpha_i \in \{0, 1, \ldots, m(m-1)\}.$$

Hence, there are at most $(m(m-1)+1)^k < m^{2k} < m^m$ distinct values of our expressions; therefore, at least two of them coincide.

Since $s_i = m^i$, this contradicts the uniqueness of representation of positive integers in the base-m system.