Slithering Graphs

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Abstract

Slither is a 2 player game where the game starts with a square grid of $m \times n$ vertices. On the first player's first turn, they connect any two vertically or horizontally adjacent vertices with an edge. Afterwards, both players take turns connecting two adjacent vertices where exactly one of the vertices is connected by another edge. Failure to do so will result in that player losing. However, each vertex may not have more than 2 edges connected to it. This project aims to investigate the winning strategies of Slither for both the original and new variations of *Slither* we created, and to provide mathematical proof for the strategies we found. Graph theory with the assistance of diagrams was used to verify and proof findings obtained. It was found that the winning strategy for most variations of Slither revolved around the idea of maximum matching, and the construction for maximum matching was proven using induction. The winning strategies of proposed variations of the game has been solved with rigorous proof.

1 Introduction

1.1 Terminology

Term	Definition
$G_{(m,n)}$	Finite Graph G with a m x n array of vertices
$v_{(x,y)}$	Vertex with coordinates x, y and $v_{(x,y)} \in G_{(m,n)}$
$v_{(x,y)}$ *is exposed*	$v_{(x,y)}$ has a degree of 0 (*i.e. no edge is incident upon* v_i)
$v_{(x,y)}$ *is covered*	$v_{(x,y)}$ has a degree more than 0 (*i.e. at least one edge is incident upon* v_i)
$M_{(m,n)}$	Maximum matching of graph $G_{(m,n)}$
v_i *and* v_j *are adjacent*	v_i is horizontally or vertically right next to v_j (i.e. if i=(x,y), j= (x-1,y), (x+1,y), (x,y-1) or (x,y+1))
v_i — v_j	An edge is incident upon v_i and v_j
snake	A sequence of edges in $G_{(m,n)}$ formed by the players in the process of the game

1.2 Description of Slither

Slither is a combinatorial game created by D. Silverman, a mathematician. The game takes its name from the snaking shape of the line formed by a path of edges chosen by the two players. Slither is played on a finite graph $G_{(m,n)}$ *of* m x n vertices which means that the game will eventually end (i.e. there are no draws). An example of a graph is shown below in Fig. 1-1.

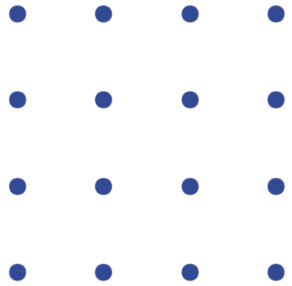

Fig. 1-1: Graph $G_{(4,4)}$

At the start of the game, the first player chooses two vertices to form an edge. Players then take turns extending the snake by joining an adjacent vertex from a vertex of either end with an edge. The snake cannot overlap itself, and the player who runs out of possible moves first loses. As can be seen in Figure 1-2 below, the snake does not overlap itself.

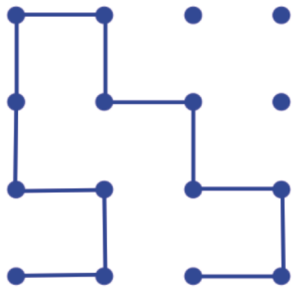

Fig. 1-2: Example of path on $G_{(4,4)}$

We make use of a cartesian graph system to name the vertices, and the distance between adjacent vertices is 1. We always set the origin at the bottom left corner vertex, with an example shown in Figure 1-3.

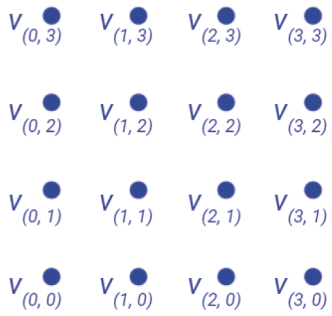

Fig. 1-3: Figure of labelled vertices in $G_{(4,4)}$

2 Literature Review

In his writings for the *Scientific American*, *Martin Gardner (1972)* introduced us to the game of Slither and its rules, and discussed the origins of the game, which was developed and named by David Silverman. Though he was unable to develop a full strategy to Slither, he discussed the possible different cases in Slither, and showed us the existence of the strategy for the game of Slither. His works served as a basis to the strategy of Slither.

William N. Anderson, Jr. (1973) developed a hypothetical strategy of Slither, and argued that why one player will win the game, if he/she knows the strategy, but did not show us how that player will win. He gave us the understanding that just like many other combinatorial games, the projected outcome of the game depends on which of the two players start first. Moreover, he discusses the relevance of maximum matching in the game of Slither, and how the strategy of the game is in line with the idea of maximum matching, which is very relevant to our project.

3 Objectives and Research Questions

The objectives of this project are to understand the mathematical solution behind the game of Slither, and investigate the effect of parameters on the game of Slither, through designing and discovering the proofs to certain variations of Slither.

The research questions are as follows:

1. What is the strategy to the original form of Slither?
2. What is the strategy to the variations of Slither?
3. What is the mathematical proof to both the original version and new versions of Slither?

4 Results

4.1 Research Question 1

What is the strategy to the original form of Slither?

The strategy to the original form of Slither involves maximum matching. A maximum matching $M_{(m,n)}$ of graph $G_{(m,n)}$ is a set of edges incident upon vertices part of the graph such that each vertice is covered and has a degree of 1.

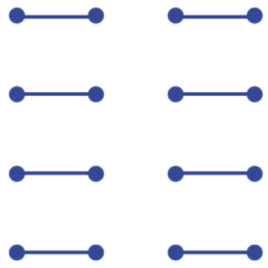

Fig. 4-1: Maximum matching in a graph with an even number of vertices

If the graph has an odd number of vertices, one vertice is excluded from the maximum matching.

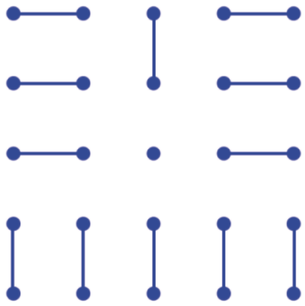

Fig. 4-2: Maximum matching in a graph with an odd number of vertices

Maximum matching is crucial to the strategy to win the game. It is visualised at the start of the player’s move and edges that are part of the maximum matching represent possible moves that player can make in response to their opponents’.

When a player visualises a maximum matching, one edge has a vertex part of the snake. For player 1, any vertex can be chosen to be part of the snake. For player 2, one of the joined vertices will be chosen to be part of the maximum matching (ignoring the existing edge). Hence, at the end of the player's move, any edge part of the maximum matching has either both vertices as part of the snake or no vertices as part of the snake.

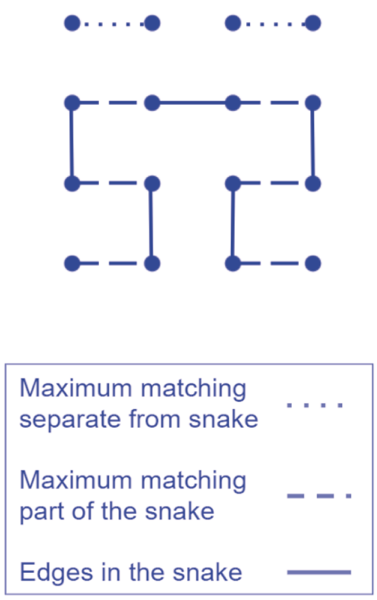

Fig. 4-3: Example of edges of maximum matching included/excluded as part of the snake

Hence, when the opponent makes the move, one vertex part of the maximum matching not included in the snake will be connected to the snake, and the player can follow-up by joining the edge the included vertex was part of in the maximum matching.

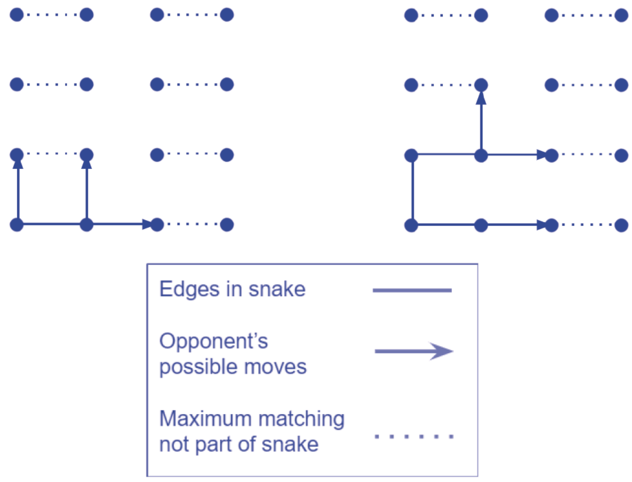

Fig 4-4 and 4-5: Examples showing how maximum matching can enable a player to win

This is because the new vertex is always part of the maximum matching by definition. Hence, the player can follow the maximum matching to make an appropriate move.

The player who is able to utilise the maximum matching will always have a follow-up move, and will not run out of moves before the opponent. As the player who can utilise the maximum matching depends on the parity of mn, it affects who is guaranteed to win as well. When mn is even, Player 1 wins. When mn is odd, player 2 wins. The proofs when mn is even and when mn is odd can be found in Appendix 8.1 and 8.2 respectively.

4.2 Research Question 2

What is the strategy to the variations of Slither?

4.2.1 Private Slither

Private Slither is a variation of Slither where each player has a separate snake. In each player's initial move, an edge will be chosen such that the two initial edges are not incident upon the same vertex. Each player's subsequent move involves extending his/her own snake, where the initial edge he/she has chosen is part of the snake. The snake formed by extending each of these edges cannot meet or intersect at any point of the game.

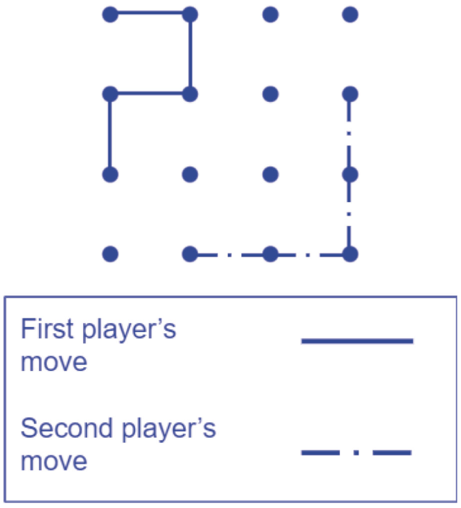

Fig 4-6: Example of private slither

The figure above shows an example of snakes formed by each player after some moves in the game of Private Slither, where the edges chosen by the first and second players are denoted by p_1 and p_2 respectively.

The separation of vertices allows the restriction of the opposing player's territory. Since it is a turn-based game, the player with the largest amount of territory is thus allowed for the most number of moves, and will not run out of available moves.

Because of this, the best strategy is to divide the board into half at the start of the match. There are three possible cases, based on the parity of m, n.

1. m, n is odd

Player 1 wins by dividing the board into half and blocking Player 2.

2. m or n is odd while the other is even

Player 1 wins by dividing the board into half and blocking Player 2.

3. m, n is even

Player 2 wins by mirroring Player 1's moves.

4.2.2 Public Slither

Public Slither is when there are two snakes, the initial edges cannot be incident upon the same vertex, and the two snakes cannot meet or intersect, similar to Private Slither. However, the difference is that after taking turns placing down the initial edges, the two players can take turns extending either snake.

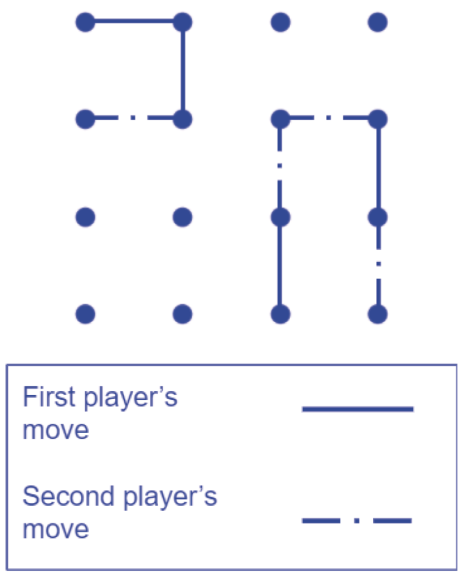

Fig 4-7: Example of Public Slither

Public Slither takes on the qualities of the Original Slither, and maximum matching can be applied. However, the splitting of one snake into two will reduce the number of exposed vertices by one after the second move, and thus will have the reverse of the results of the Original Slither.

By applying maximum matching, there are two cases.

1. Number of vertices is even

Second player wins by applying maximum matching.

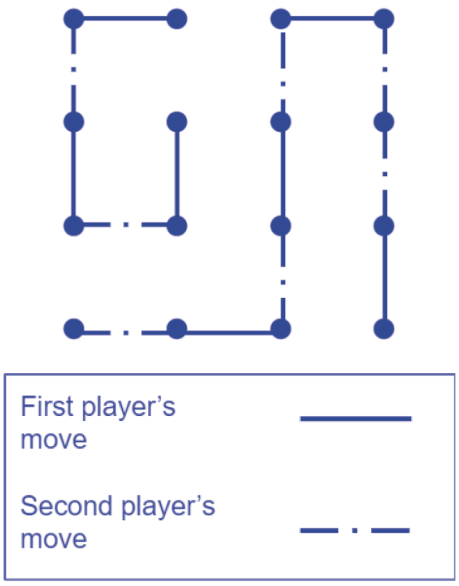

Fig. 4-8: Example of Public Slither with Even Number of Vertices

2. Number of vertices is odd

First Player wins by applying maximum matching.

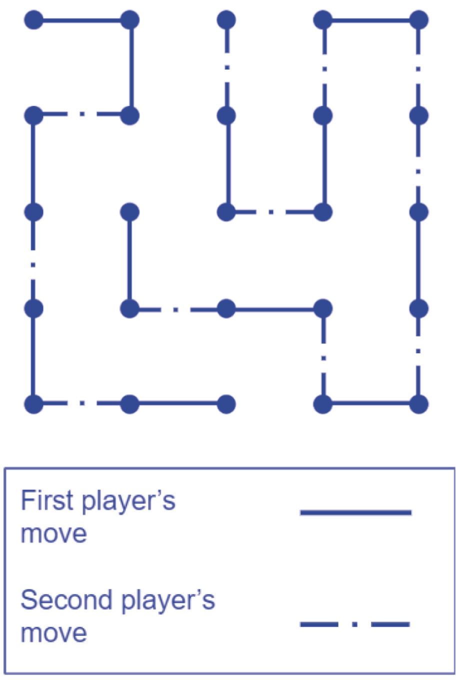

Fig. 4-9: Example of Public Slither with Odd Number of Vertices

4.2.3 3D Slither

Three-dimensional Slither mimics the Original Slither in terms of game rules. The only difference is that this form of Slither is played on a finite three-dimensional array of m x n x p vertices, and the snake can be extended heightwise, lengthwise or widthwise.

Three-dimensional Slither has a graph that takes on the qualities of multiple congruent two-dimensional graphs stacked upon each other and thus maximum matching can be applied.

By applying maximum matching, there are two cases.

1. Number of vertices is even

First player wins by applying maximum matching.

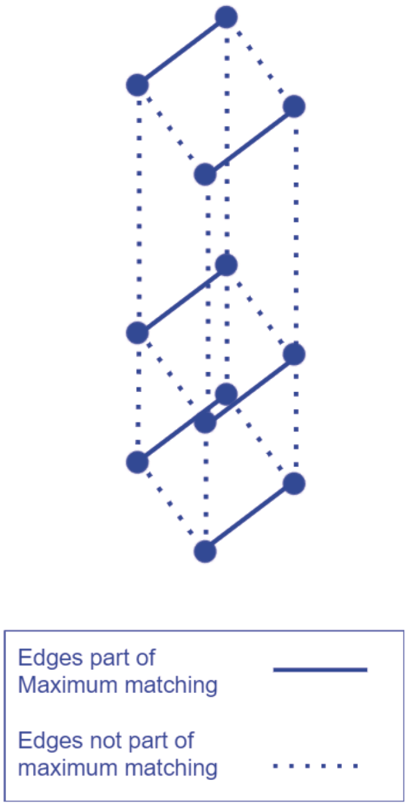

Fig 4-10 : Example of a Three-Dimensional Slither with an even number of vertices

2. Number of vertices is odd

Second Player wins by applying maximum matching.

4.2.4 Triangle Slither

Triangle Slither is played on an array of vertices arranged in the shape of an equilateral triangle with at least 3 vertices. Vertices can be connected in six directions: left-up, left-middle, left-down, right-up, right-middle and right-down.

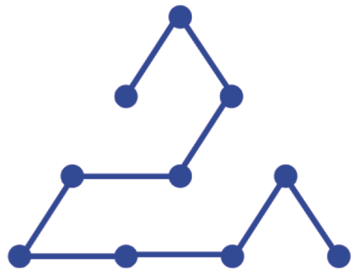

Fig 4-11 : Example of Triangle Slither

Triangle Slither can also be solved using maximum matching, just in three directions. (This is because of the six directions, there are three pairs of directions that have exactly the same qualities: left-up and right-down, left-middle and right-middle, left-down and right-up.)

By applying this new form of maximum matching, there are two cases.

1. Number of vertices is even

First player wins by maximum matching.

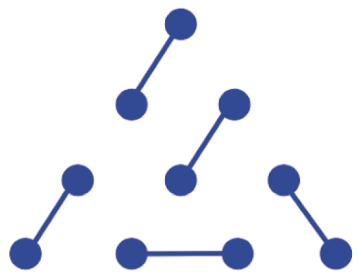

Fig. 4-12: Maximum matching with an even number of vertices

2. Number of vertices is odd

Second player wins by maximum matching.

Fig. 4-13: Maximum matching with an odd number of vertices

4.3 Research Question 3

What is the mathematical proof to both the original version and new version of Slither?

4.3.1 Proof to Original Form of Slither

In Research Question 1, we concluded that the player who will be able to utilise maximum matching is dependent on the parity of the number of vertices, mn in graph $G_{(m,n)}$. We hence claimed that when mn is even, the first player will win and when mn is odd, the second player will win. In addition, we have discussed and proven the viability of maximum matching as a strategy in Slither. At this juncture, we need to prove that for all m, n, there would always exist $M_{(m,n)}$ for $G_{(m,n)}$.

We will prove that there would always exist $M_{(m,n)}$ for $G_{(m,n)}$ for all m, n in two cases, the first being when mn is even and the second being when mn is odd.

When mn is even, we can prove that there would always exist $M_{(m,n)}$ for $G_{(m,n)}$ for all m, n through a construction that exists for all m, n.

The proof and the construction for the case when mn is even can be found in Appendix 8.1.

When mn is odd, consider the following two trivial cases: $(m,n)=(1,3)$ and $(m,n)=(3,1)$

Fig 4-14: Trivial cases where mn is odd

In both figures shown above, the second player can simply cover the last exposed vertex by connecting the path to it and win the game.

For the remaining cases ($m,n \geq 3$), we will prove that there would always exist $M_{(m,n)}$ for $G_{(m,n)}$ for the second player, independent of the first player's initial move, through induction. In our induction proof, we first prove that the base case, which is $G_{(3,3)}$ always has a maximum matching $M_{(3,3)}$. Then, we will prove in our inductive step that there always exists $M_{(m,3)}$ for $G_{(m,3)}$, and repeat a similar induction process that there always exists $M_{(m,n)}$ for $G_{(m,n)}$.

The full details of the induction proof can be found in Appendix 8.2.

4.3.2 Proof to Public Slither

In Research Question 2, we have suggested that the results obtained for Public Slither would be opposite of that of the original form of Slither, because given the same number of moves carried out and the same number of vertices in each of the two games, there will always be one more exposed vertex in the game of the original form of Slither as compared to Public Slither.

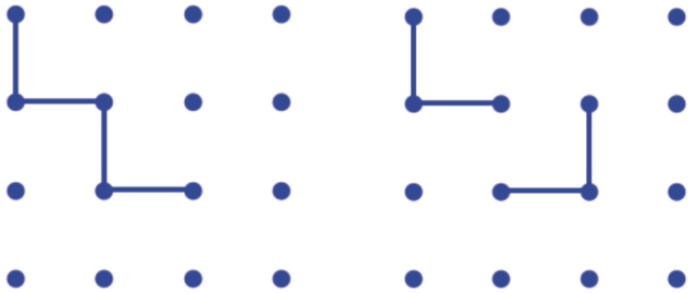

Fig 4-15 and 4-16 : Example of Slither compared to Public Slither, where there is one more exposed vertex in the game of Slither compared to Public Slither

We hence made the claim that the second player would win if mn is even and the first player would win if mn is odd. We need to prove that there would always exist $M_{(m,n)}$ for $G_{(m,n)}$ for all m, n in the game of Public Slither.

We will prove our claims in two cases, the first being when mn is even and the second being when mn is odd.

When mn is even, our proof statement can be rewritten as the need to prove that regardless of the first player's initial chosen edge, say e_1, there would always exist $M_{(m,n)}$ such that $e_1 \in M_{(m,n)}$ so that the second player, whom we project to win, can initially choose an edge e_2 such that $e_1, e_2 \in M_{(m,n)}$.

The proof and the construction for the following statement can be found in Appendix 8.3.

When mn is odd, we need to prove that after the second player's initial move, the first player can always discover a maximum matching $M_{(m,n)}$ such that $e_1 \in M_{(m,n)}$ and e_2 is incident upon the excluded vertex from $M_{(m,n)}$ so that the first player can join the other vertex e_2 is incident upon with another vertex to form $e_3 \in M_{(m,n)}$. The following can be proven by induction, in a similar but slightly different way as the induction proof in Appendix 8.2. The full proof for this case can be found in Appendix 8.4.

4.3.3 Proof to 3D Slither

In Research Question 2, we discussed how 3-Dimensional Slither on a graph $G_{(m,n,p)}$ generally takes on the quality of graphs $G_{(m,n)}$ stacked on each other p times and hence, concluded that the results of 3-Dimensional Slither will be similar to that of the original form of Slither, and involves maximum matching as well. We therefore claimed that when mnp is even, the first player wins while when mnp is odd, the second player wins. In order to complete the proof, we need to show that there would always exist $G_{(m,n,p)}$ for $M_{(m,n,p)}$ for all m, n, p.

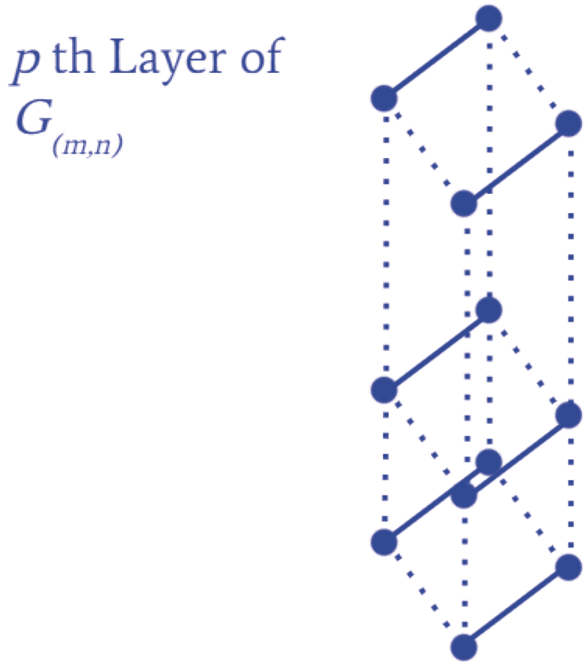

Fig 4-17 : Example of 3 Dimensional Slither showing p layers of $G_{(m,n)}$

Just like any other variation of Slither that requires maximum matching, the parity of the initial number of vertices, mnp determines which player can utilise maximum matching. Therefore, our proof of our strategy to 3-Dimensional Slither will also be divided into two cases: When mnp is even and when mnp is odd.

When mnp is even, without the loss of generality, let m be even. This suggests that mn is even and from our proof in the original form of slither, there exists $M_{(m,n)}$ for $G_{(m,n)}$, with all vertices covered. Since $G_{(m,n,p)}$, can be rewritten as p layers of $G_{(m,n)}$, there would exist a maximum matching for each of the p layers with all vertices covered, obtaining graph $G_{(m,n,p)}$ with all vertices covered, and part of the maximum matching, which concludes that there exists $M_{(m,n,p)}$ for $G_{(m,n,p)}$ when mnp is even.

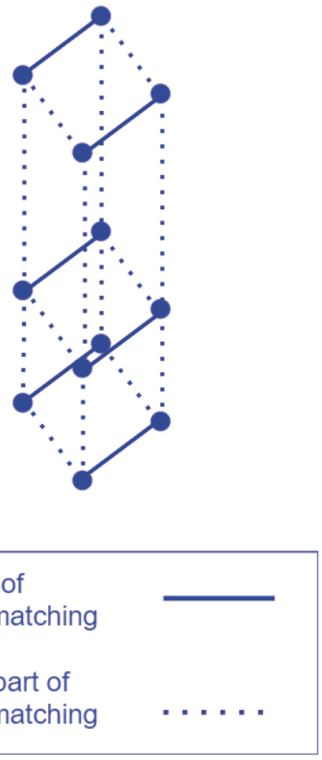

Fig 4-18 : Example that $M_{(m,n)}$ for $G_{(m,n)}$, exists with all vertices covered

When *mnp* is odd, we are unable to use a similar approach as in each layer of $M_{(m,n)}$, there will be one excluded vertex, which means that in the p layers of $M_{(m,n)}$, there will be a total of p excluded vertices, not just one excluded vertex. Therefore, we will prove that there will always exist $M_{(m,n,p)}$ for $G_{(m,n,p)}$ regardless of the first player's initial move through induction. We first prove that the base case, which is $G_{(3,3,3)}$ always has a maximum matching $M_{(3,3,3)}$. Then, we will prove in our inductive step that there always exists $M_{(m,3,3)}$ for $G_{(m,3,3)}$, and repeat a similar induction process for generalising the other two dimensions of n and p respectively to eventually prove that there will always exist $M_{(m,n,p)}$ for $G_{(m,n,p)}$ regardless of the first player's initial move.

The complete solution of the induction proof can be found in Appendix 8.5.

4.3.4 Triangle Slither

In Research Question 2, we discussed maximum matching as a strategy for Triangle Slither, with the results of a game dependent on the parity of the number of vertices in the graph. We claimed that when the number of vertices in the graph is even, the first player will win and when the number of vertices in the graph is odd, the second player will win. We need to prove that there would always exist a maximum matching for any possible triangular graph of vertices.

The number of vertices in a graph T_n is given by 1+2+3+...+n=$\frac{1}{2}n(n+1)$. In order for the number of vertices in T_n, which equals to $\frac{1}{2}n(n+1)$, to be even, n or n+1 must be divisible by 4, which concludes that when the number of vertices in T_n is even, $n \equiv 0$ or 3 (mod 4) while the number of vertices in T_n is odd if $n \equiv 1$ or 2 (mod 4).

When the number of vertices in T_n is even, $n \equiv 0$ or 3 (mod 4). In other terms, we can rewrite n as $n=4k$ and $n=4k+3$ respectively, where k is a non-negative integer. For both cases, we will prove the existence of a maximum matching through induction, with the base case of the induction proof will be $n=4$ and $n=3$ respectively.

The complete proof when the number of vertices is even can be found in Appendix 8.6.

When the number of vertices in T_n is odd, $n \equiv 1$ or 2 (mod 4). In other terms, we can rewrite n as $n=4k+1$ and $n=4k+2$ respectively, where k is a non-negative integer. For both cases, we will prove the existence of a maximum matching through induction.

The complete proof when the number of vertices is even can be found in Appendix 8.7.

5 Conclusion

For research question 1, we have discovered how maximum matching is used in the strategy to the original form of Slither, and why it is useful in preparing against any move the opponent may make. We have also determined that when mn is even, player 1 wins and when mn is odd, player 2 wins.

For research question 2, we have found the strategies to multiple variations of Slither and how these strategies work.

For research question 3, we discussed the proof that maximum matching would exist for the original form of Slither. We have also proven the existence of maximum matching in the variations of Slither which use it.

6 Future Extensions

There are three possible future extensions.

1. Allowing degree of a vertex to be greater than 2

For the forms of Slither that we investigated, none of them allowed intersection and thus the maximum degree of a vertex was limited to 2. Thus it will be interesting for us to explore a new variation where we allow the degree of a vertex to be greater than 2, as we have to find a new strategy as it can no longer involve maximum matching.

2. Allowing moving distance to be greater than 1

For the forms of Slither that we investigated, the distance between adjacent horizontal or vertical vertices was 1, and the moving distance was limited to one vertex. Thus it will be interesting for us to explore a new variation where we allow the moving distance to be more than 1, as we have to find a new strategy as it can no longer involve maximum matching.

3. Removal of a certain number of vertices

For the forms of Slither that we investigated, all of them involved complete shapes with no missing vertices in the middle. Thus it will be interesting for us to explore a new variation where we allow the removal of vertices, as we will have to apply either a whole new strategy or a different, constantly changing form of maximum matching.

7 References

1. WILLIAM N. ANDERSON JR. (1973) *Maximum Matching and the Game of Slither* Retrieved March 13, 2020 https://www.sciencedirect.com/science/article/pii/009589567490029X
2. *What is a Combinatorial Game?* American Mathematical Society Retrieved April 18, 2020 http://www.ams.org/publicoutreach/feature-column/fcarc-games2
3. Gardner, M. (1972, June). Mathematical Games. *Scientific American*, *226*(6), 117-118.

8 Appendix

8.1 Original Slither, mn is even

Without the loss of generality, let m be even. Notice that when all $v_{(2x,y)} \in G_{(m,n)}$, which satisfy the inequalities $0 \leq x \leq \frac{1}{2}(m\text{-}2)$ and $0 \leq y \leq n\text{-}1$, results in $v_{(x,y)} — v_{(2x+1,y)}$, a maximum matching $M_{(m,n)}$ is achieved for $G_{(m,n)}$ for any n and even m.

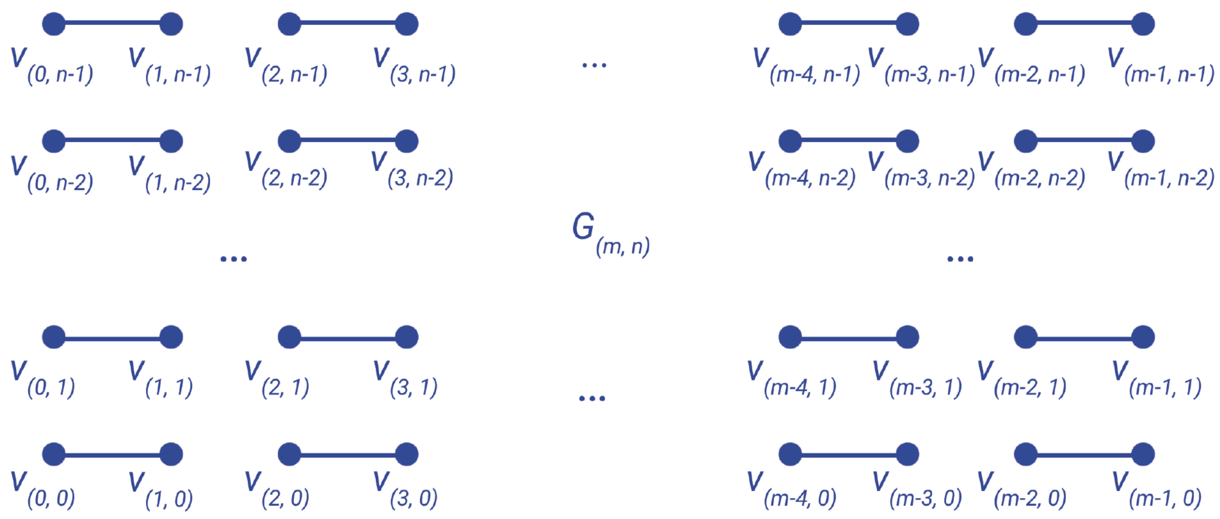

Fig. 8-1 Maximum matching construction when mn is even

8.2 Original Slither, mn is odd

We first prove that there always exists $M_{(m,3)}$ for $G_{(m,3)}$, regardless of the first player's initial move.

Consider the base case, which is $G_{(3,3)}$. We colour the adjacent vertices with different colours, say Colour 1 and Colour 2, as shown below. Note that an edge would always be incident upon two vertices of different colours. This suggests that in a maximum matching of edges, there will be an equal number of vertices coloured with each colour.

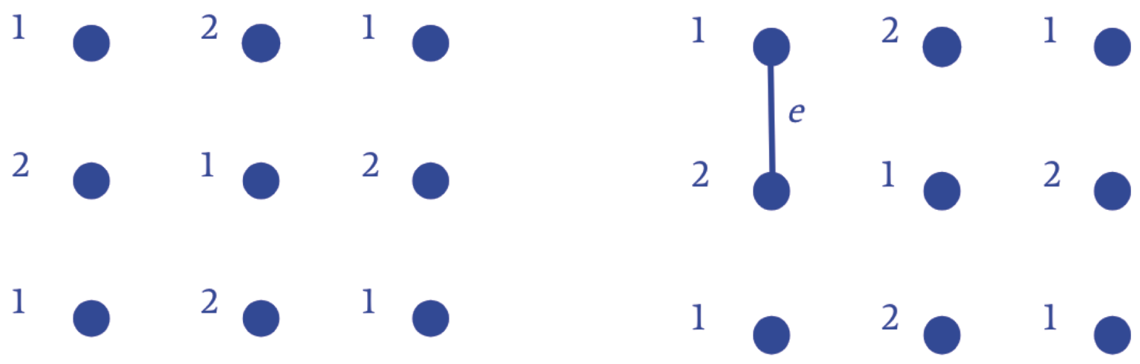

Fig. 8-2 (left) Colouring in $G_{(3,3)}$
Fig. 8-3 (right) Edge incidents upon two different coloured vertices

Based on the colouring shown above, we have a total of 5 vertices of colour 1 and 4 vertices of Colour 2. Therefore, in order to find $M_{(3,3)}$ for $G_{(3,3)}$, which requires the exclusion of one vertex from $G_{(3,3)}$, we need to exclude the vertex with Colour 1 that the edge the first player chose as his/her initial move is incident upon.

Note that the excluded vertex is either one of the corner vertex or the vertex in the centre of $G_{(3,3)}$. Fig. 8-4 shows $M_{(3,3)}$ when the central is excluded, and Fig. 8-5 shows $M_{(3,3)}$ when one of the corner vertices is excluded. (Note that the construction for excluding any of the other three can be done by rotation)

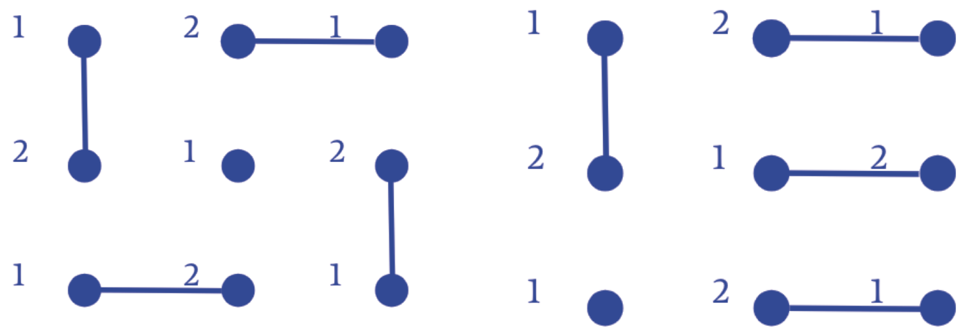

Fig. 8-4 (left) Maximum matching with central vertex excluded
Fig. 8-5 (right) Maximum matching with corner vertex excluded

This completes the base case, which is that there always exists $M_{(3,3)}$ for $G_{(3,3)}$.

We make the assumption that when $m=2k+1$ ($k \geq 1$), there always exists $M_{(2k+1,3)}$ for $G_{(2k+1,3)}$, regardless of the first player's initial move.

Consider $m=2k+3$. (Note that $m \geq 5$ since $k \geq 1$)

We can divide the graph $G_{(2k+3,3)}$ into two subgraphs $G_{(2k+1,3)}$ and $G_{(2,3)}$ in two different ways as shown below.

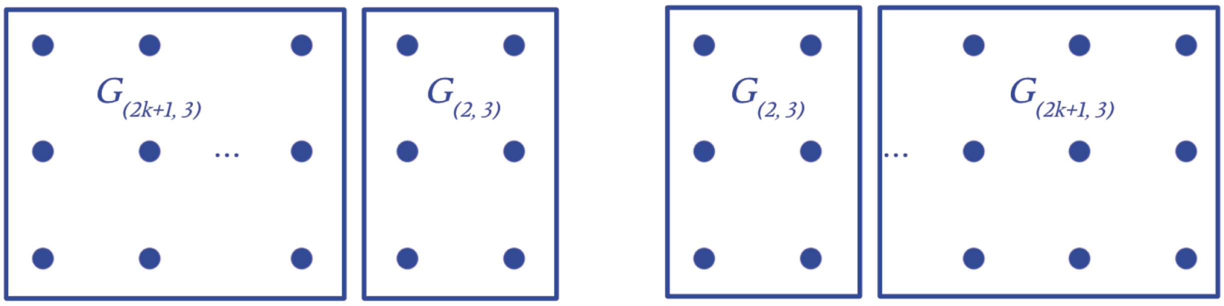

Fig. 8-6 Two ways of dividing $G_{(2k+3,3)}$ into $G_{(2k+1,3)}$ and $G_{(2,3)}$

We need to prove that $G_{(2k+3,3)}$ can always be split into the two subgraphs in a certain way, such that any possible excluded vertex, say v_1, based on the first player's initial move, can be part of subgraph $G_{(2k+1,3)}$. Note that $v_1 \in G_{(2k+1,3)}$ in at least one of the two ways we divide $G_{(2k+3,3)}$ because if we were to overlap the two possible subgraphs of $G_{(2k+1,3)}$, all the vertices in $G_{(2k+3,3)}$ will be part of either or both subgraphs of $G_{(2k+1,3)}$.

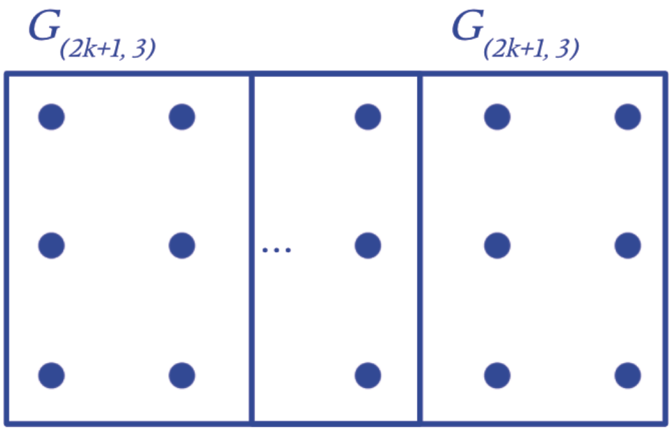

Fig. 8-7 Two overlapping $G_{(2k+1,3)}$

The following is always possible because an overlap between both subgraphs $G_{(2k+1,3)}$ will occur if and only if all the vertices in $G_{(2k+3,3)}$ will be part of either or both subgraphs $G_{(2k+1,3)}$, and an overlap will occur when $m \geq 5$, which happens for all k since $m=2k+3m\geq5$. A possible construction for the maximum matching for each way of diving $G_{(2k+3,3)}$ is shown below (v_1 will be the excluded vertex for illustration)

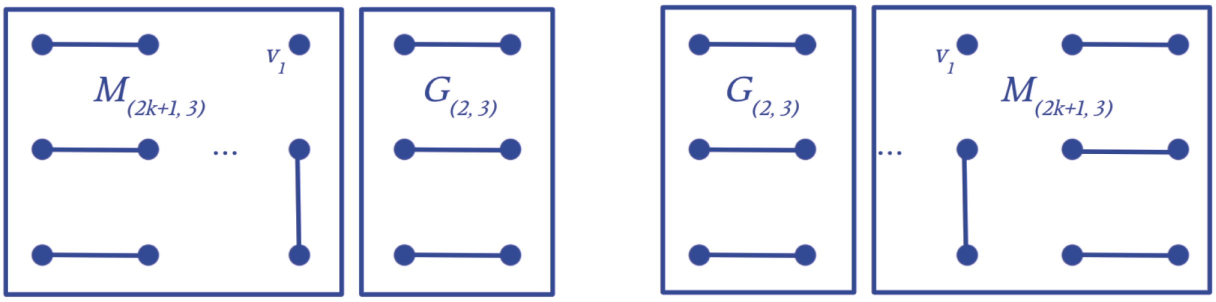

Fig. 8-8 Possible Constructions for excluded vertex v_1

This completes our proof that there always exists $M_{(m,3)}$ for $G_{(m,3)}$, regardless of the first player's initial move in the original form of Slither

We can carry out a similar inductive proof for the other dimension of the graph to ultimately show that there always exists $M_{(m,n)}$ for $G_{(m,n)}$, regardless of the first player's initial move in the original form of Slither.

8.3 Public Slither, *mn* is even

Let the first player initially choose an edge e_1 such that the two vertices e_1 is incident upon have coordinates (x_1, y_1) and (x_1+1, y_1) respectively. Fig. 8-9 illustrates the game after the first player's initial move. (Note that by a rotation, the other possible initial moves can be obtained)

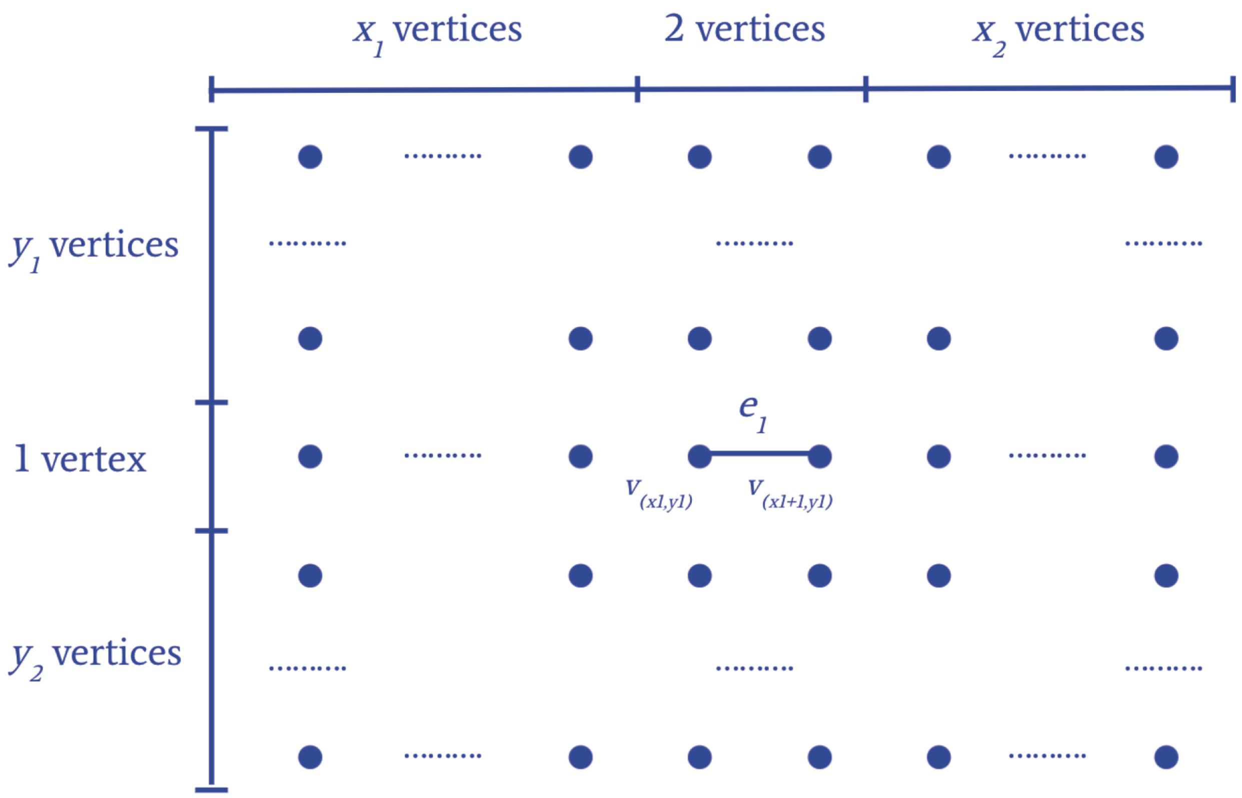

Fig. 8-9 First player's initial move

From Fig. 8-9, it can be deduced that $x_1+x_2+2=m$ and $y_1+y_2+1=n$.

We will discuss the following cases: (1) When n is even, (2) When n is odd, m is even and x_1, x_2 are even and (3) When n is odd, m is even and x_1, x_2 are odd.

Case 1: n is even

From our proof for the original form of Slither, there exists $M_{(x1,n)}$ and $M_{(x2,n)}$ without having to exclude any vertex since n is even. (Refer to section 6.1) The remaining subgraph in which we need to find a maximum matching is $G_{(2,n)}$. Notice that when all $v_{(0,k)} \in G_{(2,n)}$, which satisfy the inequalities $0 \leq k \leq n\text{-}1$, results in $v_{(0,k)} \rightharpoonup v_{(1,k)}$, a maximum matching $M_{(2,n)}$ is achieved. (Note: $e_1 \in M_{(2,n)}$) The construction is shown below in Fig. 8-10.

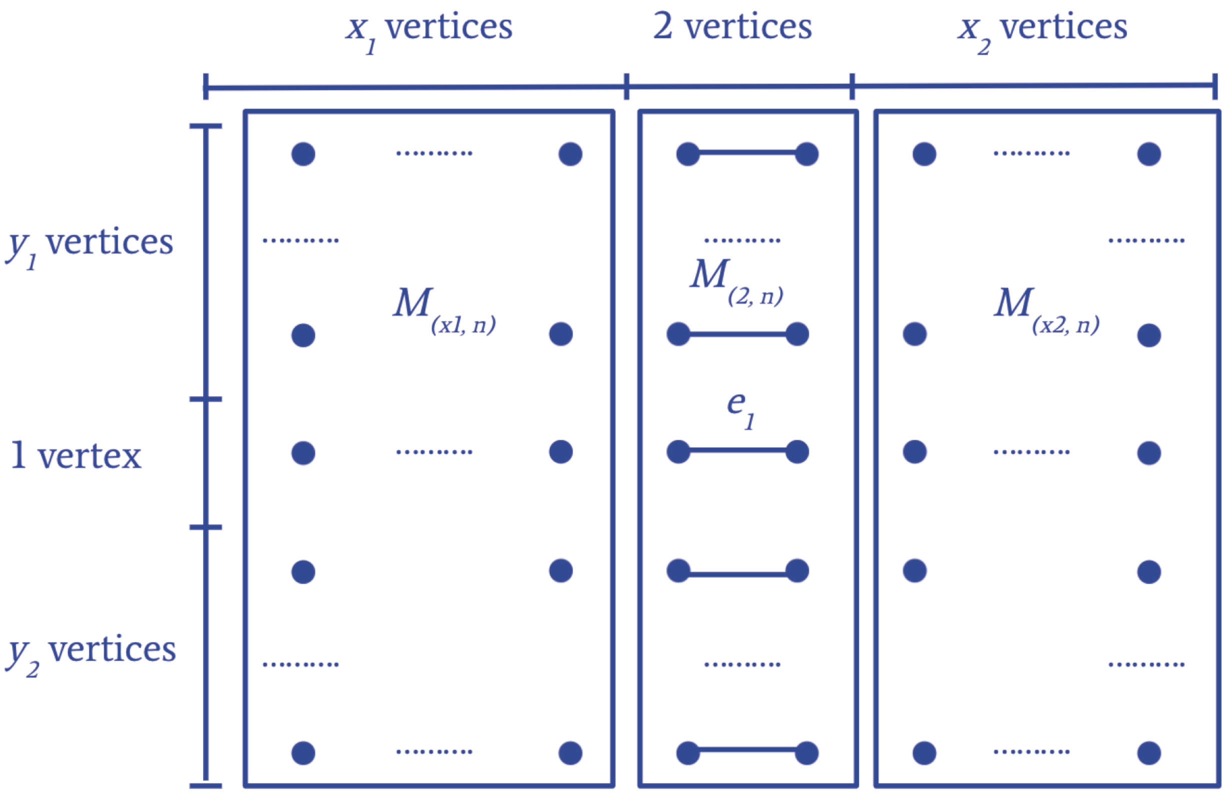

Fig. 8-10 Maximum matching construction for Case 1 and 2

Case 2: n is odd, m is even and x_1, x_2 are even

From our proof for the original form of Slither, there exists $M_{(x1,n)}$ and $M_{(x2,n)}$ without having to exclude any vertex since x_1 and x_2 are even. (Refer to section 8.1) The remaining subgraph in which we need to find a maximum matching is $G_{(2,n)}$. Using the same construction for $M_{(2,n)}$ as that in Case 1, the construction can be obtained, which corresponds to that shown in Diag. 8-10.

Case 3: n is odd, m is even and x_1, x_2 are odd.

Using alternate colouring and the results in section 8.2, there exists $M_{(x1,n)}$ and $M_{(x2,n)}$ with any one of the corner vertices exposed. Let the exposed vertices in $M_{(x1,n)}$ and $M_{(x2,n)}$ be v_1 and v_2 respectively such that both have the same y-coordinate. Our construction involves v_1 — $v_{(x1,k)}$ and $v_{(x1+1,\ k)}$ — v_2, where $0 \leq k \leq n$-1. (We can always choose k such that $k \neq y_1$). For the remaining subgraph $G_{(2,n)}$ that excludes v_1 and v_2, we have $v_{(0,z)}$ — $v_{(1,z)}$ for all $0 \leq z \leq n$-1 and $z \neq k$, and the construction is shown below in Fig. 8-11.

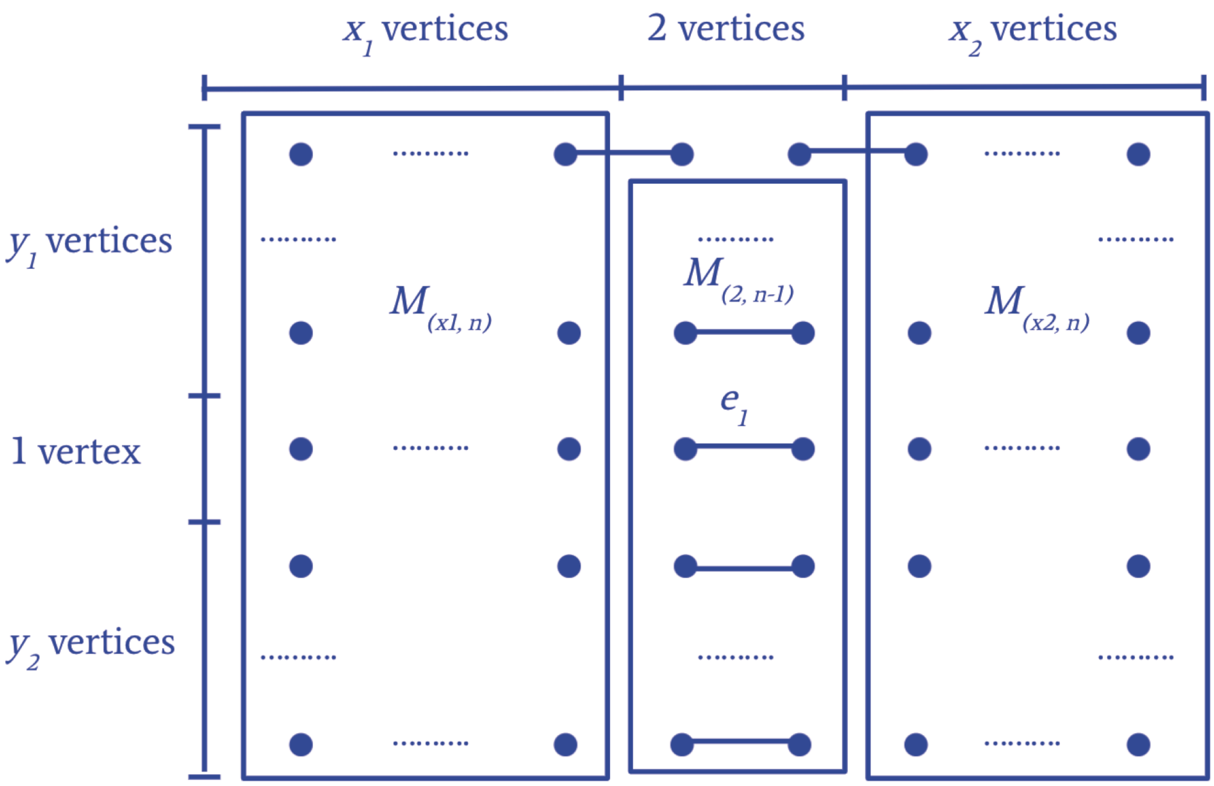

Fig. 8-11 Maximum matching construction for Case 3

8.4 Public Slither, *mn* is odd

When *mn* is odd, the first player can win by choosing an edge e_1 that is incident upon a corner vertex that is part of $G_{(m,n)}$. Note that the first player will be able to visualise the maximum matching only after the second player's initial move, as the second player's initial move determines the excluded vertex for maximum matching.

Note that when $(m,n)=(3,1)$ or $(1,3)$ as shown below, the first player will definitely win because after the first player's initial move, the second player is unable to choose his/her initial edge.

Fig. 8-12 (left) $(m,n)=(3,1)$
Fig. 8-13 (right) $(m,n)=(1,3)$

Induction proof for $m,n \geq 3$

We first prove that there always exists $M_{(m,3)}$ for $G_{(m,3)}$, regardless of the second player's initial move.

Consider the base case, which is $G_{(3,3)}$. Using the results from the base case of the induction proof in section 8.2, we can conclude that there exists $M_{(3,3)}$ for $G_{(3,3)}$ regardless of the second player's initial move. Note that in both construction cases, a different edge is connected to each included corner vertex and e_1 can always be one of such edges. This completes our verification for the base case.

We make the assumption that when $m=2k+1$ ($k \geq 1$), there always exists $M_{(2k+1,3)}$ for $G_{(2k+1,3)}$, regardless of the second player's initial move.

The same argument can be produced as that in the inductive step of the proof in section 6.2 when $m=2k+3$. (Note that $m \geq 5$ since $k \geq 1$) We will divide $G_{(2k+1,3)}$ into two subgraphs $G_{(2k+1,3)}$ and $G_{(2,3)}$ in the two different ways discussed in section 8.2 as shown below. Without the loss of generality, let e_1 chosen by the first player in his/her initial move be incident upon $v_{(0,2)}$ amd $v_{(1,2)}$ as shown below (The following assumption can be made since $v_{(0,2)}$ is a corner vertex of $G_{(2k+1,3)}$)

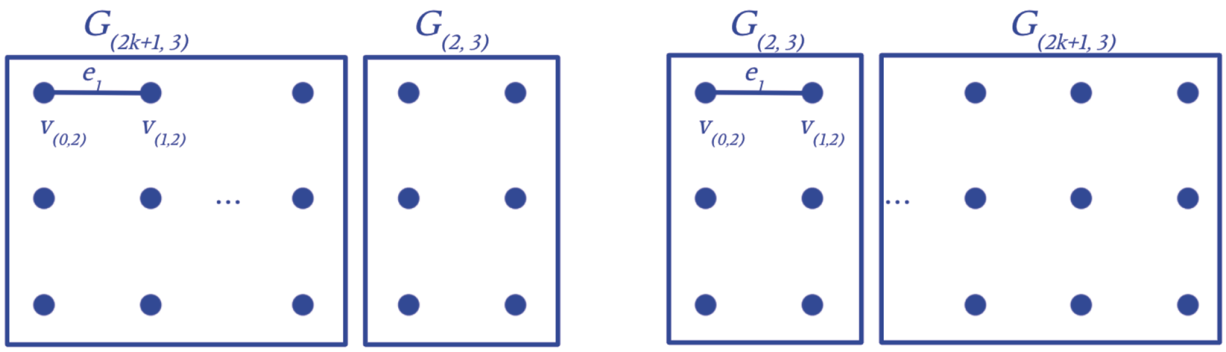

Fig. 8-14 Two ways of dividing $G_{(2k+3,3)}$ into $G_{(2k+1,3)}$ and $G_{(2,3)}$

The construction of the maximum matching will be similar as that discussed and suggested in the proof in section 8.2. If e_1 is part of $G_{(2k+1,3)}$, the maximum matching $M_{(2k+3,3)}$ is shown below in Fig. 8-15 (For illustrative purposes, let v_1 be the excluded vertex)

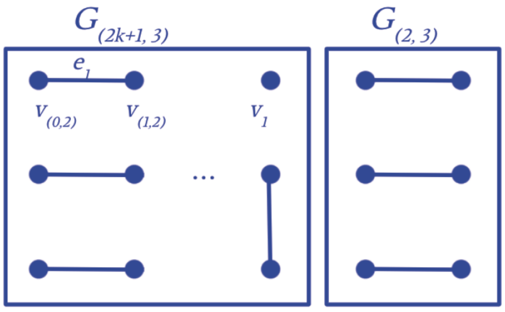

Fig. 8-15 Maximum matching when $e_1 \in G_{(2k+1,3)}$

When e_1 is part of $G_{(2,3)}$, the maximum matching $M_{(2k+3,3)}$ is shown below in Fig. 8-16 (For illustrative purposes, let v_2 be the excluded vertex)

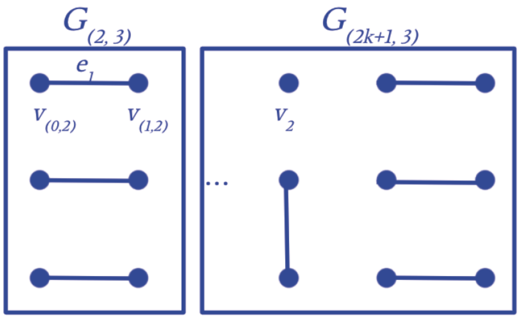

Fig. 8-16 Maximum matching when $e_1 \in G_{(2,3)}$

This completes our proof that there always exists $M_{(m,3)}$ for $G_{(m,3)}$, regardless of the second player's initial move in Public Slither.

We can carry out a similar inductive proof for the other dimension of the graph to ultimately show that there always exists $M_{(m,n)}$ for $G_{(m,n)}$, regardless of the second player's initial move in Public Slither.

8.5 3-Dimensional Slither, *mnp* is odd

We first prove that there always exists $M_{(m,3,3)}$ for $G_{(m,3,3)}$, regardless of the first player's initial move.

The base case is to consider maximum matching in $G_{(3,3,3)}$. We colour the vertices part of $G_{(3,3,3)}$ with Colour 1 and Colour 2 such that no adjacent vertices are of the same colour as shown below, where vertices coloured with Colour 1 are larger than those coloured with Colour 2. Note that an edge connects two different coloured vertices and it can be thus deduced that a maximum matching involves an equal number of vertices coloured with each colour.

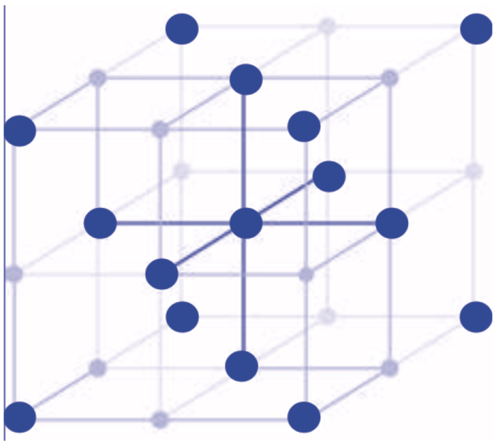

Fig. 8-17. Colouring in $G_{(3,3,3)}$

Based on the colouring shown above, we have a total of 14 vertices of colour 1 and 13 vertices of Colour 2. Therefore, in order to find $M_{(3,3,3)}$ for $G_{(3,3,3)}$, which requires the exclusion of one vertex from $G_{(3,3,3)}$, we need to exclude the vertex with Colour 1 that the edge the first player chose as his/her initial move is incident upon.

Since the central vertex of $G_{(3,3,3)}$ is coloured with Colour 2, it must be included in maximum matching. Therefore, we can always assume that the excluded vertex is on the top layer of $G_{(3,3,3)}$ by rotation.

The removed vertex is either a corner or the central vertex of the top layer $G_{(3,3,1)}$. Fig. 8-18 shows $M_{(3,3,3)}$ when the central vertex of the top layer is excluded, and Fig. 8-19 shows $M_{(3,3,3)}$ when one of the corner vertices of the top layer is excluded. (Note that the construction for excluding any of the other three can be done by rotation)

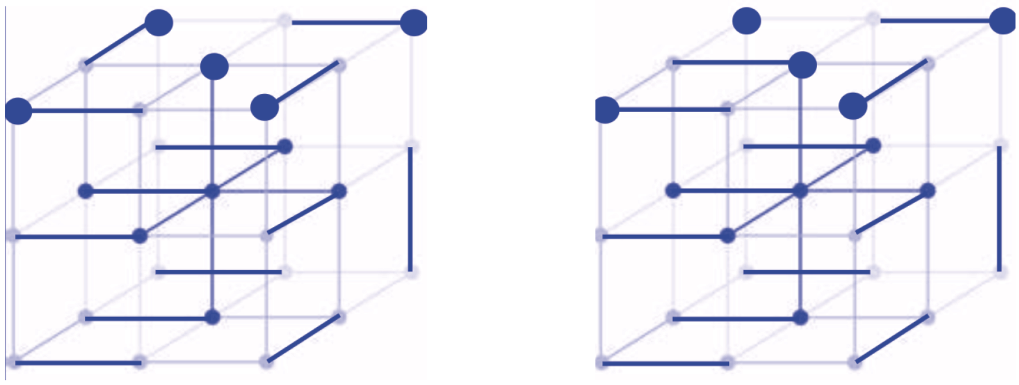

Fig. 8-18 (left) Maximum matching when central vertex is excluded
Fig. 8-19 (right) Maximum matching when corner vertex is excluded

Assume that when m=2k+1 ($k \geq 1$), there always exists $M_{(2k+1,3,3)}$ for $G_{(2k+1,3,3)}$, regardless of the first player's initial move.

When m=2k+3, consider splitting $G_{(2k+3,3,3)}$ into two subgraphs $G_{(2k+1,3,3)}$ and $G_{(2,3,3)}$ in two different ways as shown in the cross diagram below.

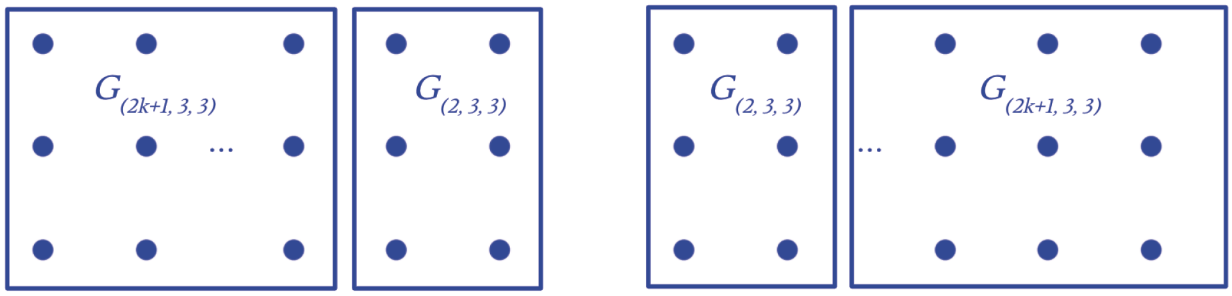

Fig. 8-20 Two ways of splitting $G_{(2k+3,3,3)}$ into $G_{(2k+1,3,3)}$ and $G_{(2,3,3)}$ (Cross Section)

We need to prove that $G_{(2k+3,3,3)}$ can always be split into the two subgraphs in a certain way, such that any possible excluded vertex, say v_1, based on the first player's initial move, can be part of subgraph $G_{(2k+1,3,3)}$. Note that $v_1 \in G_{(2k+1,3,3)}$ in at least one of the two ways we divide $G_{(2k+3,3)}$, and this can be verified with the argument given in section 8.2. An example is shown below, where v_1 is not part of $G_{(2k+1,3,3)}$ in the first way of dividing but part of $G_{(2k+1,3,3)}$ in the second way of dividing.

This completes our proof that there always exists $M_{(m,3,3)}$ for $G_{(m,3,3)}$, regardless of the first player's initial move in 3-Dimensional Slither.

We can carry out a similar inductive proof for the other two dimensions of the graph to ultimately show that there always exists $M_{(m,n,p)}$ for $G_{(m,n,p)}$, regardless of the first player's initial move in the original form of Slither.

8.6 Triangle Slither, number of vertices even

In a graph of Triangle Slither, we label each vertex $v_{(a,b)}$ such that $v_{(a,b)}$ is the *bth* vertex from the left in the *ath* row. An example of vertex labelling in T_4 is shown in Fig. 8-21.

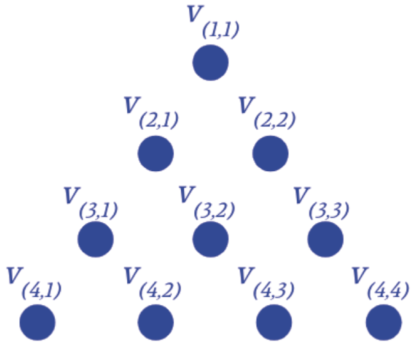

Fig. 8-21 Vertices labelling in T_4

This suggests that a vertex $v_{(a,b)}$ can be connected to either $v_{(a-1,b-1)}$, $v_{(a-1,b)}$, $v_{(a,b-1)}$, $v_{(a,b+1)}$, $v_{(a+1,b)}$ or $v_{(a+1,b+1)}$ to form an edge during a player's turn.

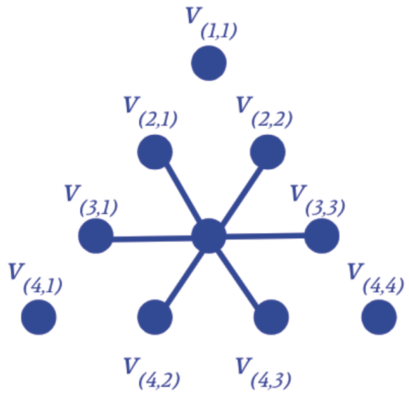

Fig. 8-22 Possible moves from $v_{(3,2)}$

Case 1: When $n \equiv 3 \pmod 4$

We first consider the base case n=3, which is to prove that there exists a maximum matching for T_3. This can be simply proven by construction, as shown below.

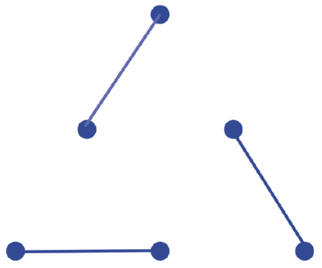

Fig. 8-23 Maximum matching in T_3

Suppose that when n=4k+3 ($k \geq 0$), there exists a maximum matching for T_{4k+3}.

When n=4(k+1)+3=4k+7, from the assumption above, there will exist a maximum matching for T_{4k+3}. In order to prove that there exists a maximum matching for T_{4k+7}, we need to prove that there exists a maximum matching for T_{4k+7} - T_{4k+3}.

The following construction below shows a maximum matching for T_{4k+7} - T_{4k+3}. This can be done by having $v_{(4k+4,n)} \rightharpoondown v_{(4k+5,n)}$ and $v_{(4k+6,n)} \rightharpoondown v_{(4k+7,n)}$ for all n that satisfy $0 \leq n \leq 4k+4$. For the remaining six vertices, which are arranged in the shape of T_3, we can follow the maximum matching given in the base case.

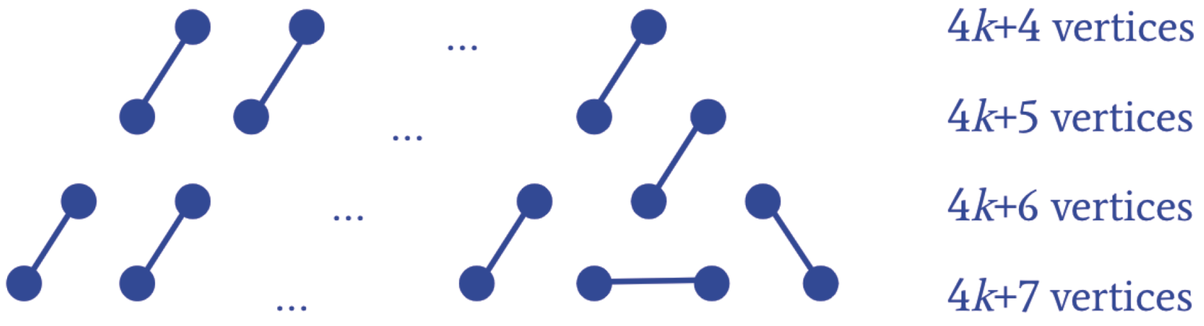

Fig. 8-24 Maximum matching for T_{4k+7} - T_{4k+3}.

This completes our induction proof, and therefore there always exists a maximum matching for T_{4k+3} when $k \geq 0$.

Case 2: When $n \equiv 0 \pmod 4$)

The base case, which is to prove that there exists a maximum matching for T_4 can be proven through construction as shown below.

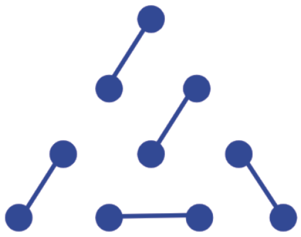

Fig. 8-25 Maximum matching in T_3

Now, we suppose that when $n=4k$ ($k \geq 1$), there exists a maximum matching for T_{4k}.

When $n=4(k+1)=4k+4$, we need to prove that there exists a maximum matching for T_{4k+4} - T_{4k} under the assumption that there exists a maximum matching for T_{4k}.

Similar to the previous case, a construction can be obtained by removing 3 vertices from each row from the construction in Case 1. The removed vertices can form a maximum matching as shown below (and displayed in dotted lines), and we can hence obtain a maximum matching for T_{4k+4} - T_{4k}.

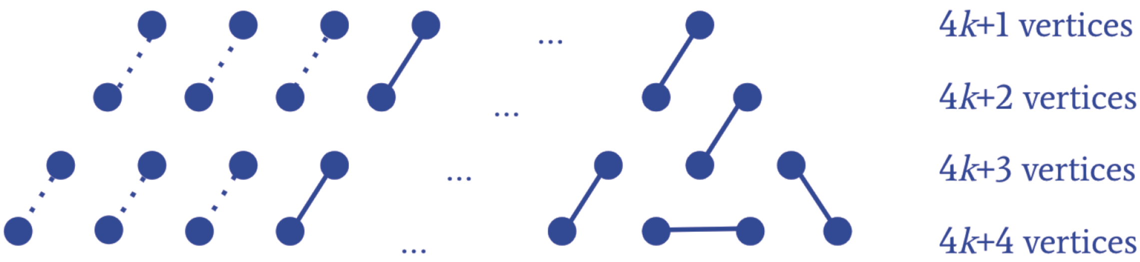

Fig. 8-26 Maximum matching in T_{4k+4} - T_{4k}

This completes our induction proof, and therefore there always exists a maximum matching for T_{4k} when $k \geq 1$.

8.7 Triangle Slither, number of vertices odd

To prove the results for the following case, we introduce a lemma below, which is useful in solving the problem.

Lemma:
If all vertices $v \in T_n$ also satisfies any of $v \in T_{n-4}$, $v \in T'_{n-4}$, $v \in T''_{n-4}$, where:

(1) T_{n-4}, T'_{n-4} and T''_{n-4} indicates the three triangles of size n-4 with $v_1 \in T_{n-4}$, $v_2 \in T'_{n-4}$, $v_3 \in T''_{n-4}$;

(2) v_1, v_2, v_3 are the three corner vertices of T_n,

then $n > 12$.

Proof:

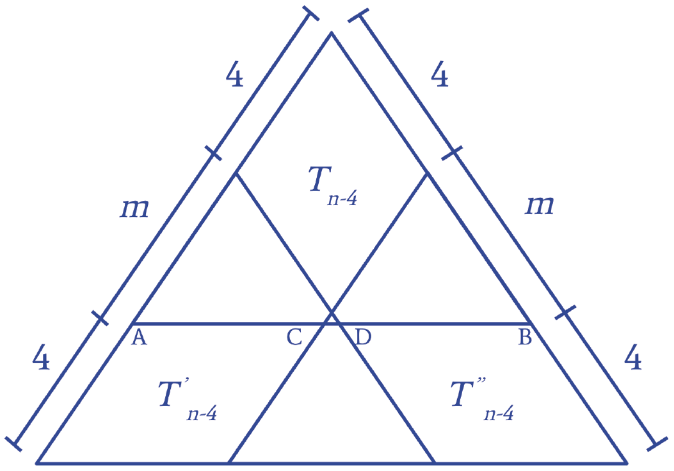

Fig. 8-27

We suppose that n is sufficiently large such that all vertices $v \in T_n$ also satisfy any of $v \in T_{n-4}$, $v \in T'_{n-4}$, $v \in T''_{n-4}$ as illustrated in Fig. 8-27. We define the magnitude of the arbitrary length indicated in Fig. 8-27 as the number of vertices within the region. Let m be the length of the region of vertices that is formed by the overlap of any two of T_{n-4}, T'_{n-4}, T''_{n-4}.

It suffices to deduce that (1) $m=n-8$; (2) AD=BC=m; (3) AB=m+4=n-4. For all vertices $v \in T_n$ to also satisfy any of $v \in T_{n-4}$, $v \in T'_{n-4}$, $v \in T''_{n-4}$, CD > 0 as illustrated in Fig. 8-27, that is, AD+BC > AB.

Therefore, the conclusion follows that AB+CD=m+m=2n-16 > AB=n-4 which yields the inequality $n > 12$, as desired.

Case 1: When n ≡ 1 (mod 4)

In Appendix 8.6, we have already proven that for any non-negative integer k, there is a maximum matching for T_{4k+4} - T_{4k}. In the same way, there must also be a maximum matching for T_{4k+5} - T_{4k+1}.

We will prove the following case using induction.

From the lemma, we have already shown that all vertices $v \in T_n$ also satisfy any of $v \in T_{n-4}$, $v \in T'_{n-4}$, $v \in T''_{n-4}$ when $n > 12$. Therefore, we would consider two base cases where $n < 13$: n=5 and n=9.

We first consider the base case n=5, which is to prove that there exists a maximum matching for T_5. This can be proven by construction. Note that one vertex is excluded from the maximum matching as there is an odd number of vertices in the triangle. There are four cases for discussion, which depends on the vertex that we exclude from maximum matching. There are four possible vertices that can be removed, as illustrated in Fig. 8-28 below. (Note that in each of the figures below, each of the black coloured vertices can be obtained by a reflection or rotation of any other black vertices and hence the exclusion of any would remain to be the discussion of the same case)

Fig. 8-28 The four possible types of position of the excluded vertex

We follow up the proof by providing a maximum matching for each of these cases in Fig. 8-29.

Fig. 8-29 Construction for n=5

Now, consider n=9. Since there exists a maximum matching for T_{4k+5} - T_{4k+1} there hence exists a maximum matching for T_9 - T_5. If the vertex excluded in maximum matching is part of one of the three T_5 as shown below in Fig. 8-30, then there exists a maximum matching for both T_9 - T_5 and T_5, and hence, T_9. However, if the vertex excluded is not part of one of the three T_5, we need to consider a separate construction for the following case, shown in Fig. 8-31.

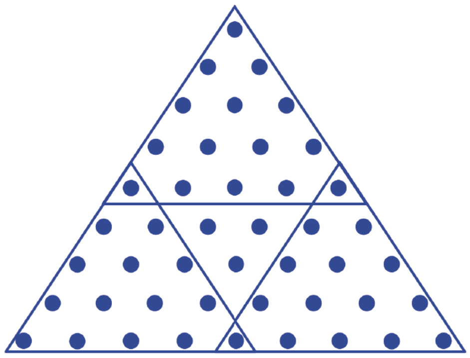

Fig. 8-30 Three T_5, with three vertices in the center not part of any of the three

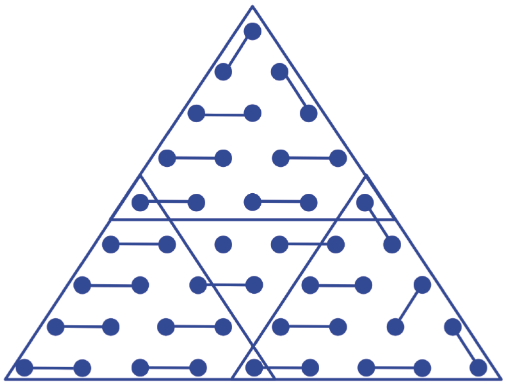

Fig. 8-31 Construction for the separate case

Suppose that when n=4k+1 ($k \geq 2$), there exists a maximum matching for all possible vertices excluded from maximum matching.

Consider n=4(k+1)+1=4k+5. Since $k \geq 2$, 4k+5 $\geq$ 13. Based on our lemma, all vertices $v \in T_n$ satisfy at least one of $v \in T_{n-4}$, $v \in T'_{n-4}$, $v \in T''_{n-4}$ when $n > 12$. Since there exists a maximum matching for T_{4k+5} - T_{4k+1} and T_{4k+1} for any $v \in T_n$ excluded from maximum matching, there must exist a maximum matching for T_{4k+5} regardless of the vertex excluded.

This completes our inductive step, and therefore there always exists a maximum matching for T_{4k+1} when $k \geq 1$.

Case 2: When n ≡ 2 (mod 4)

From the lemma, we have already shown that all vertices $v \in T_n$ also satisfy any of $v \in T_{n-4}$, $v \in T'_{n-4}$, $v \in T''_{n-4}$ when $n > 12$. Therefore, we would consider three base cases where $n < 13$: n=2, n=6 and n=10.

When n=2, it is obvious that the second player wins.

We first consider the base case n=6, which is to prove that there exists a maximum matching for T_6. This can be proven by construction. Note that one vertex is excluded from the maximum matching as there is an odd number of vertices in the triangle. There are five cases for discussion, which depends on the vertex that we exclude from maximum matching. There are five possible vertices that can be removed, as illustrated in Fig. 8-32 below. (Note that in each of the figures below, each of the black coloured vertices can be obtained by a reflection or rotation of any other black vertices and hence the exclusion of any would remain to be the discussion of the same case)

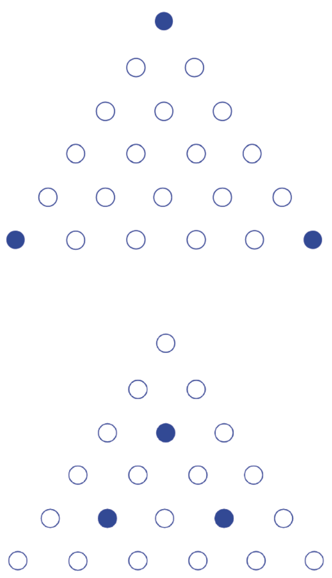

Fig. 8-32 The five possible types of position of the excluded vertex

We follow up the proof by providing a maximum matching for each of these cases in Fig. 8-33.

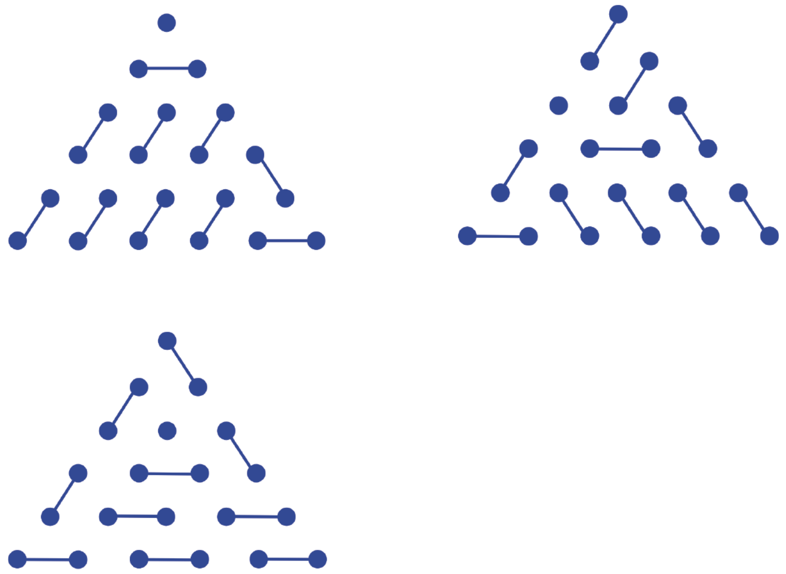

Fig. 8-33 Construction for n=6

Now, consider n=10. Since there exists a maximum matching for T_{4k+6} - T_{4k+2} there hence exists a maximum matching for T_{10} - T_6. If the vertex excluded in maximum matching is part of one of the three T_6 as shown below in Fig. 8-34, then there exists a maximum matching for both T_{10} - T_6 and T_{10}, and hence, T_{10}. However, if the vertex excluded is not part of one of the three T_6, we need to consider a separate construction for the following case, shown in Fig. 8-35.

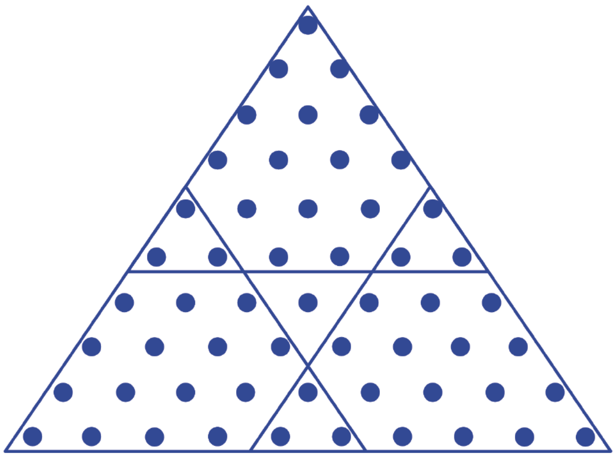

Fig. 8-34 Three T_6, with one vertex in the center not part of any of the three

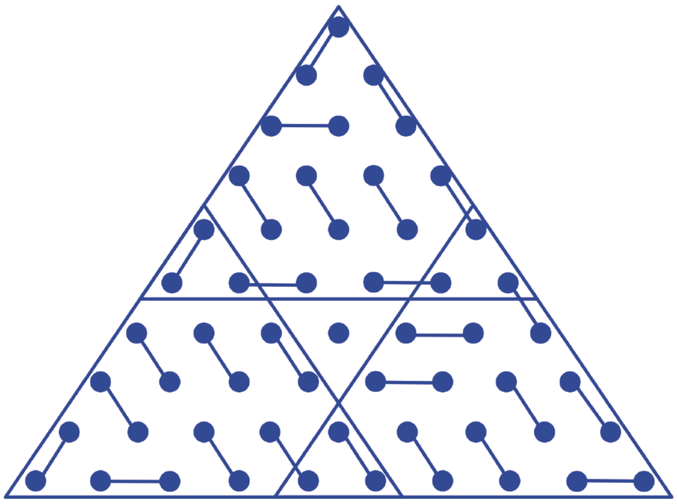

Fig. 8-35 Construction for the separate case

Suppose that when n=4k+2 ($k \geq 2$), there exists a maximum matching for all possible vertices excluded from maximum matching.

Consider n=4(k+1)+2=4k+6. Since $k \geq 2$, 4k+6 $\geq$ 13. Based on our lemma, all vertices $v \in T_n$ satisfy at least one of $v \in T_{n-4}$, $v \in T'_{n-4}$, $v \in T''_{n-4}$ when $n > 12$. Since there exists a maximum matching for T_{4k+6} - T_{4k+2} and T_{4k+2} for any $v \in T_n$ excluded from maximum matching, there must exist a maximum matching for T_{4k+6} regardless of the vertex excluded.

This completes our inductive step, and therefore there always exists a maximum matching for T_{4k+2} when $k \geq 0$.