Investigation of Shoelace Method in Higher Dimensions

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1. Motivation and Objective

This project is inspired by our previous math project, 'Investigation of the Shoelace Method', which focuses on the proof and geometrical representation of the Shoelace Method. When applying the Shoelace Method, vertices of a polygon are arranged in anti-clockwise manner. To briefly demonstrate:

Let the coordinates of vertex A, ..., D be (x_A, y_A), ..., (x_D, y_D).

The area of the quadrilateral ABCD is

$$\frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_D & x_A \\ y_A & y_B & y_C & y_D & y_A \end{vmatrix} = \frac{1}{2}(x_A y_B + x_B y_C + x_C y_D + x_D y_A - x_B y_A - x_C y_B - x_D y_C - x_A y_D)$$

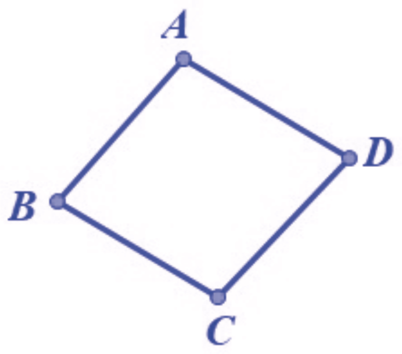


Figure 1.1

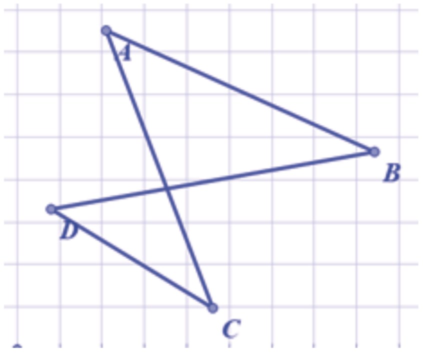


Figure 1.2

We also found out that, even if the points inside the shoelace representation are not arranged in anti-clockwise manner, the representation still has geometric meaning.

For example, if we connect A, B, C, D in such a manner that AC, BD intercept and two triangles are formed (Figure 1.2)

The Shoelace Representation will be:

$$\frac{1}{2}\begin{vmatrix} x_A & x_C & x_D & x_B & x_A \\ y_A & y_C & y_D & y_B & y_A \end{vmatrix} = \frac{1}{2}(x_A y_C + x_C y_D + x_D y_B + x_B y_A - x_C y_A - x_D y_C - x_B y_D - x_A y_B)$$

We proved that the area calculated using Shoelace Method is the difference of the areas of the two triangles formed.

We observe that two rows in the shoelace method corresponds to the 2-dimensions vertex coordinates. We wonder, if we extend the method to 3-dimensions vertex coordinates, with 3 rows in the shoelace expression instead of two rows, would it relate to the volume of the 3-dimensional figure?

In this project, we extend the use of "shoelace method" on 3-dimension vertex coordinates. We aim to establish the relationship between the result of "shoelace method" on 3-dimension figure to its volume.

2. Introduction

In our previous project, we already found a way to calculate the volume of the most fundamental polyhedron--tetrahedron. If we put one of the four vertices at the origin and construct a 3D Cartesian coordinate system, the coordinates of the other three points will be $(x_1, y_1, z_1), (x_2, y_2, z_2), and\ (x_3, y_3, z_3)$. It is known that the volume of such a tetrahedron is $\left|\frac{1}{6}[x_1y_2z_3 - x_1y_3z_2 - x_2y_1z_3 + x_3y_1z_2 - x_3y_2z_3 + x_2y_3z_1]\right|$. This equals to one sixth of the determinant of the other three vertices.

$$\text{Therefore, } V_{tet} = \frac{1}{6} \times \left| \det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \right|$$

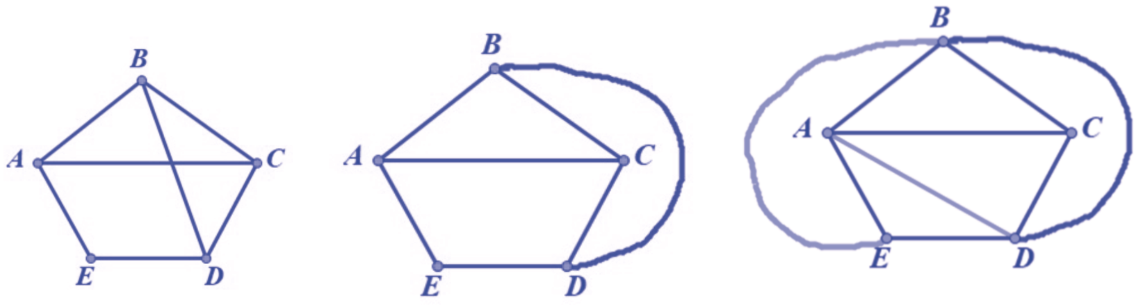
If we triangulate all surfaces of a polyhedron such that each face can be represented by a set of triangles, we can calculate its volume by adding or subtracting volumes of tetrahedrons formed by these triangles and the origin, in a similar sense to what we did for the 2-dimensional Shoelace Method. In this report, we will make use of graph theory to determine which volume to add and which ones to subtract.

In short, we will use Steinitz's Theorem [1] which allows us to construct a planar graph with the vertices, edges and faces identical to the input polyhedron, hence allowing us to manipulate the input information much more efficiently using graph theory. We will introduce a specific type of walk on maximal planar graphs called Petrie Walk to help us determine the rotation of determinants in respect to the polyhedron's interior. In a Petrie Walk, we are altering rotation while walking on the simplicial polyhedron in a zigzag manner. We can, hence, solve for the volume of the polyhedron by correcting of alternating rotation of its Petrie Walks, using a method similar to Shoelace Method in 2D.

3. Preliminaries

3.1 Vertices and Edges of Graphs

Figure 3.1a) is a graph. Points on the graph (such as A, B, ..., E) are called vertices (singular: vertex). There are five vertices. Edges are the line segments connecting the vertices, such as AB, AC, BD, etc. Hence there are seven edges in total. The relative positions of the vertices do not matter, as only the adjacency of the vertices is of importance. Therefore, Figure 3.1a) and Figure 3.1b) are equivalent, or isomorphic.



3.2 Incidence of Edges and Vertices, Adjacent Edges and Walks on Graphs

An edge is only incident to the two vertices that define it. For example, the edge AC is only incident to the vertices A and C.

Edges which share a common vertex are called adjacent edges. For example, in Figure 3.1a), Edges AC and AE are adjacent, while AC and BD are not.

A walk on a graph consists of an alternating sequence of vertices and edges, consecutive elements of which must be incident. A walk starts and ends with vertices. For example, a walk on figure 3.1a) can be A—AC—C—CD—D. Another walk can be B—BC—C—CD—D—DE—E—EA—A—AB—B.

Since the edges of a walk are defined by their incident vertices, the notation of a walk can be simplified by only writing out the sequence of vertices in the walk. Hence, the first example in the previous paragraph could be rewritten as ACD, and the second one could be rewritten as BCDEAB.

3.3 Planar Graphs and Their Faces

Graphs that have a planar embedding are called planar graphs. That means for a graph G to be planar, there must exist at least one graph G' isomorphic to G, which satisfies the fact that no two edges cross each other. From studying the properties of planar graphs, mathematicians are able to define planar graphs as such:

If a graph G does not contain a K_5(fig 3.2) or a $K_{3,3}$(fig 3.3) subgraph, it is a planar graph.

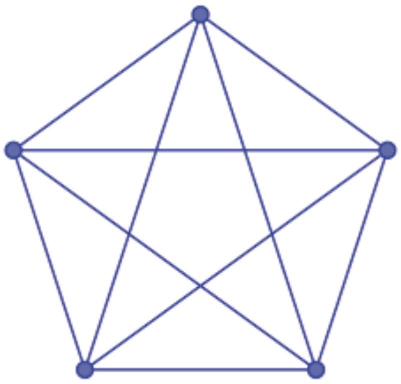


figure 3.2

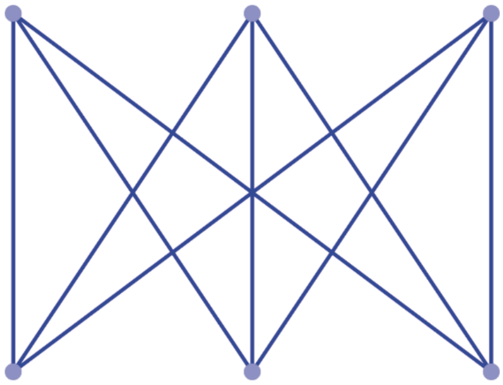


figure 3.3

We call the graph G' with the fact that no two edges cross each other the planar embedding of G. Figure 3.1b) is a planar embedding of figure 3.1a). Note that figure 3.1a) and 3.1b) are isomorphic to each other. All 'planar graphs' mentioned in this article will be in its planar embedding.

Faces of a planar graph refers to different parts of the plane which are separated by the edges of the planar graph. For example, Figure 3.1b) has 4 faces, namely the face surrounded by AB, BC, and AC; the face surrounded by AC, CD, DE, and EA; the face surrounded by BC, CD, DB; and finally, the face surrounded by AB, BD, DE, EA. Note that the fourth face refers to the 'exterior' part of the plane, 'surrounded' all the external edges of the planar embedding.

All of the faces of a planar graph should fill the entire 2D plane that the planar graph is situated in.

3.4 Maximization of Planar Graph

Maximizing a planar graph means adding extra edges until all faces of the planar graph are surrounded by only three edges. For example, if we look the planar embedding Figure 3.1b), the graph is obviously not maximized because the face AEDC and outer face ABDE are surrounded by four edges. There are multiple ways to maximize the graph, and figure 3.1c) demonstrates one of them. The idea of maximization is essential and will be frequently used in our following discussions.

3.5 Triangulation of Convex Polyhedron

The idea of triangulation of convex polyhedrons is closely related to the maximization of planar graphs. It is known that each face of a convex polyhedron is a polygon. When triangulating a convex polyhedron, we simply triangulate all of the faces of the polyhedron, which is equivalent to triangulating separate polygons. Polygon triangulation is the decomposition of a polygon into a set of coplanar triangles.

3.6 Euler's Formula

One famous property of the planar graph is:

Assume $V,\ E,\ F$ are the number of the vertices, edges and faces of a planar graph G respectively, then

$$V - E + F = 2$$

It was discovered by Leonhard Euler, hence named Euler's formula. To test this formula, we can look at Figure 3.1b) once again, and see that the number of its vertices, edges and faces are 5, 7, and 4 respectively, which satisfies $V - E + F = 2\ (5 - 7 + 4 = 2)$

4. Maximal Planar Graphs (MPGs) and Petrie Walk

4.1 Steinitz's Theorem

According to Steinitz's Theorem [1], each simplicial convex polyhedron corresponds to a maximal planar graph and each maximal planar graph can be constructed into a convex polyhedron. Each face, edge and vertex of the polyhedron will have its planar graph counterpart and vice versa. Figure 4.1 is a planar graph corresponding to a tetrahedron, and Figure 4.2 is a planar graph corresponding to an octahedron. As we can see, the number of edges, vertices and facets of the planar graphs are both identical to their polytopial counterparts. Hence, we can analyze some properties of polyhedra by looking at their corresponding properties of planar graphs.

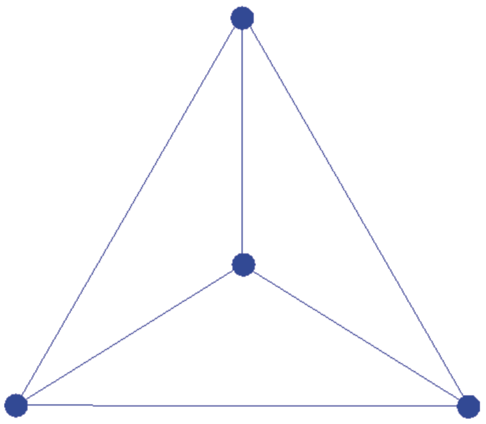


Figure 4.1

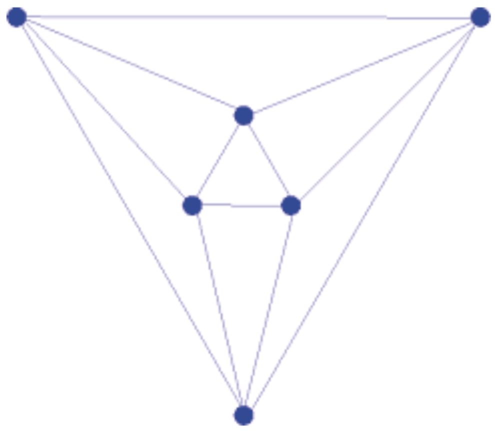


Figure 4.2

4.2 Maximal Planar Graphs (MPGs)

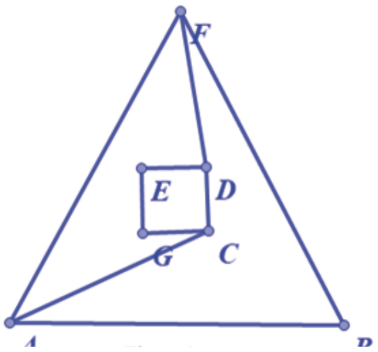
From the previous section, we know that for each simplicial(triangulated) polyhedron, we can obtain a corresponding MPG.

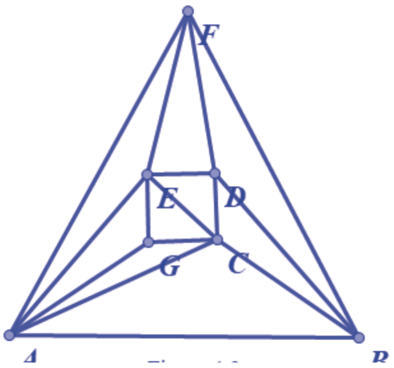
Note This is equivalent to making sure that every face of a planar graph is only surrounded by 3 edges.

Figure 3.1c) is one possible Maximal Planar Graph obtained by maximizing figure3.1b). Note that every face that has more than 3 edges surrounding it are maximized, including the external face in which had four edges surrounding it.

Another more complicated example would be figure 4.3a), a typical planar graph. Figure 4.3b) is one possible Maximal Planar Graph obtained by triangulating Figure 4.3a).

Maximal Planar Graphs are crucial to the project. Do note that each simplicial convex polyhedron corresponds to at least one MPG with four or more vertices.





4.3 Definition of the Petrie Walk

This section will discuss on a special walk on planar embeddings of MPGs called a *Petrie Walk*. Petrie Walks are crucial to volume calculation as they have special properties which helps us to effectively gather information about the relative positions of the vertices, edges and faces of the MPG.

But before we get into the definition of the Petrie Walk, I would first like to introduce the concept of a Consecutive Edge Pair (CEP). A CEP is a pair of edges on an MPG with are connected by a common vertex. Note that every CEP corresponds to a face of the MPG. Each distinct CEP is identified by the set of the two consecutive edges that it contains (i.e., there can be only one CEP which contains a certain pair of consecutive edges in the entire MPG). As an example, in figure 4.35, the CEP highlighted in red is non-equivalent compared to the CEP highlighted in blue, since the red CEP is identified by {CB, BD} but the blue CEP is identified by {BD, DA} and {CB, BD} ≠ {BD, DA}.

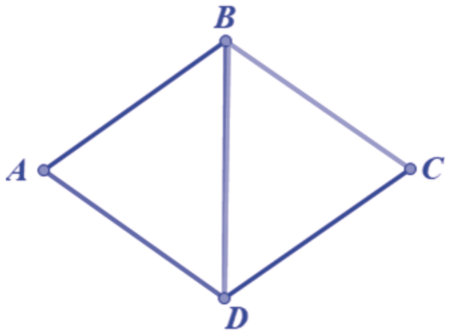


Figure 4.35

Definition of the Petrie Walk

A Petrie walk can be found by following the rules below on a planar embedding of a planar graph G:

1. Start with any vertex u and a face f_1 that contains vertices u, and other two vertices v and w. Starting from u, trace a CEP corresponding to f_1, in anti-clockwise manner with respect to f_1. This CEP can be represented as {uv, vw}.

2. Find the facethat share the most recently traced edge with f_1, and name it f_2. Trace the edge which:

    a. belongs to f_2, and

b. is connected with the previous edge.

Note that the last two edges you traced will always be a CEP corresponding to f_2 (the most recently selected face)

3. Repeat step 3 with the newly found face until we trace the edge uv again. However, do not trace uv and stop at u.

Note that a Petrie walk starts and ends at the same vertex.

The set of Petrie walks of an MPG is denoted as W, while each non-equivalent Petrie walk is denoted with w_i. $w_i \in W$. Two Petrie walks are equivalent if and only if the set of edges contained in the two walks are equal (i.e., every edge contained in the first walk is also contained in the second walk and vice versa).

To determine if two Petrie walks, namely w_1 and w_2, are equivalent, we simply need to take any CEP belonging to w_1, and then iterate w_2 to see if the chosen CEP also belongs to w_2. If yes, w_1 and w_2 are equivalent, and vice versa.

The length of a Petrie walk is denoted as $< w >$. It is equal to the number of edges contained in the walk.

When we have traced all possible Petrie Walks on an MPG (i.e., no more Petrie walk which is non-equivalent from all the pre-existing walks can be traced on the MPG), we say that the MPG is 'Petrie Completed'.

Figure 4.4 and 4.5 trace out two possible Petrie walks on figure 4.3, where an edge is marked pink if it is traced once and blue if traced twice. Figure 4.4 starts with face ABC and vertex A, while Figure 4.5 starts with face ABC and vertex C, where $< w_1 > = 6$ and $< w_2 > = 24$.

With the two Petrie Walks traced, the MPG is now Petrie Completed, as no more non-equivalent walks can be traced on it.

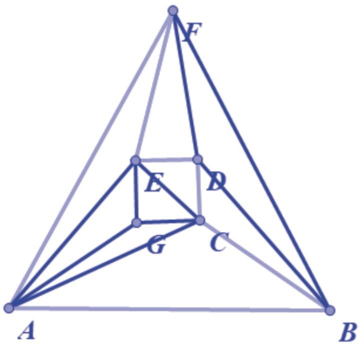


Figure 4.4
w1: ABCDEF(A)

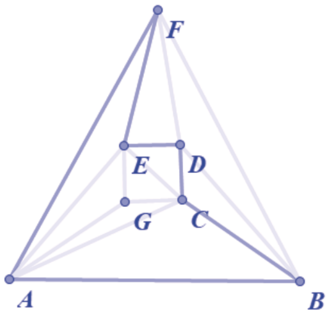


Figure 4.5
w2: CABFDECGAEFDBCAGECDBFAED(C)

4.4. Properties of the Petrie Walk

Property 1: The direction of consecutive CEPs with respect to their corresponding face in a Petrie Walk alternate between anticlockwise and clockwise, with the first CEP defined to be anticlockwise with respect to the first face.

Proof:

For every edge traversed in the walk other than the first edge, its direction with respect to the two faces that share it must not be the same due to the fact that the two faces are on different sides of the edge. Hence, the direction reverses with every edge, causing the alternating directions in the walk.

Property 2: For a Petrie Walk w_1, if we choose any CEP from the walk, preserve its direction, and use it (the CEP with direction) as the initial conditions for a new walk w_1', then w_1 and w_1' are equivalent.

Proof:

The Petrie Walk is deterministic, since every next edge of w_1 is induced from the previous CEP. In addition, the walk is cyclic, with the walk ending only when the initial condition and repeated. Hence, starting from anywhere in w_1, we would be able to trace the rest of w_1.

Note: This property can be further interpreted as 'from any one CEP of a Petrie Walk w_1, we can obtain the entire walk'.

Property 3: Petrie Walks are retraceable in reverse.

Proof:

Firstly, note that the inputs and outputs of step 2 of the definition of Petrie walk are one-to-one, meaning that if we use the original output of step 2 as input, we will get the original input (to step 2) as output. As the walk only ends when the very first edge(uv) is repeated, we can reverse the entire tracing process by starting with the output of the very last time step 2 was iterated (namely vu, and the last face that was obtained by step 2) tracing the entire Petrie walk in reverse.

Corollary 1: For any Petrie Completed MPG G of order n, $\sum < w_i > \ = 6n - 12$

Proof:

For this proof only, we do not consider the direction of Petrie Walks. Since G is Petrie Completed, we should not be able to find any Petrie Walk which is non-equivalent to pre-existing Petrie walks in G. According to Property 2, this is equivalent to not finding any CEP which is not a part of an already-traced Petrie Walk. Hence, all CEPs of G should be traced at least once.

Furthermore, a CEP can only belong to one Petrie Walk. This is because if a CEP belongs to two non-equivalent Petrie Walks at the same time, namely w_1 and w_2, according to property 2, we should be able to use the aforementioned CEP and retrace one and only one Petrie Walk, but this walk cannot be equivalent to both w_1 and w_2 at the same time, and we arrive at a contradiction. Hence, all CEPs of G can at most be traced once.

Combining the two results above, we can see that each non-equivalent CEP of the graph is traced exactly once. For each face of the MPG, there are three non-equivalent CEPs, and the number of non-equivalent CEPs in a Petrie walk is equal to the length of it as each CEP contributes one new edge to the walk. This would mean that the sum of the lengths of the Petrie Walks of G is thrice the number of faces, $\sum < w_i > \ = 3F$. According to Euler's formula, $V - E + F = 2$, and it is known that for a planar graph, $F = 2V - 4$. Hence, $\sum < w_i > \ = 6V - 12$, and the corollary follows.

MPGs and medial graphs.

- For every connected plane graph, we can draw only one medial graph.
- For every medial graph, we can deduce only two connected plane graphs. The two planes graphs must be dual graphs.
- Dual graph of MPGs are not MPGs (except for K4), hence the relation between medial graphs and MPGs are one to one.

Theorem: A MPG is a PCMPG if and only if its corresponding medial graph is a straight Eulerian graph.

5. Petrie Complete Maximal Planar Graph (PCMPG) and Medial Graph

5.1 Definition of PCMPG

Definition: Let G be an MPG, and G' be a planar embedding of G. Then, G is a Petrie-complete MPG if and only if there is exactly one unique Petrie walk in G'. (i.e. |W(G')| = 1)

The graph of a triangular bipyramid (Figure 6.4.2) is Petrie-complete, while the tetrahedral graph

(Figure 4.1) is non-Petrie-complete.

5.2 Definition of Dual Graph

The dual graph of a plane graph G is a graph that has a vertex for each face of G. The dual graph has an edge whenever two faces of G are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge.

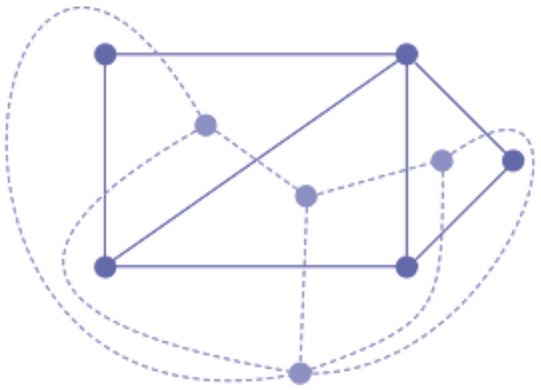


Figure 5.2.1 An example of a dual graph of a planar graph

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5.3 Definition of Medial Graph

Definition: Let G' be a planar embedding of an MPG, G, and Med(G') be the medial graph of G'. Med(G') shares the same set of vertices as the line graph of G. Two vertices in Med(G') are adjacent if and only if the corresponding two edges in G' are of a common face.

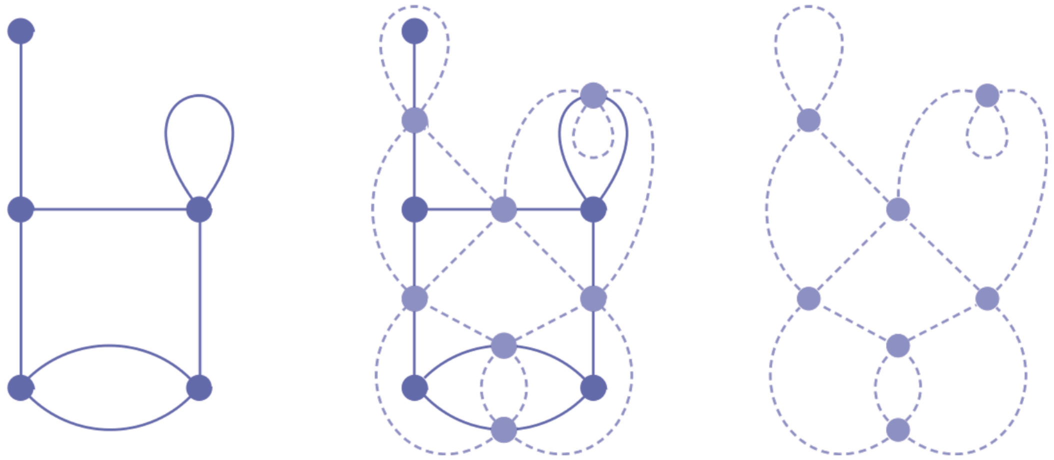


Figure 5.3.1 An example of medial graph of a connected plane graph

For any connected plane, we can only draw one medial graph.

5.4 Medial Graph of Maximal Planar graph

Medial Graph of a MPG must be a 4-regular plane graph. In an MPG, each edge is shared by exactly two faces, and each face has exactly three edges. For each edge, there are four edges that it shares a common face with. According to the definition of Medial Graph, the medial graph of a MPG must be a 4-regular plane graph. Given any MPG G, the medial graph of both G and its dual graph are isomorphic. We also know that the dual graph of an MPG cannot be an MPG (Except K4,4 which is self-dual). Therefore we can conclude that medial graph and MPG have a one-to-one relationship.

5.5 Petrie Walk on PCMPG

Let G' be a planar embedding of an MPG, G. A Petrie walk in M(G') is a Eulerian circuit if and only if G is a Petrie-complete MPG.

Proof: There is an equivalence in tracing a Petrie walk in a MPG, G, and its medial graph, M(G'). From the proof of Proposition 3.2 we know that for a Petrie walk in an MPG:

i) No three consecutive points is mentioned twice in the walk in the same order.

ii) If a surface triangle P1P2P3 is mentioned in order of "P1P2P3" in the walk, it will not be mentioned in the reverse order of "P3P2P1" in the same walk.

We can therefore conclude that in M(G'), no edge will be travelled twice in a Petrie walk. Also, it is known that G must contain 3f/2 edges since G is an MPG, where f is the number of faces of G. Hence, M(G') must contain 3f/2 vertices. Since M(G') is 4-regular, it contains 3f edges by handshaking lemma.

From Proposition 3.2, G is a Petrie-complete MPG of order n, if and only if wÎW(G') is of length 3f. That's also equivalent to travelling 3f edges in M(G'), creating a Eulerian circuit.

5.6 Enumeration of PCMPG

5.6.1 The Verification Algorithm

Using Corollary 1, we know that the length of the Petrie Walk belonging to a PCMPG must be 6n-12. Now we have a way to test whether an MPG is Petrie-complete. This allows computer-run programs to effectively verify PCMPGs at a faster pace. The verification algorithm contains two major steps: finding the facets (steps 1-3) and tracing out the walk (step 4). The algorithm is as follows:

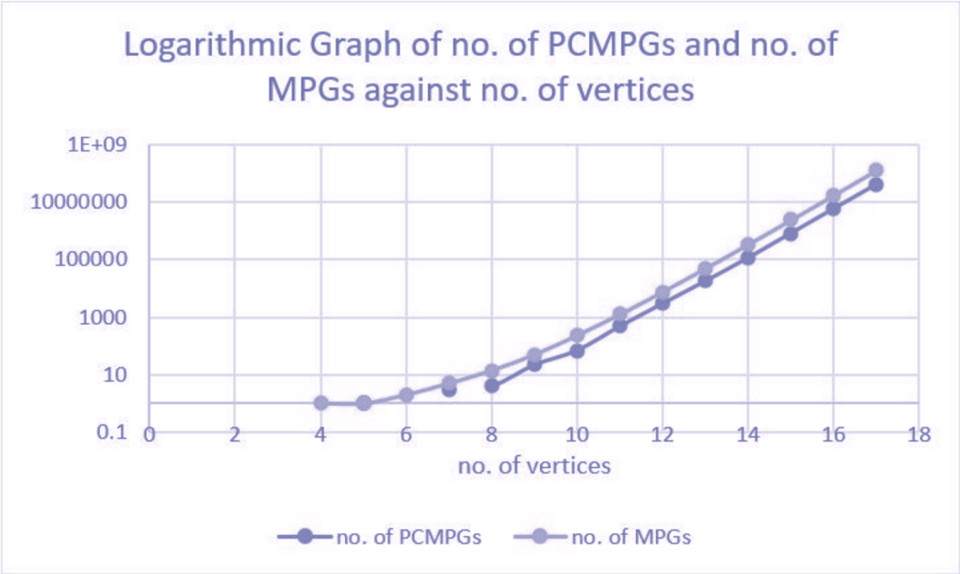
1. List out all the C_3 subgraphs of the input graph.
2. Remove each C_3 subgraph from the input, and test whether the resultant graph is connected. If yes, the C_3 subgraph is a facet, and vice versa. Store the two types of C_3 subgraphs in two separate lists.
3. Looking at the facets, we map out a relation between an edge(identified by its two vertices) and the two vertices it shares a facet with.
4. Using the mapping in step 3, we can effectively determine a Petrie walk in the given MPG, starting from any facet. Therefore, we can check whether the length of the resultant walk is 6V-12. If yes, then the input MPG is a PCMPG, and vice versa.

5.6.2 The Trend of the Number of Order n MPGs and PCMPGs

Using the algorithm mentioned in section 5.3, we can enumerate the number of PCMPGs, given the enumeration of MPGs, for order 4 to 17. Below is the result such enumeration.

No. Of vertices	*No. Of MPGs*	*No. Of PCMPGs*
4	*1*	*0*
5	*1*	*1*
6	*2*	*0*
7	*5*	*3*
8	*14*	*4*

9	50	22
10	233	70
11	1249	482
12	7595	2955
13	49566	17907
14	339722	114642
15	2406841	825097
16	17490241	5843386
17	129644753	41036436



From the Scatter graph above, we can see that the logarithmic graph of the number of PCMPGs and the number of MPGs are both approximately linear for 8≤number of vertices≤17. Since the number of the PCMPGs is approximately a third of the number of MPGs, we can make a based hypothesis that:

$$\lim_{n\to\infty}\left(\frac{No.\ of\ order\ n\ PCMPGs}{No.\ of\ order\ n\ MPGs}\right)=\frac{1}{3}$$

6. The Shoelace Method in Three Dimensions

6.1 Defining Shoelace Cross Multiplication and its polarity

The Fundamental Block of Shoelace Cross Multiplication:

In 2D Shoelace Method, we defined the 2-row Shoelace Cross Multiplication to calculate the area of polygons. A 2-by-2 block of the cross-multiplication representation is:

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$$

Note that the absolute value of this block of cross multiplication equals to twice of the area of a triangle with vertices $(x_1,\ y_1),\ (x_2, y_2)\ and\ (0,0)$

For the Extended Shoelace Method in 3D, we can use a similar notation. We define a 3-by-3 Shoelace Cross Multiplication block to be:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1y_2z_3 - x_3y_2z_1$$

Where the three points are the vertices of a consecutive edge pair. Note that each 3-by-3 Shoelace block corresponds to a CEP. As we will show later, the sum of three 3 by 3 blocks is equal to one sixth of the volume of a tetrahedron.

However, this definition presents a problem: What if the direction at which the CEP is traced in reversed? It should not have an effect on the volume calculated, but the change of direction of the CEP will cause a column exchange in the 3-by-3 block (the 1st and 3rd column), which will negate the value of the 3-by-3 block corresponding to the CEP. To solve to problem, we would like to introduce polarity of a CEP.

Polarity:

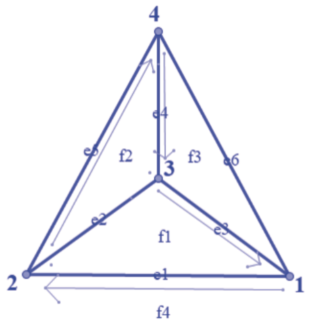
As we can trace Petrie Walks in two different directions on the graph, we can also trace the same consecutive edge pair in two different directions. The value of the 3-by-3 block corresponding to it is also different. We shall add a characteristic of each CEP called polarity to differentiate the two directions. "+" is used when the CEP is traced in an anti-clockwise manner with respect to its face, while "-" is for clockwise.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^{+} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1y_2z_3 - x_3y_2z_1$$

$$\begin{vmatrix} x_3 & x_2 & x_1 \\ y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \end{vmatrix}^{-} = \begin{vmatrix} x_3 & x_2 & x_1 \\ y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \end{vmatrix} = x_1y_2z_3 - x_3y_2z_1$$

With polarity defined, the value of each 3-by-3 block corresponding to each CEP no matter in which direction it is traced, will always be equivalent to the value obtained when the same CEP is traced in the anti-clockwise order.

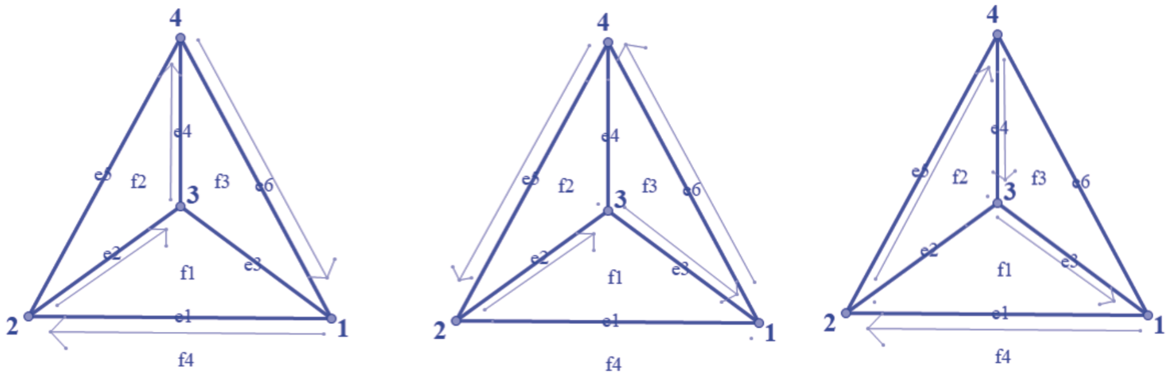
We will take the planar graph of a tetrahedron as an example. The CEP (e_3, e_1) is traced in a clockwise manner with respect to f1, so we will use "-" for the 3-by-3 block corresponding to it.



6.2 Shoelace Cross Multiplication and Determinant

In the Introduction section of this article, we have mentioned that the volume of a tetrahedron formed with one of its vertices as origin equals to the absolute value of the determinant of the 3 by 3 matrix formed with its other three vertices. We know that the determinant of a 3 by 3 matrix is comprised of 6 terms in the form of '$x_A y_B z_C$' where A, B, C are points corresponding to the determinant. Observe that the result of a 3-by-3 block of the Shoelace Cross Multiplication of a CEP (which consists of two terms in the form '$x_A y_B z_C$') must be two distinct terms from the 3 by 3 determinant formed by the three points which make up the CEP.

Let us go back to the planar graph of tetrahedron example. We shall focus on the face f_3. Three CEPs of the face are all traced exactly once by Petrie Walks. They are negative $\{e_4, e_6\}$, positive $\{e_3, e_6\}$ and positive $\{e_4, e_3\}$.



w_1 w_2 w_3

If we add the result of the three 3-by-3 Shoelace Cross Multiplication blocks together:

$$\begin{vmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{vmatrix}^+ + \begin{vmatrix} x_3 & x_4 & x_1 \\ y_3 & y_4 & y_1 \\ z_3 & z_4 & z_1 \end{vmatrix}^- + \begin{vmatrix} x_4 & x_3 & x_1 \\ y_4 & y_3 & y_1 \\ z_4 & z_3 & z_1 \end{vmatrix}^+$$

$$= x_3y_1z_4 - x_4y_1z_3 + x_1y_4z_3 - x_3y_4z_1 + x_4y_3z_1 - x_1y_3z_4$$

This exactly equals to the determinant formed by the three points:

$$\det\begin{pmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{pmatrix}$$

Similarly, we can also calculate the determinants of every other face of the MPG by summing up the 3-by-3 blocks corresponding to each non-equivalent CEP. Each determinant represents for the volume of a tetrahedron with the face as its base and the origin as the fourth vertex. Summing up all the determinants, we will obtain the volume of the entire convex polyhedron.

Note that the determinant can be both positive and negative. This matches with our initial idea at the end of our previous project where we get the volume of the input convex polyhedron can be obtained by adding the volume of some tetrahedron and subjecting others.

6.3 Volume Computation of Convex Polyhedrons

Let us go back again to the planar graph of tetrahedron example, we can write Petrie Walk w_1 as

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 & x_2 \\ y_1 & y_2 & y_3 & y_4 & y_1 & y_2 \\ z_1 & z_2 & z_3 & z_4 & z_1 & z_2 \end{vmatrix}$$

In vertex notation. Note that an extra edge is added to w_1 when it is inserted into the shoelace expression. This edge is always the same edge as the very first edge in this Petrie walk. This is necessary for all Petrie walks when they are inserted into the shoelace expression. This is to ensure that the shoelace expression has the same number of 3 by 3 blocks as the length of the Petrie walk.

We then can split the Petrie Walk into 4 CEPs on 4 different faces. According to Property 1 of Petrie Walk, the order of tracing with respect to consecutive faces always alternates, hence the polarity of the 3-by-3 blocks should also alternate. Hence, the shoelace expression can be split into:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^- + \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}^+ + \begin{vmatrix} x_3 & x_4 & x_1 \\ y_3 & y_4 & y_1 \\ z_3 & z_4 & z_1 \end{vmatrix}^- + \begin{vmatrix} x_4 & x_1 & x_2 \\ y_4 & y_1 & y_2 \\ z_4 & z_1 & z_2 \end{vmatrix}^+$$

We then perform Shoelace Cross Multiplication to Petrie Walks w_2 and w_3, add all of them together and regroup them according to faces, we will get the sum of determinants of matrices formed by all faces, which is 6 times the volume of the tetrahedron.

Since for a Petrie Completed graph, every CEP is traced exactly once (see proof of corollary 1), for any face on the MPG, we will always find all three of its CEPs in the 3-by-3 blocks obtained by breaking down the Petrie Walk Shoelace expressions.

If we denote the shoelace multiplication of a Petrie walk w to be $\|w\|$, and the set of Petrie Walks of a planar graph G to be W(G), then:

$$\text{Volume of the polyhedron corresponding to planar graph G} = \frac{1}{6}\sum_{i=1}^{|W(G)|}\|w_i\|$$

6.4 An Example of Volume Calculation Using the Extended Shoelace Method

Let us consider a bipyramid ABCDE (Figure 5.4.1):

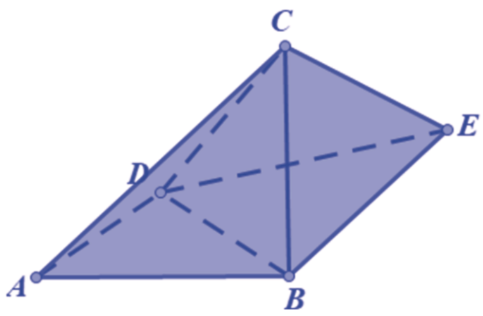


Figure 6.4.1

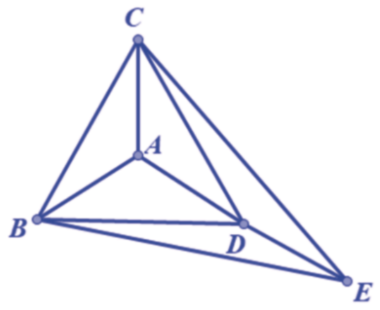


Figure 6.4.2

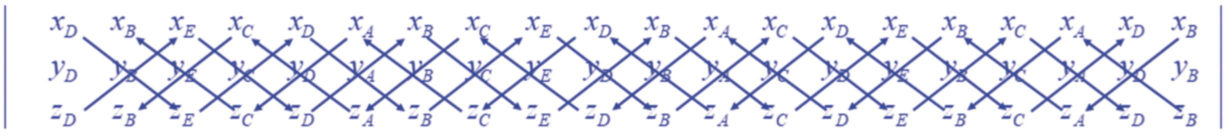
We can draw the corresponding planar graph of the bipyramid (Figure 5.4.2)

Trace the Petrie Walk on the planar graph starting from vertex A and the CEP {DB, BE} corresponding to face DBE, we will find out that there is only one distinct Petrie Walk on the graph, then we can form the Shoelace Cross Multiplication.

The coordinates of the points are: A (-1, -1,0), B (-1,1,0), C (-1, 1,2), D (1, 1, 0), E (0, 3, 1)

Petrie Walk: DBECDABCEDBACDEBCA(D)

Shoelace Cross Multiplication:



Note: Each three consecutive vertices are vertices of a CEP. We can see that unlike the usual 2D Shoelace Representation, the direction of Shoelace Cross Multiplication keeps changing. This is due to **Property 1** of Petrie Walk: the direction of consecutive CEPs with respect to their corresponding face in a Petrie

Walk alternate between anticlockwise and clockwise, with the first CEP defined to be anticlockwise with respect to the first face. (From Section 4.4)

If we plug in the coordinates and divide the answer by 6, we get that the volume of this bipyramid: $\frac{8}{3}$

To verify the answer calculated using our method, we will calculate the volume of bipyramid using geometry.

BCD is perpendicular to the x-y plane, we can use it as the base of the two pyramids.

$$\mathrm{V} = V_{CDBE} + V_{CDBA} = \frac{1}{3} \times A_{\Delta CDB} \times \left(h_1 + h_2\right) = \frac{1}{3} \times \left(\frac{1}{2} \times 2 \times 2\right) \times (2+2) = \frac{8}{3}$$

Which is the same as the result we got using Extended Shoelace Method.

It may seem that in this case, our method is much more complicated than just directly using the formula $\mathrm{V} = \frac{1}{3} \times A_{Base} \times h$, but if we simply rotate the bipyramid inside the coordinate system, you will find that it becomes extremely troublesome to find out base area and height. In that case our method will be more convenient to use.

7. Conclusion

In our project, we utilized the Petrie walk to effectively line out every simplicial surface of the input polyhedron and used the alternating shoelace multiplication to elegantly calculate and sum each tetrahedron corresponding to each surface. The result of our project is what we call the Extended shoelace method. Our algorithm may pale in comparison with other computational algorithms with the same purpose in term of time efficiency, but our method can be done relatively easily by hand. In addition, we believe that computing the volume of polyhedra with graph theory is a new and interesting approach.

Looking at the extended shoelace method at a grand scale, it can be split into two portions: The graph theory analysis and the shoelace expression calculation. The graph theory analysis takes in the **adjacency information** of the polyhedron (the set of information which describes whether two vertices are directly connected by an edge, for all possible pairs of vertices), and outputs the Petrie walks, which is equivalent to the complete set of CEPs that the planar graph has. Each CEP represents three vertices from the same triangular face (each set of three such vertices can be permutated in three ways, corresponding to the three non-equivalent CEPs in a face on the MPG). Hence, the Petrie Walk effectively collects all possible permutation of the points of a face, for all faces of the given simplicial polyhedron. After that, we insert **the positions of all of the vertices**, described by their coordinates, using the shoelace expression. Finally, the shoelace expressions produce the volume of the convex polyhedron of interest.

The combination of the **adjacency information of the vertices of a certain convex polyhedron**, and the **positions of each vertex of the polyhedron** are just enough for us to identify a unique convex polyhedron. Hence, it makes sense for the extended shoelace method to be able to obtain the volume of the polyhedron using the same combination of information as input.

4. Future Studies

In the future, we can try to extend the input of our algorithm into all polyhedra (including convex and concave polyhedra)

We can also try to look for trends in the Shoelace Method in even higher dimensions.

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