

Singapore International Mathematical Olympiad 2020
National Team Selection Test
Day 0

1. Let $u_1, u_2, \dots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \dots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \dots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \dots, u_{2019})$ and $b = \max(u_1, u_2, \dots, u_{2019})$. Prove that

$$ab \leq -\frac{1}{2019}.$$

2. Let $\mathcal{S} = \{(x, y) \mid x, y \in \mathbb{Q}, 0 \leq x, y \leq 1\}$, where \mathbb{Q} is the set of all rational numbers. Given a set of lines and a set of marked points in \mathcal{S} , Euclid can do one of two moves:

- (i) Draw a line connecting two marked points, or
- (ii) Mark a point in \mathcal{S} which lies on at least two drawn lines.

At first, the five distinct points $A(0, 0), B(1, 0), C(1, 1), D(0, 1)$ and $P \in \mathcal{S}$ are marked. Find all such points P such that Euclid can mark any point in \mathcal{S} after finitely many moves.

3. Let ABC be a triangle. Circle Γ passes through A , meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G . The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T . Suppose the points A and T are distinct. Prove that AT is parallel to BC .
4. We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \dots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \dots + a_nx^n$ are also in S . Find all rootiful sets of integers that contains all numbers of the form $2^a - 2^b$ for positive integers a and b .

Time allowed: 4 hours

Singapore International Mathematical Olympiad 2020
National Team Selection Test
Day 1

5. The infinite sequence a_0, a_1, a_2, \dots of (not necessarily different) integers has the following properties: $0 \leq a_i \leq i$ for all integers $i \geq 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \dots + \binom{k}{a_k} = 2^k$$

for all integers $k \geq 0$.

Prove that all integers $N \geq 0$ occur in the sequence (that is, for all $N \geq 0$, there exists $i \geq 0$ with $a_i = N$).

6. Let P be a point inside triangle ABC . Let AP meet BC at A_1 , let BP meet CA at B_1 , and let CP meet AB at C_1 . Let A_2 be the point such that A_1 is the midpoint of PA_2 , let B_2 be the point such that B_1 is the midpoint of PB_2 , and let C_2 be the point such that C_1 is the midpoint of PC_2 . Prove that points A_2, B_2 , and C_2 cannot all lie strictly inside the circumcircle of triangle ABC .
7. Let $n > 1$ be an integer. Suppose we are given $2n$ points in a plane such that no three of them are collinear. The points are to be labelled A_1, A_2, \dots, A_{2n} in some way. We then consider the $2n$ angles $\angle A_1 A_2 A_3, \angle A_2 A_3 A_4, \dots, \angle A_{2n-2} A_{2n-1} A_{2n}, \angle A_{2n-1} A_{2n} A_1, \angle A_{2n} A_1 A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists a labelling of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

Time allowed: 4.5 hours



Singapore International Mathematical Olympiad 2020
National Team Selection Test
Day 2

Singapore International Mathematical Olympiad 2020
National Team Selection Test
Day 2

8. Given a regular $(6n + 3)$ -gon, $3n$ of its vertices are used to form n acute triangles with distinct vertices. Prove that the other $3n + 3$ vertices can be used to form $n + 1$ acute triangles with distinct vertices.
9. Let a be a positive integer. We say that a positive integer b is a -good if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geq b$. Suppose b is a positive integer such that b is a -good, but $b + 2$ is not a -good. Prove that $b + 1$ is prime.
10. Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z .)

Time allowed: 4.5 hours

Solutions to NTST 2020

1. Let $u_1, u_2, \dots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \dots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \dots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \dots, u_{2019})$ and $b = \max(u_1, u_2, \dots, u_{2019})$. Prove that

$$ab \leq -\frac{1}{2019}.$$

IMO 2019 Shortlist.

Solution. Notice first that $b > 0$ and $a < 0$. Indeed, since $\sum_{i=1}^{2019} u_i^2 = 1$, the variables u_i cannot be all zero, and since $\sum_{i=1}^{2019} u_i = 0$, the nonzero elements cannot be all positive or all negative.

Let $P = \{i \mid u_i > 0\}$ and $N = \{i \mid u_i \leq 0\}$ be the indices of positive and nonpositive elements in the sequence, and let $p = |P|$ and $n = |N|$ be the sizes of these sets; then $p + n = 2019$. By the condition $\sum_{i=1}^{2019} u_i = 0$, we have $0 = \sum_{i=1}^{2019} u_i = \sum_{i \in P} u_i - \sum_{i \in N} |u_i|$, so

$$\sum_{i \in P} u_i = \sum_{i \in N} |u_i|. \quad (1.1)$$

Now we estimate the sum of squares of the positive and nonpositive elements.

$$\sum_{i \in P} u_i^2 \leq b \sum_{i \in P} u_i = b \sum_{i \in N} |u_i| \leq b \sum_{i \in N} |a| = -nab, \quad (1.2)$$

$$\sum_{i \in N} u_i^2 \leq |a| \sum_{i \in N} |u_i| = |a| \sum_{i \in P} u_i \leq |a| \sum_{i \in P} b = -pab. \quad (1.3)$$

Thus $1 = \sum_{i=1}^{2019} u_i^2 = \sum_{i \in P} u_i^2 + \sum_{i \in N} u_i^2 \leq -(p+n)ab = -2019ab$. That is $ab \leq -\frac{1}{2019}$.

2. Let $\mathcal{S} = \{(x, y) \mid x, y \in \mathbb{Q}, 0 \leq x, y \leq 1\}$, where \mathbb{Q} is the set of all rational numbers. Given a set of lines and a set of marked points in \mathcal{S} , Euclid can do one of two moves:

- (i) Draw a line connecting two marked points, or
- (ii) Mark a point in \mathcal{S} which lies on at least two drawn lines.

At first, the five distinct points $A(0,0), B(1,0), C(1,1), D(0,1)$ and $P \in \mathcal{S}$ are marked. Find all such points P such that Euclid can mark any point in \mathcal{S} after finitely many moves.

Proposed by Glen Lim.

Solution. All points P other than $(0.5, 0.5)$ work. First, if $P(0.5, 0.5)$, then drawing all possible lines, we have $y = 0, y = 1, x = 0, x = 1, y = x, y = 1 - x$ which do not allow Euclid to mark any more points.



Claim 1. $P(0.5, 0)$ works.

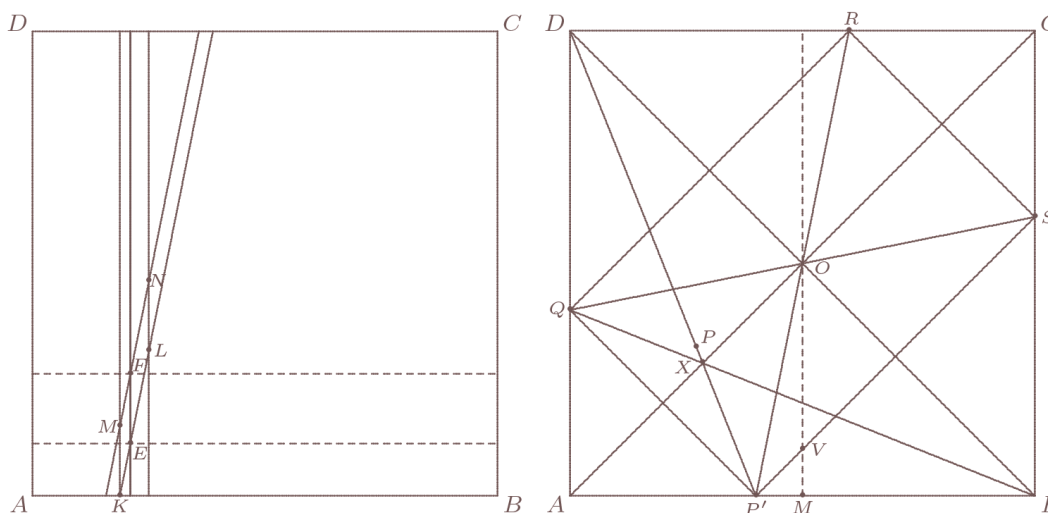
Proof. We shall prove that Euclid can construct any lines $x = q, y = q, \forall q \in \mathbb{Q}$. Any point in \mathcal{S} can thus be marked as the intersection of two such lines. First note that Euclid can mark $O(0.5, 0.5)$ by intersecting AC and BD . Drawing OP gives the line $x = 0.5$, which lets Euclid mark the point $Q(0.5, 1)$. Intersecting AQ and DP , as well as BQ and CP give points $R(0.25, 0.5)$ and $S(0.75, 0.5)$ respectively. Drawing RS thus gives $y = 0.5$, allowing Euclid to mark points $P'(0, 0.5)$ and $Q'(1, 0.5)$.

Claim 1a. Euclid can draw any line of the form $x = \frac{n}{2^m}, n, m \in \mathbb{N}, n \leq 2^m$.

Proof. We proceed by induction on m . The case $m = 1$ has already been proven. Now assume the claim is true for $m = k$. It suffices to show it is true for $m = k + 1$. Then we are done for all even n . If n is odd, Consider the points $U(\frac{n-1}{2^{k+1}}, 0), V(\frac{n-1}{2^{k+1}}, 0.5), W(\frac{n-1}{2^{k+1}}, 1), X(\frac{n+1}{2^{k+1}}, 0), Y(\frac{n+1}{2^{k+1}}, 0.5), Z(\frac{n+1}{2^{k+1}}, 1)$, all of which can be marked by the induction hypothesis. Intersecting UY and VX , as well as VZ and WY give, respectively, the points $(\frac{n}{2^{k+1}}, 0.25)$ and $(\frac{n}{2^{k+1}}, 0.75)$, the line through which is $x = \frac{n}{2^{k+1}}$, completing the induction. Hence, Claim 1a is true. Similarly, Euclid can draw any line of the form $y = \frac{n}{2^m}, n, m \in \mathbb{N}, n \leq 2^m$.

Now we shall complete the proof to Claim 1. We shall show that Euclid can construct any lines $x = q \forall q \in \mathbb{Q}$ (the case for $y = q$ follows similarly).

Let $q = \frac{a}{b}$, where a, b are coprime. WLOG let b not be a power of 2. Let m be an integer such that $2^m > b$. Then let $k = \lfloor q2^m \rfloor$. Note that $q - \frac{k}{2^m}$ is of the form $\frac{r}{b2^m}$, where r is an integer, and since $q2^m - 1 < k < q2^m$, we have $0 < q - \frac{k}{2^m} < \frac{1}{2^m} = \frac{b}{b2^m}$, so $0 < r < b < 2^m$. Now consider the points $K(\frac{k}{2^m}, 0), L(\frac{k+1}{2^m}, \frac{b}{2^m}), M(\frac{k}{2^m}, \frac{1}{2^m}), N(\frac{k+1}{2^m}, \frac{b+1}{2^m})$, which all can be marked by Claim 1a. The line KL has the equation $y = b(x - \frac{k}{2^m})$, which, when intersected with $y = \frac{r}{2^m}$, gives $E(q, \frac{r}{2^m})$. Similarly, intersecting MN with $y = \frac{r+1}{2^m}$ gives $F(q, \frac{r+1}{2^m})$, so joining these two points gives the line $x = q$ as required. Hence, Claim 1 is proven.



Generalisation to Claim 1. Similarly, given any rectangle with marked vertices in \mathcal{S} and one of the midpoints of its sides is marked, all points (whose coordinates

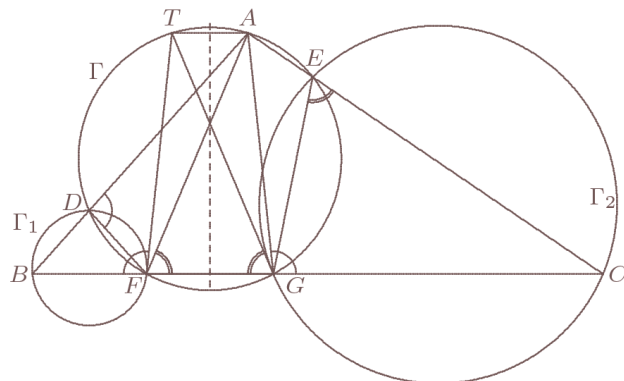
are rational multiples of its corresponding side lengths) in \mathcal{S} within or on the boundary of the rectangle can be marked. (This is just the case for Claim 1, but skewed and rotated.)

It now suffices to show that for any P other than $O(0.5, 0.5)$, the point $M(0.5, 0)$ can be marked. WLOG let P be in the bottom left of the square $ABCD$. Then DP intersects AB at some point P' . Letting X be the intersection of DP' and AC , and then intersecting BX and AD , we get Q , the reflection of P' across AC . Extending $P'O$ and QO to meet the opposite sides of the square gives R and S , the reflections of P' and Q across O respectively. Now note that Euclid can mark the vertices of the rectangle $P'QRS$, and draw its axes of symmetry, so by the generalisation of Claim 1, he can mark another point $V(0.5, v)$ within the rectangle with $v \neq 0.5$, allowing him to draw $x = 0.5$, and thus marking the point $M(0.5, 0)$.

For instance, if $X = (\alpha, \alpha)$, then $P'V : VS = 1 - \frac{2\alpha}{1-\alpha}$ which is rational. Alternatively, if $P = (p_1, p_2)$, then $P'V : VS = 1 - \frac{2p_1}{1-p_2}$ which is rational. Hence, Euclid can mark any point in \mathcal{S} , given that P is not $(0.5, 0.5)$.

3. Let ABC be a triangle. Circle Γ passes through A , meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G . The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T . Suppose the points A and T are distinct. Prove that AT is parallel to BC . *IMO 2019 Shortlist.*

Solution 1.



Notice that $\angle TFB = \angle FDA$ as FT is tangent to the circle BDF , and $\angle FDA = \angle CGA$ since the quadrilateral $ADFG$ is cyclic. Similarly, $\angle TGB = \angle GEC$ because GT is tangent to circle CEG , and $\angle GEC = \angle CFA$. Hence, $\angle TFB = \angle CGA$ and $\angle TGB = \angle CFA$. Triangles FGA and GFT have a common side FG , and their angles at F, G are the same, so these two triangles are congruent. Thus their altitudes starting at A and T , respectively, are equal and hence AT is parallel to BC .

Solution 2. We use barycentric coordinates. Let AFG be the reference triangle. That is $A = (1 : 0 : 0)$, $F = (0 : 1 : 0)$, $G = (0 : 0 : 1)$, $FG = a$, $GA = b$, $AF = c$. Let $B = (0 : 1 + u : -u)$, $u > 0$, $C = (0 : 1 + v : -v)$, $v < -1$. Thus Γ has the equation $a^2yz + b^2zx + c^2xy = 0$.

A parametric equation for the line AB is $(1-t : (1+u)t : -ut)$. Substituting this into the equation of Γ , we have $-a^2u(1+u)t^2 - b^2ut(1-t) + c^2(1+u)t(1-t) = 0$. Solving for t , we have $t = 0$ and $t = \frac{-b^2u + c^2(1+u)}{a^2(1+u)u - b^2u + c^2(1+u)}$. Thus

$$D = (a^2u(1+u) : (1+u)(-b^2u + c^2(1+u)) : -u(-b^2u + c^2(1+u))).$$

Then line $DF = [-b^2u + c^2(1+u) : 0 : a^2(1+u)]$. Thus the circumcircle Γ_1 of BDF has the equation of the form $a^2yz + b^2zx + c^2xy + \lambda(x+y+z)((-b^2u + c^2(1+u))x + a^2(1+u)z) = 0$, for some λ , since DF is the radical axis of Γ and Γ_1 . Substituting the coordinates of B , we get $\lambda = -1$. Therefore the equation of Γ_1 is

$$a^2yz + b^2zx + c^2xy - (x+y+z)((-b^2u + c^2(1+u))x + a^2(1+u)z) = 0.$$

From this, the equation of the tangent to Γ_1 at $F(0 : 1 : 0)$ is

$$a^2z + c^2x - ((-b^2u + c^2(1+u))x + a^2(1+u)z) = 0.$$

That is $[b^2 - c^2 : 0 : -a^2]$.

Replacing u by v , we get

$$E = (a^2v(1+v) : (1+v)(-b^2v + c^2(1+v)) : -v(-b^2v + c^2(1+v))).$$

Then line $GE = [-b^2v + c^2(1+v) : -a^2v : 0]$. Thus the circumcircle Γ_2 of GEC has the equation of the form $a^2yz + b^2zx + c^2xy + \mu(x+y+z)((-b^2v + c^2(1+v))x - a^2vy) = 0$, for some μ . Substituting the coordinates of C , we get $\mu = -1$. Thus Γ_2 has the equation

$$a^2yz + b^2zx + c^2xy - (x+y+z)((-b^2v + c^2(1+v))x - a^2vy) = 0.$$

From this, the equation of tangent to Γ_2 at $G(0 : 0 : 1)$ is $[b^2 - c^2 : a^2 : 0]$.

Then the intersection of these two tangents is $T = (a^2 : c^2 - b^2 : b^2 - c^2) = (1 : \frac{c^2 - b^2}{a^2} : \frac{b^2 - c^2}{a^2})$.

Thus $TA = (0 : \frac{c^2 - b^2}{a^2} : \frac{b^2 - c^2}{a^2}) = \frac{b^2 - c^2}{a^2}(0 : -1 : 1) = \frac{b^2 - c^2}{a^2}FG$. This shows that TA is parallel to BC .

Remark. This also shows that $TA = \frac{|b^2 - c^2|}{a}$.

4. We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \dots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \dots + a_nx^n$ are also in S . Find all rootiful sets of integers that contains all numbers of the form $2^a - 2^b$ for positive integers a and b . *IMO 2019 Shortlist.*

Solution. We shall show that the set \mathbb{Z} of integers is the only such rootiful set. Clearly the set of all integers \mathbb{Z} is rootiful. Let S be a rootiful set that contains all numbers of the form $2^a - 2^b$ for positive integers a and b . We shall prove that $S = \mathbb{Z}$.

First, note that $0 = 2^1 - 2^1 \in S$ and $2 = 2^2 - 2^1 \in S$. Now $-1 \in S$, since it is a root of $2x + 2$, and $1 \in S$, since it is a root of $2x^2 - x - 1$. Also if $n \in S$, then $-n$ is a root of $x + n$, so it suffices to prove that all positive integers must be in S .

Now we claim that any positive integer n has a multiple in S . Indeed, suppose that $n = 2^{\alpha}t$ for $\alpha \in \mathbb{Z}_{\geq 0}$ and t odd. Then by Euler's theorem, $t|2^{\phi(t)} - 1$, so $n|2^{\alpha+\phi(t)+1} - 2^{\alpha+1}$. Moreover, $2^{\alpha+\phi(t)+1} - 2^{\alpha+1} \in S$, and so S contains a multiple of every positive integer n .

We shall now prove by induction that all positive integers are in S . Suppose that $0, 1, \dots, n-1 \in S$; furthermore, let N be a multiple of n in S . Consider the base- n expansion of N , say $N = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$. Since $0 \leq a_i < n$ for each a_i , we have that all the a_i are in S . Furthermore, $a_0 = 0$ since N is a multiple of n . Therefore, $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n - N = 0$, so n is a root of a polynomial with coefficients in S . This shows that $n \in S$, completing the induction.

5. The infinite sequence a_0, a_1, a_2, \dots of (not necessarily different) integers has the following properties: $0 \leq a_i \leq i$ for all integers $i \geq 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \dots + \binom{k}{a_k} = 2^k$$

for all integers $k \geq 0$.

Prove that all integers $N \geq 0$ occur in the sequence (that is, for all $N \geq 0$, there exists $i \geq 0$ with $a_i = N$). *IMO 2019 Shortlist.*

Solution. We prove by induction on k that every initial segment of the sequence a_0, a_1, \dots, a_k , consists of the following elements (counted with multiplicity, and not necessarily in order), for some $\ell \geq 0$ with $2\ell \leq k+1$:

$$0, 1, \dots, \ell-1, \quad 0, 1, \dots, k-\ell.$$

For $k=0$ we have $a_0=0$, which is of this form. Now suppose that for $k=m$ the elements a_0, a_1, \dots, a_m are $0, 0, 1, 1, 2, 2, \dots, \ell-1, \ell-1, \ell, \ell+1, \dots, m-\ell-1, m-\ell$ for some ℓ with $0 \leq 2\ell \leq m+1$. It is given that

$$\binom{m+1}{a_0} + \binom{m+1}{a_1} + \dots + \binom{m+1}{a_m} + \binom{m+1}{a_{m+1}} = 2^{m+1},$$

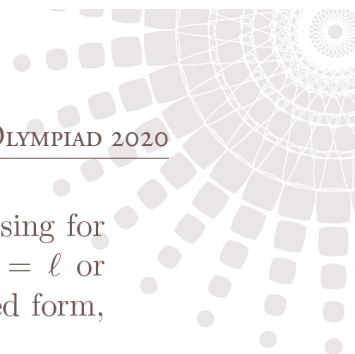
which becomes

$$\left(\binom{m+1}{0} + \binom{m+1}{1} + \dots + \binom{m+1}{\ell-1} \right) + \left(\binom{m+1}{0} + \binom{m+1}{1} + \dots + \binom{m+1}{m-\ell} \right) + \binom{m+1}{a_{m+1}} = 2^{m+1},$$

or, using $\binom{m+1}{i} = \binom{m+1}{m+1-i}$, that

$$\left(\binom{m+1}{0} + \binom{m+1}{1} + \dots + \binom{m+1}{\ell-1} \right) + \left(\binom{m+1}{m+1} + \binom{m+1}{m} + \dots + \binom{m+1}{\ell+1} \right) + \binom{m+1}{a_{m+1}} = 2^{m+1},$$

On the other hand, it is well-known that $\binom{m+1}{0} + \binom{m+1}{1} + \dots + \binom{m+1}{m+1} = 2^{m+1}$, and so by subtracting, we get $\binom{m+1}{a_{m+1}} = \binom{m+1}{\ell}$.



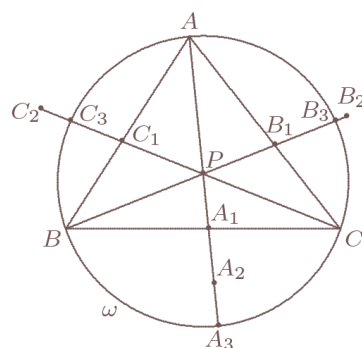
From this, using the fact that the binomial coefficients $\binom{m+1}{i}$ are increasing for $i \leq \frac{m+1}{2}$ and decreasing for $i \geq \frac{m+1}{2}$, we conclude that either $a_{m+1} = \ell$ or $a_{m+1} = m + 1 - \ell$. In either case, a_0, a_1, \dots, a_{m+1} is again of the claimed form, which concludes the induction.

As a result of this description, any integer $N \geq 0$ appears as a term of the sequence a_i for some $0 \leq i \leq 2N$.

6. Let P be a point inside triangle ABC . Let AP meet BC at A_1 , let BP meet CA at B_1 , and let CP meet AB at C_1 . Let A_2 be the point such that A_1 is the midpoint of PA_2 , let B_2 be the point such that B_1 is the midpoint of PB_2 , and let C_2 be the point such that C_1 is the midpoint of PC_2 . Prove that points A_2, B_2 , and C_2 cannot all lie strictly inside the circumcircle of triangle ABC .

IMO 2019 Shortlist.

Solution 1.



Since $\angle APB + \angle BPC + \angle CPA = 2\pi = (\pi - \angle ACB) + (\pi - \angle BAC) + (\pi - \angle CBA)$, at least one of the following inequalities holds:

$$\angle APB \geq \pi - \angle ACB, \quad \angle BPC \geq \pi - \angle BAC, \quad \angle CPA \geq \pi - \angle CBA.$$

Without loss of generality, we assume that $\angle BPC \geq \pi - \angle BAC$. Then $\angle BPC > \angle BAC$ since P is a point inside the triangle ABC . So $\angle BPC \geq \max(\angle BAC, \pi - \angle BAC)$ and hence

$$\sin \angle BPC \leq \sin \angle BAC. \tag{6.1}$$

Let the ray AP, BP, CP cross the circumcircle ω again at A_3, B_3, C_3 , respectively. We shall prove that at least one of the ratios $\frac{PB_1}{B_1B_3}$ and $\frac{PC_1}{C_1C_3}$ is at least 1, which yields that one of the points B_2 and C_2 does not strictly lie inside ω .

Because A, B, C, B_3 lies on a circle, the triangles CB_1B_3 and BB_1A are similar, so $\frac{CB_1}{B_1B_3} = \frac{BB_1}{B_1A}$. By sine rule, we have

$$\frac{PB_1}{B_1B_3} = \frac{PB_1}{CB_1} \frac{CB_1}{B_1B_3} = \frac{PB_1}{CB_1} \frac{BB_1}{B_1A} = \frac{\sin \angle ACP \sin \angle BAC}{\sin \angle BPC \sin \angle PBA}.$$

Similarly,

$$\frac{PC_1}{C_1C_3} = \frac{\sin \angle PBA \sin \angle BAC}{\sin \angle BPC \sin \angle ACP}.$$

Multiplying these two equations, we get

$$\frac{PB_1}{B_1B_3} \frac{PC_1}{C_1C_3} = \frac{\sin^2 \angle BAC}{\sin^2 \angle BPC} \geq 1,$$

by (6.1), which yields the desired conclusion.

Solution 2. We use barycentric coordinates. For basic notations and related formulas, refer to the remarks at the end. Let $P = (u : v : w)$, with $u + v + w = 1$ and $0 < u, v, w < 1$. Then $A_1 = (0 : \frac{v}{1-u} : \frac{w}{1-u})$ and $A_2 = (-u : \frac{v(1+u)}{1-u} : \frac{w(1+u)}{1-u})$. As $O = (\frac{a^2 S_A}{2S^2} : \frac{b^2 S_B}{2S^2} : \frac{c^2 S_C}{2S^2})$, we have

$$OA_2 = (-u - \frac{a^2 S_A}{2S^2} : \frac{v(1+u)}{1-u} - \frac{b^2 S_B}{2S^2} : \frac{w(1+u)}{1-u} - \frac{c^2 S_C}{2S^2}).$$

$$\begin{aligned} OA_2^2 &= -a^2 \left(\frac{v(1+u)}{1-u} - \frac{b^2 S_B}{2S^2} \right) \left(\frac{w(1+u)}{1-u} - \frac{c^2 S_C}{2S^2} \right) - b^2 \left(-u - \frac{a^2 S_A}{2S^2} \right) \left(\frac{w(1+u)}{1-u} - \frac{c^2 S_C}{2S^2} \right) \\ &\quad - c^2 \left(-u - \frac{a^2 S_A}{2S^2} \right) \left(\frac{v(1+u)}{1-u} - \frac{b^2 S_B}{2S^2} \right) \\ &= -a^2 \left(\frac{1+u}{1-u} \right)^2 vw + b^2 \frac{1+u}{1-u} wu + c^2 \frac{1+u}{1-u} uv \\ &\quad + \frac{1+u}{1-u} \frac{v}{2S^2} a^2 b^2 c^2 + \frac{1+u}{1-u} \frac{w}{2S^2} a^2 b^2 c^2 - \frac{u}{2S^2} a^2 b^2 c^2 - \frac{a^2 b^2 c^2}{4S^2} \\ &= -a^2 \left(\frac{1+u}{1-u} \right)^2 vw + b^2 \frac{1+u}{1-u} wu + c^2 \frac{1+u}{1-u} uv + \frac{a^2 b^2 c^2}{4S^2} \\ &= -a^2 \left(\frac{1+u}{1-u} \right)^2 vw + b^2 \frac{1+u}{1-u} wu + c^2 \frac{1+u}{1-u} uv + R^2. \end{aligned}$$

$$\begin{aligned} \text{Then } \frac{(1-u)^2}{(1+u)} (OA_2^2 - R^2) &= -a^2 vw(1+u) + b^2(1-u)wu + c^2(1-u)uv \\ &= -a^2 vw + b^2 wu + c^2 uv - (a^2 vw + b^2 wu + c^2 uv)u. \end{aligned}$$

$$\text{Similarly, } \frac{(1-v)^2}{(1+v)} (OB_2^2 - R^2) = a^2 vw - b^2 wu + c^2 uv - (a^2 vw + b^2 wu + c^2 uv)v,$$

$$\text{and } \frac{(1-w)^2}{(1+w)} (OC_2^2 - R^2) = a^2 vw + b^2 wu - c^2 uv - (a^2 vw + b^2 wu + c^2 uv)w.$$

$$\text{Consequently, } \frac{(1-u)^2}{(1+u)} (OA_2^2 - R^2) + \frac{(1-v)^2}{(1+v)} (OB_2^2 - R^2) + \frac{(1-w)^2}{(1+w)} (OC_2^2 - R^2) = 0.$$

Therefore, $OA_2 < R, OB_2 < R, OC_2 < R$ cannot hold simultaneously.

Remark. Here $S_A = \frac{1}{2}(b^2 + c^2 - a^2)$, etc, and we have the following formulas: $S_A + S_B = c^2$, $a^2 S_A + b^2 S_B + c^2 S_C = 2S^2$, $S_A S_B + S_B S_C + S_C S_A = S^2$, where S is twice the area of the triangle ABC . Also O is the circumcenter, R is the circumradius and $R = \frac{abc}{2S}$.

7. Let $n > 1$ be an integer. Suppose we are given $2n$ points in a plane such that no three of them are collinear. The points are to be labelled A_1, A_2, \dots, A_{2n} in some way. We then consider the $2n$ angles $\angle A_1 A_2 A_3, \angle A_2 A_3 A_4, \dots, \angle A_{2n-2} A_{2n-1} A_{2n}, \angle A_{2n-1} A_{2n} A_1, \angle A_{2n} A_1 A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists a labelling of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.
- IMO 2019 Shortlist.*

Solution. Let ℓ be a line separating the points into two groups (L and R) with n points in each. Label the points A_1, A_2, \dots, A_{2n} so that $L = \{A_1, A_3, \dots, A_{2n-1}\}$. We claim that this labelling works. Take a line $s = A_{2n} A_1$.



- (a) Rotate s around A_1 until it passes through A_2 ; the rotation is performed in a direction such that s is never parallel to ℓ .
- (b) Then rotate the new s around A_2 until it passes through A_3 in a similar manner.
- (c) Perform $2n - 2$ more such steps, after which s returns to its initial position.

The total (directed) rotation angle Θ of s is clearly a multiple of 180° . On the other hand, s was never parallel to ℓ , which is possible only if $\Theta = 0$. Now it remains to partition all the $2n$ angles into those where s is rotated anticlockwise, and the others.

8. Given a regular $(6n + 3)$ -gon, $3n$ of its vertices are used to form n acute triangles with distinct vertices. Prove that the other $3n + 3$ vertices can be used to form $n + 1$ acute triangles with distinct vertices. *Proposed by Lim Jeck.*

Solution. Inscribe the regular $(6n + 3)$ -gon in a circle. Set $k = n + 1$. Let v_1, v_2, \dots, v_{3k} be the remaining vertices, numbered clockwise. Split them into triples of the form (v_i, v_{i+k}, v_{i+2k}) . We claim that this partition works.

Arguing indirectly, assume say that the angle $\angle v_i v_{i+2k} v_{i+k}$ is obtuse. Along the semi-circle running clockwise from v_i to the point opposite to v_i (which is not one of the $6n + 3$ vertices), there are $3n + 1$ vertices (excluding v_i) of the polygon; and at most $k - 1 = n$ of these vertices are v_j 's. Thus there are at least $2n + 1$ vertices on this semicircle which are chosen initially. It follows that this semicircle must contain 3 vertices of one of the initially chosen triangles, which means that it is obtuse, and this is a contradiction.

9. Let a be a positive integer. We say that a positive integer b is a -good if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geq b$. Suppose b is a positive integer such that b is a -good, but $b + 2$ is not a -good. Prove that $b + 1$ is prime. *IMO 2019 Shortlist.*

Solution. For a prime p and n a nonzero integer, we write $v_p(n)$ for p -adic valuation of n : the largest integer t such that $p^t \mid n$.

We first show that b is a -good if and only if b is even and $p \mid a$ for all primes $p \leq b$.

To start with, the condition that $an + 1 \mid \binom{an}{b} - 1$ can be rewritten as saying that

$$\frac{an(an-1)\cdots(an-b+1)}{b!} \equiv 1 \pmod{an+1}. \quad (9.1)$$

Suppose, on the one hand, there is a prime $p \leq b$ with $p \nmid a$. Take $t = v_p(b!)$. Then there exist positive integers c such that $ac \equiv 1 \pmod{p^{t+1}}$. If we take c big enough, and then take $n = (p-1)c$, then $an = a(p-1)c \equiv p-1 \pmod{p^{t+1}}$ and $an \geq b$. Since $p \leq b$, one of the terms of the numerator $an(an-1)\cdots(an-b+1)$ is $an-p+1$, which is divisible by p^{t+1} . Hence the p -adic valuation of the numerator

is at least $t + 1$, but that of the denominator is exactly t . This means $p \mid \binom{an}{b}$, so $p \nmid \binom{an}{b} - 1$. As $p \mid an + 1$, we get that $an + 1 \nmid \binom{an}{b} - 1$, so b is not a -good.

On the other hand, if for all primes $p \leq b$ we have $p \mid a$, then every factor of $b!$ is coprime to $an + 1$, and hence invertible modulo $an + 1$: hence $b!$ is also invertible modulo $an + 1$. Then equation (9.1) reduces to

$$an(an - 1) \cdots (an - b + 1) \equiv b! \pmod{an + 1}.$$

However, we can write the left-hand side as follow:

$$an(an - 1) \cdots (an - b + 1) \equiv (-1)(-2) \cdots (-b) \equiv (-1)^b b! \pmod{an + 1}.$$

Provided that $an > 1$, if b is even we deduce that $(-1)^b b! \equiv b!$ as needed. On the other hand, if b is odd, and we take $an + 1 > 2(b!)$, then we will not have $(-1)^b b! \equiv b!$, so b is not a -good. This completes the proof of the claim.

To conclude from here, suppose that b is a -good, but $b + 2$ is not. Then b is even, and $p \mid a$ for all primes $p \leq b$, but there is a prime $q \leq b + 2$ for which $q \nmid a$: so $q = b + 1$ or $b + 2$. We cannot have $q = b + 2$, as that is even too, so we have $q = b + 1$; in other words, $b + 1$ is prime.

10. Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z .)

IMO 2019 Shortlist.

Solution. Suppose on the contrary that $a^2 + \left\lceil \frac{4a^2}{b} \right\rceil = (a + k)^2$, or equivalently $\left\lceil \frac{(2a)^2}{b} \right\rceil = (2a + k)k$. Clearly $k \geq 1$. Thus the equation

$$\left\lceil \frac{c^2}{b} \right\rceil = (c + k)k \tag{10.1}$$

has a positive integer solution (c, k) , with an even value of c .

Choose a positive integer solution of (10.1) with minimal value of k , without regard to the parity of c . From $\frac{c^2}{b} > \left\lceil \frac{c^2}{b} \right\rceil - 1 = ck + k^2 - 1 \geq ck$, and $\frac{(c-k)(c+k)}{b} < \frac{c^2}{b} \leq \left\lceil \frac{c^2}{b} \right\rceil = (c + k)k$, it can be seen that $c > bk > c - k$, so $c = kb + r$ with some $0 < r < k$. Substituting this into (10.1), we get $\left\lceil \frac{c^2}{b} \right\rceil = \left\lceil \frac{(bk+r)^2}{b} \right\rceil = k^2b + 2kr + \left\lceil \frac{r^2}{b} \right\rceil$ and $(c + k)k = (kb + r + k)k = k^2b + 2kr + k(k - r)$, so

$$\left\lceil \frac{r^2}{b} \right\rceil = k(k - r). \tag{10.2}$$

Notice that (10.2) provides another positive integer solution of (10.1), namely $c' = r$ and $k' = k - r$, with $c' > 0$ and $0 < k' < k$. That contradicts the minimality of k , and hence finishes the solution.