The 9-Digit Problem

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In arithmetic a non-zero digit is one of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$. Are there three 3-digit numbers abc, def, ghi, where a, b, c, d, e, f, g, h, i are distinct non-zero digits, such that

$$\mathrm{abc} + \mathrm{def} = \mathrm{ghi}?$$

By trial and error we find

$$128 + 439 = 567,$$
$$354 + 627 = 981,$$
$$173 + 286 = 459.$$

If we try hard enough, we will find many more such equations. Let us call each of the above equations a solution. The 9-digit problem is this: How many solutions are there?

One way to find an answer is by finding all solutions explicitly by whatever method we have in hand. Trial and error is such a method but is rather laborious if we do not know beforehand roughly how many solutions there are. Or we can write a computer program to search for all solutions and list them. In this note we shall show that there is a way to find all solutions by first finding a much smaller subset of solutions satisfying certain conditions.

Let abc + def = ghi be a solution. Then we have the addition diagram

$$\begin{array}{r} \mathrm{abc} \\ +\ \mathrm{def} \\ \hline \mathrm{ghi} \end{array}$$

which means the same thing. In what follows we shall call the addition diagram a solution too.

Given any two 3-digit numbers abc and def, how do we find by hand its sum abc + def? We first display abc + def as

$$\begin{array}{r} \mathrm{abc} \\ +\ \mathrm{def} \end{array}$$

and then carry out the additions c + f, b + e and a + d starting from c + f. Since we are looking for abc and def such that abc + def is another 3-digit number ghi, we can conclude immediately that $a + d < 10$, i.e. there is no carry in a + d. Now for c + f we have either $c + f = i < 10$ or $c + f = i + 10$ for some digit i, i.e. there is either no carry or one carry in c + f. Similarly, in b + e we have either no carry or one carry. Then the following four cases can occur:

there is no carry in c + f and b + e (C1),

there is a carry in c + f and no carry in b + e (C2),

there is no carry in c + f and one carry in b + e (C3),

there is one carry in c + f and one in b + e (C4).

These four cases can all occur if we drop the condition that the digits are distinct. For example, we have

234	234	243	234
+ 451	+ 458	+ 576	+ 597
685	692	819	831
no carry	carry in c + f only	carry in b + e only	carry in c + f and b + e

If all the digits are distinct, then the situation is different. Only cases (C2) and (C3) can occur. We now proceed to prove this and two other results which we shall need.

Result 1: Case (C1) cannot occur.

Proof. Suppose that (C1) can occur. Then we have

$$a + d = g,$$
$$b + e = h,$$
$$c + f = i.$$

Adding the equations and rearranging terms we get

$$a + b + c + d + e + f = g + h + i.$$

Adding g + h + i to both sides we get

$$a + b + c + d + e + f + g + h + i = 2 \times (g + h + i).$$

Since

$$a + b + c + d + e + f + g + h + i$$

is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$$

in some order and

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45,$$

we have

$$45 = 2 \times (g + h + i),$$

which is a contradiction. This contradiction proves Result 1.

Result 2: Case (C4) cannot occur.

Proof. Suppose Case (C4) can occur. Then we have

$$c + f = i + 10$$
$$1 + b + e = h + 10$$
$$1 + a + d = g.$$

Then it follows from a similar argument that

$$45 = 2 \times (g + h + i) + 18,$$

which is a contradiction. This contradiction proves Result 2.

Result 3: The formula

$$g + h + i = 18 \qquad \text{(F1)}$$

is valid for all solutions.
Proof. Result 1 and Result 2 together imply that in any solution either Case (C2) or Case (C3) must hold. Therefore we have either

$$\begin{aligned} c + f &= i + 10 \\ 1 + b + e &= h \\ a + d &= g. \end{aligned}$$

or

$$\begin{aligned} c + f &= i \\ b + e &= h + 10 \\ 1 + a + d &= g. \end{aligned}$$

In either case a similar argument will give us

$$45 = 2 \times (g + h + i) + 9$$

from which the formula (F1) follows.

It will be seen that the formula (F1) will play an important role in our effort to find all solutions.

To state our next result we need to define a solution satisfying certain conditions.

Definition. A solution

$$\begin{array}{r} abc \\ +\ def \\ \hline ghi \end{array}$$

is a basic solution if and only if the digits satisfy the conditions

$$a < d,\ b < e,\ c < f \qquad \text{(P1)},$$

$$c + f = i + 10,\ 1 + b + e = h,\ a + d = g \qquad \text{(P2)}.$$

By this definition, only the first of the following three solutions

$$\begin{array}{r} 128 \\ +\ 439 \\ \hline 567 \end{array} \qquad \begin{array}{r} 654 \\ +\ 327 \\ \hline 981 \end{array} \qquad \begin{array}{r} 276 \\ +\ 183 \\ \hline 459 \end{array}$$

is a basic solution.

We can now state the result crucial to our effort to find all solutions.

Result 4: Every solution can be obtained from a basic solution.

Proof. Let

$$\begin{array}{r} abc \\ +\ def \\ \hline ghi \end{array}$$

be a solution. We first swop one or more digits in the first row with those in the second row in similar positions so that the condition (P1) for a basic solution is satisfied. If by so doing the solution obtained is a basic solution, we have nothing more to do. If not, the right most column must be c + f = i. Then we obtain a basic solution by shifting this column to the left most position. Thus from any solution we can obtain a basic solution by swopping digits and shifting columns. From this basic solution we can retrieve the original solution by carrying out the same procedure except that shifting now means moving the left most column to the right most position. This proves Result 4.

We illustrate this procedure with two examples. The solution

$$\begin{array}{r} 654 \\ +\ 327 \\ \hline 981 \end{array}$$

is not a basic solution. We first swop 6 with 3 and then 5 with 2 to obtain the solution

$$\begin{array}{r} 324 \\ +\ 657 \\ \hline 981 \end{array}$$

which is a basic solution.

The solution

$$\begin{array}{r} 276 \\ +\ 183 \\ \hline 459 \end{array}$$

is not a basic solution. We swop 2 with 1 and 6 with 3 to obtain the solution

$$\begin{array}{r} 173 \\ +\ 286 \\ \hline 459 \end{array}$$

This is still not a basic solution. We now shift the right most column to the left most position to obtain

$$\begin{array}{r} 317 \\ +\ 628 \\ \hline 945 \end{array}$$

which is a basic solution.

It follows from Result 4 that all solutions can be found if we have first found all basic solutions. We now explain how to find a basic solution step-by-step.

Suppose that

$$\begin{array}{r} \text{abc} \\ +\ \text{def} \\ \hline \text{ghi} \end{array}$$

is a basic solution. Then there are digits c, f and i such that c $<$ f and c + f = i + 10.

We start with i = 1. The following table lists all possible c and f such that c + f = 1+ 10:

c	f
2	9
3	8
4	7
5	6

We then use the formula (F1) $g + h + i = 18$ to find all possible g and h such that $g + h + 1 = 18$. In this case we get either $g = 8$, $h = 9$ or $g = 9$, $h = 8$.

Let us choose $g = 8$, $h = 9$. Then the choice $c = 2$, $f = 9$ will give us $h = f = 9$, contradicting the fact that all the digits in a solution are distinct. The choice $g = 8$, $h = 9$ and $c = 3$, $f = 8$ will give us $g = f = 8$, again a contradiction. Likewise the choice $g = 9$, $h = 8$ will lead to contradictions. This means that basic solutions do not exist for $c = 2$, $f = 9$ and $c = 3$, $f = 8$.

What about $c = 4$ and $f = 7$? As before, we have $g = 8$, $h = 9$ or $g = 9$, $h = 8$. Let us choose $g = 8$, $h = 9$. In this case there is no contradiction like above. Trial and error can now be used to determine the digits a, b, d, e in

$$\begin{array}{r} ab4 \\ +\ de7 \\ \hline 891 \end{array}$$

We give here full details how to do this. We first determine b. Clearly $b = 4$, $b = 5$ cannot hold. Since b cannot be 1, 4, 7, 8 or 9, only $b = 2$ and $b = 3$ are possible. Try $b = 2$. Then we have $e = 6$ and $a = 3$, $d = 5$ follow. Hence a basic solution is found. Next try $b = 3$. Then $e = 5$ followed by $a = 2$, $d = 6$. Again a basic solution is found. Hence $c = 4$, $f = 7$ and $g = 8$, $h = 9$ gives us the following two basic solutions:

$$\begin{array}{r} 324 \\ +\ 567 \\ \hline 891 \end{array} \qquad \begin{array}{r} 234 \\ +\ 657 \\ \hline 891 \end{array}$$

Similarly for the choice $g = 9$, $h = 8$ we first determine b and use the trial and error method to determine the other digits. We see at once that $b = 2$ is the only possibility and from this we get $e = 5$ followed by $a = 3$, $d = 6$. Therefore another basic solution is found, namely

$$\begin{array}{r} 324 \\ +\ 657 \\ \hline 981 \end{array}$$

In the above table we still have the choice $c = 5$, $f = 6$ left. The trial and error method will give us the following as the only basic solution

$$\begin{array}{r} 235 \\ +\ 746 \\ \hline 981 \end{array}$$

So in total $i = 1$ gives us 4 basic solutions.

We now choose i from 2, 3, 4, 6 or 7 and for each i we find all c and f such that $c < f$ and $c + f = i + 10$ and all g and h such that $g + h + i = 18$. Trial and error will find us all the basic solutions for this i bearing in mind that we may find no solutions in some cases. Thus when i ranges through 2 to 7 the rest of the solutions will all be found.

What all this means is that to find all basic solutions we need first to compile a list of c and f for which basic solutions exist. This is the same as saying we need first to compile a list of c and f and from each c, f find all digits g, h from g + h + (c + f − 10) = 18 such that c, f, g, h are distinct. Then trial and error will then find the possible basic solutions.

Table A lists all c and f for which no solutions exist and Table B lists all c and f for which basic solutions exist and the number of solutions for each such c and f.

Table A

c	f
2	9
3	8
3	9
5	7
6	7
6	9

Table B

c	f	i	Number of Basic solutions
4	7	1	3
4	8	2	2
4	9	3	3
5	6	1	1
5	8	3	2
5	9	4	1
6	8	4	2
7	8	5	2
7	9	6	2
8	9	7	3
Total			21

We are now ready to give an answer to the 9-digit problem: How many solutions are there?

From each basic solution we can obtain 7 more solutions by swopping the digits as described earlier. From each of these 8 solutions we can obtain 1 more solution by shifting the left most column to the right most position. Therefore we can obtain 16 solutions (including itself) from any basic solution. It is clear that distinct basic solutions gives us distinct solutions. It follows from Table B that there are altogether 21×16 solutions. The answer to the 9-digit problem is therefore 336.

If abc + def = ghi is a solution, then so is def + abc = ghi. Some people may insist that every solution must have abc < def. If we impose this additional condition on solutions, then it is clear that only 1/2 of the 336 solutions satisfy this condition and the answer to the 9-digit problem is $168 = 336 \times 1/2$. However, the number of basic solutions remains at 21.

We end this note by remarking that if you wish to compile Table B by yourself you will find that the trial and error method is easy to implement and gets progressively easier as you go down the list. Try it and have fun.

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