IMO 2019
60TH INTERNATIONAL MATHEMATICAL OLYMPIAD

English (eng), day 1

Tuesday, July 16, 2019

Problem 1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Problem 2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Problem 3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

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60TH INTERNATIONAL MATHEMATICAL OLYMPIAD

English (eng), day 2

Wednesday, July 17, 2019

Problem 4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

Problem 6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Solutions

1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,
$$f(2a) + 2f(b) = f(f(a + b)).$$

Solution. Let
$$f(2a) + 2f(b) = f(f(a + b)). \tag{1.1}$$

Let $a = 0$ in (1.1), we have $f(0) + 2f(b) = f(f(b))$. In this equation, replacing b by a, we have
$$f(0) + 2f(a) = f(f(a)). \tag{1.2}$$

Substituting $b = 0$ in (1.1), we have
$$f(2a) + 2f(0) = f(f(a)). \tag{1.3}$$

Equating (1.2) and (1.3), we have
$$f(2a) = 2f(a) - f(0). \tag{1.4}$$

Substituting (1.4) into (1.1), we have $2f(a)-f(0)+2f(b) = f(f(a+b))$. Using (1.2), we have $f(0)+2f(a+b) = f(f(a+b))$. Thus $2f(a)-f(0)+2f(b) = f(0)+2f(a+b)$. That is $f(a+b) = f(a)+f(b)-f(0)$. This means the function $f(x)-f(0)$ is linear. Thus $f(x) = kx + f(0)$ for some integer k.

As $3f(0) = f(f(0))$, we have $3f(0) = kf(0) + f(0)$. That is $(k-2)f(0) = 0$ so that $k = 2$ or $f(0) = 0$.

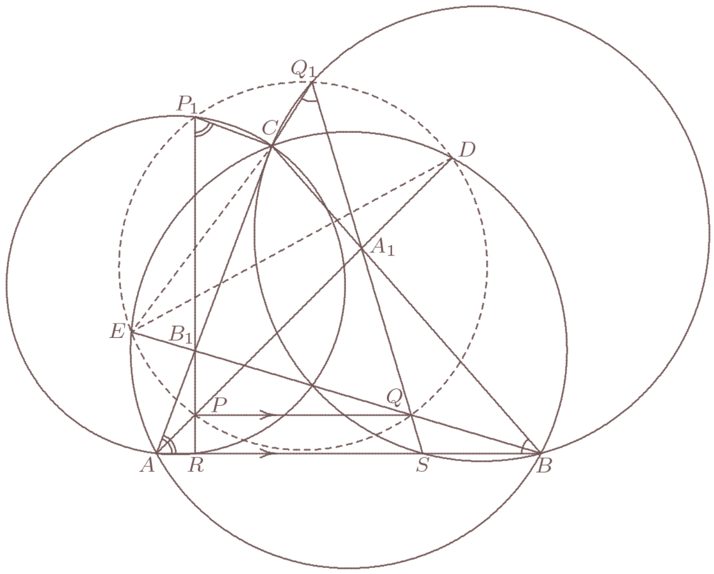
If $k = 2$, then $f(x) = 2x+C$, where $C \in \mathbb{Z}$, is indeed a solution by direct checking. If $f(0) = 0$, then $f(x) = kx$. Substituting this into (1.1), we have $2k(a+b) = k^2(a+b)$ so that $k = 0$ or 2.

Consequently, the solutions are $f(x) = 0$ for all $x \in \mathbb{Z}$ or $f(x) = 2x + C$.

2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Solution. Let the line P_1P meet the line AB at R. As $\angle PP_1C = \angle BAC$, the points A, P_1, C, R are concyclic. Similarly let the line Q_1Q meet the line AB at S. $\angle CQ_1Q = \angle CBA$, the points C, S, B, Q_1 are concyclic. Let the lines AA_1 and BB_1 meet the circumcircle at D and E, respectively. Then $\angle DPQ = \angle DAB = \angle DEB = \angle DEQ$ so that P, Q, D, E are concyclic. As $\angle B_1EC = \angle BEC = \angle BAC = \angle RP_1C = \angle B_1P_1C$, the points B_1, E, P_1, C are concyclic. It follows that $\angle EP_1P = \angle EP_1B_1 = \angle ECB_1 = \angle ECA = \angle EDP$ so that P_1, E, P, D are concyclic. Thus the five points P, Q, D, P_1, E lies on a common circle. Similarly, Q_1, E, Q, D are concyclic. Consequently, all six points P, Q, D, Q_1, P_1, E lie on a common circle.



3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

Solution. First we rephrase the problem in terms of graph theory. Consider a graph G with 2019 vertices, 1010 of which have degree 1009 and 1009 of which have degree 1010. One is allowed to perform operations on G of the following kind:

> *Suppose that vertex A is adjacent to 2 distinct vertices B and C which are not adjacent to each other. Then one may remove the edges AB and AC from G and add the edge BC into G.*

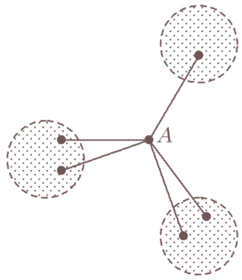
Call such an operation a *refriending*. One wants to prove that, via a sequence of such refriending, one can reach a graph which is a disjoint union of single edges and vertices.

Note that the given graph is connected, since the total degree of any 2 vertices is at least 2018 and hence they are either adjacent or have one neighbour in common. Hence the given graph satisfies the following condition:

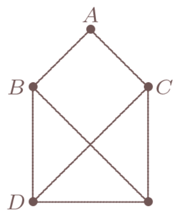
> *Every connected component of G with at least 3 vertices is not complete and has a vertex of odd degree.* (3.1)

We shall show that if a graph G satisfies condition (3.1) and has a vertex of degree at least 2, then there is a refriending on G that preserves condition (3.1). Since refriendings decrease the total number of edges of G, by using sequence of such refriendings, we must reach a graph G with maximal degree at most 1, so we are done.

Pick a vertex A of degree at least 2 in a connected component G' of G. Since no component of G with at least three vertices is complete we may assume that not all of the neighbours of A are adjacent to one another. (For example, pick a maximal complete subgraph K of G'. Some vertex A of K has a neighbour outside K, and this neighbour is not adjacent to every vertex of K by maximality.) Removing A from G splits G' into smaller connected components $G_1, \ldots, G_k$ (possibly with $k = 1$), to each of which A is connected by at least one edge. We divide into several cases.



Removing A from G splits G' into smaller connected components.



Case 4: Removing edges BA and BD and adding in edge AD does not disconnect G'.

Case 1: $k \geq 2$ and A is connected to some G_i by at least 2 edges.

Choose a vertex B of G_i, adjacent to A, and a vertex C in another component G_j adjacent to A. The vertices B and C are not adjacent, and hence removing edges AB and AC and adding edge BC does not disconnect G'. It is easy to see that this preserves the condition, since the refriending does not change the parity of the degrees of vertices.

Case 2: $k \geq 2$ and A is connected to each G_i by exactly 1 edge.

Consider the induced subgraph on any G_i, and the vertex A. The vertex A has degree 1 in this subgraph; since the number of odd-degree vertices of a graph is always even, we see that G_i has a vertex of odd degree (in G). Thus if we let B and C be any distinct neighbours of A, then removing edges AB and AC and adding in edge BC preserves the above condition; the refriending creates two new components, and if either of these components has at least three vertices, then it cannot be complete and must contain a vertex of odd degree (since each G_i does).

Case 3: $k = 1$ and A is connected to G_1 by at least 3 edges.

By assumption, A has two neighbours B and C which are not adjacent to one another. Removing edges AB and AC and adding in edge BC does not disconnect G'. We are then done as in case 1.

Case 4: $k = 1$ and A is connected to G_1 by exactly 2 edges.

Let B and C be the two neighbours of A, which are not adjacent. Removing edges AB and AC and adding in edge BC results in two new components: one consisting of a single vertex; and the other containing a vertex of odd degree. We are done unless this second component would be a complete graph on at least 3 vertices. But in this case, G_1 would be a complete graph minus the single edge BC, and hence has at least 4 vertices since G' is not a 4-cycle. if we let D be a third vertex of G_1, then removing edges BA and BD and adding in edge AD does not disconnect G'. We are then done as in case 1.

Note that (3.1) is the condition that precisely characterises those graphs which can be reduced to a graph of maximal degree ≥ 1 by a sequences of refriendings.

4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}).$$

Solution. First we have $v_2(k!) = \sum_{i=1}^{\infty} \lfloor \frac{k}{2^i} \rfloor < \sum_{i=1}^{\infty} \frac{k}{2^i} = k$. On the other hand,

$$v_2(RHS) = 0 + 1 + \cdots + n - 1 = \frac{n(n-1)}{2}.$$

Thus $k > \dfrac{n(n-1)}{2}$. Therefore

$$2^{n^2} > RHS = k! > \left(\frac{n(n-1)}{2}\right)! \geq 1 \cdot 2 \cdot 2 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot \underbrace{8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdots 8}_{\frac{1}{2}n(n-1)-7 \text{ terms}}.$$

Thus

$$2^{n^2} > 2^{2+8+3((n(n-1)/2-7)}.$$

That is

$$n^2 > 2 + 8 + 3(n(n-1)/2 - 7).$$

This holds only for $n < 7$. Checking $n = 1, 2, 3, 4, 5, 6$, we have $(n, k) = (1, 1), (2, 3)$.

5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the kth coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

   (a) Show that, for each initial configuration, Harry stops after a finite number of operations.

   (b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

Solution. We consider what happens with configurations depending on the coins they start and end with.

- If a configuration starts with H, the last $n-1$ coins follow the given rules, as if they were all the coins, until they are all T, then the first coin is turned over.

- If a configuration ends with T, the last coin will never be turned over, and the first $n-1$ coins follow the given rules, as if they were all the coins.
- If a configuration starts with T and ends with H, the middle $n-2$ coins follow the given rules, as if they were all the coins, until they are all T. After that, there are $2n-1$ more steps: first coins $1, 2, \ldots, n-1$ are turned over in that order, then coins $n, n-1, \ldots, 1$ are turned over in that order.

As this covers all configurations, and the number of steps is clearly finite for 0 or 1 coins, it follows by induction on n that the number of steps is always finite.

We define $E_{AB}(n)$, where A and B are each one of H, T or $*$, to be the average number of steps over configurations of length n restricted to those that start with A, if A is not $*$, and that end with B, if B is not $*$ (so $*$ represents either "H or T"). The above observations tell us that, for $n \geq 2$:

- $E_{H*}(n) = E(n-1) + 1$.
- $E_{*T}(n) = E(n-1)$.
- $E_{HT}(n) = E(n-2) + 1$ (by using both observations for $H*$ and $*T$).
- $E_{TH}(n) = E(n-2) + 2n - 1$.

Now $E_{H*}(n) = \frac{1}{2}(E_{HH}(n) + E_{HT}(n))$, so $E_{HH}(n) = 2E(n-1) - E(n-2) + 1$. Similarly, $E_{TT} = 2E(n-1) - E(n-2) - 1$. So

$$E(n) = \frac{1}{4}(E_{HT}(n) + E_{HH}(n) + E_{TT}(n) + E_{TH}(n)) = E(n-1) + \frac{n}{2}.$$

We have $E(0) = 0$ and $E(1) = \frac{1}{2}$, so by induction on n we have $E(n) = \frac{1}{4}n(n+1)$.

6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

   Prove that lines DI and PQ meet on the line through A perpendicular to AI.

Solution. Let Ω be the circumcircle of $\triangle BIC$, and let QP intersect BC at Z. Let the angles at A, B, C in $\triangle ABC$ be α, β, γ respectively. Without loss of generality, we may assume $\beta > \gamma$. It is easy to show that $\angle BIC = 90° + \frac{\alpha}{2}, \angle EDF = 90° - \frac{\alpha}{2}, \angle DEF = 90° - \frac{\beta}{2}, \angle DFE = 90° - \frac{\gamma}{2}$. Let AI meet BC at D'. Let L be the intersection of the line ID and the external bisector $\angle BAC$. We use directed angles for our calculations. We have

$$\angle ALD = 90° - \angle LIA = \angle AD'D = \frac{\alpha}{2} + \gamma.$$

Also

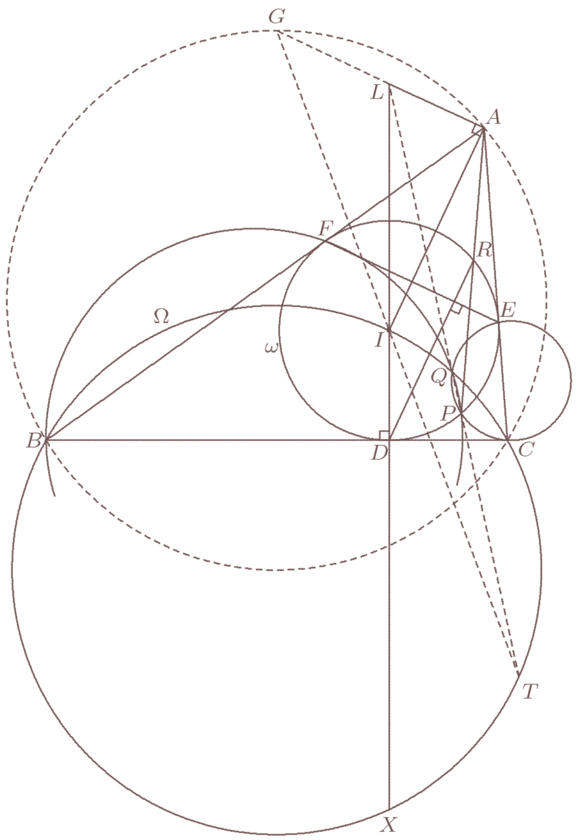
$$\begin{aligned} \angle DPA = \angle DPR = \angle DFE + \angle RFE = \angle DFE + \angle RDE \\ = \frac{\alpha}{2} + \frac{\beta}{2} + 90° - \frac{\alpha}{2} - \frac{\gamma}{2} = \frac{\alpha}{2} + \beta. \end{aligned}$$

Thus $\angle ALD + \angle DPA = 180°$. This means L, A, P, D are concyclic.

Next we have $\angle BQZ = \angle BFP$ and $\angle CQZ = \angle CEP$ so that

$$\angle BQC = \angle BFP + \angle CEP = \angle PEF + \angle PFE$$
$$= 180^\circ - \angle EPF = 180^\circ - \angle EDF = 90^\circ + \frac{\alpha}{2} = \angle BIC,$$

showing that Q lies on Ω.



A spiral symmetry taking X, T, B, I, C to D, P, F, R, E respectively.

Let T, X be the second intersection points of the lines PQ, ID with Ω. We will show that there is a spiral symmetry taking X, T, B, I, C to D, P, F, R, E respectively.

First we have $\angle FER = \angle FDR = 90^\circ - \angle DFE = \gamma/2 = \angle BCI$. Similarly, $\angle EFR = \angle CBI$ so that $\triangle BIC \sim \triangle FRE$. Thus the spiral symmetry taking $\triangle FRE$ to $\triangle BIC$ also takes D to X as D, X lies on ω, Ω respectively.

Let G be the midpoint of the arc BAC on the circumcircle of $\triangle ABC$. Note that AG is in fact the external bisector of $\angle BAC$. Sine $GB = GC$, $\angle BGC = \angle FAE$, we have $\triangle FAE \sim \triangle BGC$ so the same spiral symmetry takes A to G.

Now since $\angle PEF = \angle PFB = \angle TQB = \angle TCB$ and similarly, $\angle TBC = \angle PFE$ we know that $\triangle TBC \sim \triangle PFE$ so that spiral symmetry takes P to T. Spiral symmetries map lines to lines so that G, I, T are collinear since A, P, R are. So now $\angle GLX = \angle APD = \angle RPD = \angle ITX = \angle GTX$ showing G, L, T, X are concyclic. Finally, using that, the spiral symmetry and the fact that $LAPD$ is cyclic, we have that $\angle PLD = \angle PAD = \angle TGX = \angle TLX$ hence T, L, P are collinear and thus PQ, DI meet on the external bisector of $\angle BAC$.