On a Class of Diophantine Equations with Floor Function

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Abstract

This paper considers the following equation:

$$\left\lfloor \frac{N}{p_1} \right\rfloor + \cdots + \left\lfloor \frac{N}{p_n} \right\rfloor = N \tag{1}$$

Here, we fix $n \in \mathbb{Z}^+$, $p_1, \ldots, p_n \in \mathbb{Z}^+ \setminus \{1\}$, and we want to solve for unknown N. This problem can be found in several mathematical competitions, in which it mostly comes in the form of simple and specific cases, such as p_i following a certain sequence of numbers, or small values of n. We wish to further examine this equation and the properties of its solutions.

1 Introduction

For integers N, k where $k \geq 2$, $\left\lfloor \frac{N}{k} \right\rfloor$ gives us various useful information. The trivial information would be the integer component of the possibly non-integer value of $\frac{N}{k}$. If $N > 0$, then $\left\lfloor \frac{N}{k} \right\rfloor$ is exactly the number of positive multiples of k less than or equal to N. As such, examining $\left\lfloor \frac{N}{p_i} \right\rfloor + \cdots + \left\lfloor \frac{N}{p_n} \right\rfloor$ would be much similar to understanding the distribution of various multiples of $p_1, \ldots, p_n$.

While it may be unobvious, there is a possible application to this problem: Consider forming a community with N people (where N is to be determined later), with the following requirements:

1. The community contains n services, $S_1, \ldots, S_n$, to be provided by members of the community. Each person providing the service S_i serve exactly p_i people (so k people providing service S_i will serve exactly kp_i people).

2. Each person in the community is to provide exactly one service, and receives each service S_i at most once.

3. Each service in the community serves the maximum number of people possible. The maximum number of people here served is precisely $p_i \left\lfloor \frac{N}{p_i} \right\rfloor$, so we must have $\left\lfloor \frac{N}{p_i} \right\rfloor$ people allocated to providing service S_i.

Since everyone in the community provides exactly one service each, we have $\left\lfloor \frac{N}{p_1} \right\rfloor + \cdots + \left\lfloor \frac{N}{p_n} \right\rfloor = N$. Determining the possible number of people in the community would be equivalent to solving equation (1).

This paper thus serves to establish general results to the equation.

2 Definitions and Basic Results

To examine this equation, we first establish a few important results that would prove to be crucial for many subsequent results.

Definition 2.1. Let $N \in \mathbb{Z}$. We say that N is a **solution** iff $\sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor = N$. Furthermore, we say that N is a **non-trivial solution** iff $N \neq 0$.

Theorem 2.2. *Let $l := \text{lcm}(p_1, \ldots, p_n)$. Then:*

1. *If $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, then (1) has infinitely many solutions, given by multiples of l.*
2. *If $\sum_{i=1}^{n} \frac{1}{p_i} < 1$, then (1) has no positive solutions.*
3. *If $\sum_{i=1}^{n} \frac{1}{p_i} > 1$, then (1) has no negative solutions.*
4. *If $\sum_{i=1}^{n} \frac{1}{p_i} \neq 1$, then (1) has at most l solutions.*

Proof. For the whole of this proof, if we assume that N is a solution to this equation, then we let $N = kl + x$, where $k, x \in \mathbb{Z}$, $0 \leq x < l$.

1. Note that if $0 < x < l$, we must have that $\left\lfloor \frac{x}{p_i} \right\rfloor < \frac{x}{p_i}$ for some i, as otherwise $p_i \mid x$ for all $i \Rightarrow l \mid x$, contradicting that $0 < x < l$. Thus, if $0 < x < l$, we have:

$$\begin{aligned}
\sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor &= \sum_{i=1}^{n} \left\lfloor \frac{kl + x}{p_i} \right\rfloor \\
&= \sum_{i=1}^{n} \frac{kl}{p_i} + \sum_{i=1}^{n} \left\lfloor \frac{x}{p_i} \right\rfloor \\
&= kl + \sum_{i=1}^{n} \left\lfloor \frac{x}{p_i} \right\rfloor \\
&< kl + \sum_{i=1}^{n} \frac{x}{p_i} \\
&= N
\end{aligned}$$

Therefore, all integers which are not multiples of l cannot be solutions to (1). Since $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, when $N = kl$ we have:

$$\sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor = \sum_{i=1}^{n} \left\lfloor \frac{kl}{p_i} \right\rfloor = \sum_{i=1}^{n} \frac{kl}{p_i} = kl = N$$

2. We suppose that $N > 0$ is a solution. Then:

$$N = \sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor \leq \sum_{i=1}^{n} \frac{N}{p_i} < N$$

which is a contradiction.

3. Similarly, if we suppose that $N < 0$ is a solution:

$$N = \sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor \geq \sum_{i=1}^{n} \frac{N}{p_i} > N$$

which is a contradiction.

4. It suffices to show that there does not exist two solutions N_1, N_2 such that $N_1 \equiv N_2 \pmod{l}$. Suppose that N_1 is a solution for (1), and $N_2 = k'l + N_1$. Then:

$$\begin{aligned}
\sum_{i=1}^{n} \left\lfloor \frac{N_2}{p_i} \right\rfloor &= \sum_{i=1}^{n} \left\lfloor \frac{k'l + N_1}{p_i} \right\rfloor \\
&= \sum_{i=1}^{n} \frac{k'l}{p_i} + \sum_{i=1}^{n} \left\lfloor \frac{N_1}{p_i} \right\rfloor \\
&= k'l \sum_{i=1}^{n} \frac{1}{p_i} + N_1 \\
&\neq k'l + N_1 \\
&= N_2
\end{aligned}$$

and thus N_2 cannot be a solution to (1).

□

Naturally, we would like to examine the equation from other various angles. Here, we shall examine the properties of the following function f:

$$f(N) = \sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor - N$$

Note that f will be defined as such for all subsequent sections, unless otherwise stated. In this section we define $s := \sum_{i=1}^{n} \frac{1}{p_i}$.

Lemma 2.3. *Let $x, y \in \mathbb{Z}$ and $y \geq 1$. Then* $\left\lfloor \frac{x}{y} \right\rfloor \geq \frac{x+1}{y} - 1$.

Proof. We shall first show that this is true for $x \in \{0, \ldots, y-1\}$. Since $0 \leq \frac{x}{y} < 1$, $\left\lfloor \frac{x}{y} \right\rfloor = 0$. Furthermore, since $0 \leq x \leq y-1$, $\frac{x+1}{y} - 1 \leq \frac{y}{y} - 1 = 0$. As such, the inequality holds for such x.

Now suppose $x = ky + r$ for some $k, r \in \mathbb{Z}$, $0 \leq r < y$. Then $\left\lfloor \frac{x}{y} \right\rfloor = k + \left\lfloor \frac{r}{y} \right\rfloor \geq k + \frac{r+1}{y} - 1 = \frac{ky+r+1}{y} - 1 = \frac{x+1}{y} - 1$. Therefore, the inequality holds for any integer x. □

Proposition 2.4. $(s-1)N - n + s \leq f(N) \leq (s-1)N$.

Proof. For the right inequality:

$$f(N) = \sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor - N \leq \sum_{i=1}^{n} \frac{N}{p_i} - N = (s-1)N$$

For the left inequality, we invoke Lemma 2.3:

$$f(N) = \sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor - N \geq \sum_{i=1}^{n} \frac{N+1}{p_i} - n - N = (s-1)N - n + s$$

□

Corollary 2.5 (Bound of Solutions)**.** *Suppose $s \neq 1$. If $f(N) = 0$, then $|N| \leq \frac{n-s}{|s-1|}$.*

Proof. Since $f(N) = 0$, $(s-1)N - (n-s) \leq 0 \Rightarrow (s-1)N \leq n - s$, by Theorem 2.2(2) and Theorem 2.2(3), we have that $(s-1)N \geq 0$. Therefore, $|(s-1)N| \leq n - s \Rightarrow |N| \leq \frac{n-s}{|s-1|}$. □

Corollary 2.6. *Suppose $s \neq 1$. Then (1) has finitely many solutions.*

Proposition 2.7. *Let $l := \operatorname{lcm}(p_1, \ldots, p_n)$. Then $f(N + l) = f(N) + (s-1)l$.*

Proof.

$$\begin{aligned} f(N+l) &= \sum_{i=1}^{n} \left\lfloor \frac{N+l}{p_i} \right\rfloor - (N+l) \\ &= \sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor + l \sum_{i=1}^{n} \frac{1}{p_i} - N - l \\ &= \sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor - N + sl - l \\ &= f(N) + (s-1)l \end{aligned}$$

□

Remark. Both bounds in Proposition 2.4 are the sharpest linear bounds, as equality on the right is always attained at $N = 0$, and equality on the left is always attained at $N = -1$, and they are then attained infinitely many times due to Proposition 2.7.

An alternate point of view to this problem would be to view it as the following recurrence relation:

$$x_{k+1} = \sum_{i=1}^{n} \left\lfloor \frac{x_k}{p_i} \right\rfloor, \; x_1 \in \mathbb{R} \tag{2}$$

This section uses the proven inequality $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proposition 2.8. *$(x_m)_{m \in \mathbb{Z}^+}$ is a monotone sequence.*

Proof. We prove by induction. If $x_2 = x_1$, then the sequence is constant. Suppose $x_2 > x_1$, and let $d \in \mathbb{Z}^+$ such that $x_2 = x_1 + d$. Then:

$$\begin{aligned} x_3 = \sum_{i=1}^{n} \left\lfloor \frac{x_2}{p_i} \right\rfloor &= \sum_{i=1}^{n} \left\lfloor \frac{x_1 + d}{p_i} \right\rfloor \\ &\geq \sum_{i=1}^{n} \left\lfloor \frac{x_1}{p_i} \right\rfloor + \sum_{i=1}^{n} \left\lfloor \frac{d}{p_i} \right\rfloor \\ &\geq \sum_{i=1}^{n} \left\lfloor \frac{x_1}{p_i} \right\rfloor \\ &= x_2 \end{aligned}$$

If instead $x_2 < x_1$, then suppose $x_2 = x_1 - d$, and:

$$\begin{aligned} x_3 = \sum_{i=1}^{n} \left\lfloor \frac{x_2}{p_i} \right\rfloor &= \sum_{i=1}^{n} \left\lfloor \frac{x_1 - d}{p_i} \right\rfloor \\ &\leq \sum_{i=1}^{n} \left\lfloor \frac{x_1}{p_i} \right\rfloor + \sum_{i=1}^{n} \left\lfloor \frac{-d}{p_i} \right\rfloor + n \\ &\leq \sum_{i=1}^{n} \left\lfloor \frac{x_1}{p_i} \right\rfloor + \sum_{i=1}^{n} -1 + n \\ &= x_2 \end{aligned}$$

We can repeat this argument for all of x_m, and thus the sequence is monotone. □

Lemma 2.9. *Suppose $(x_m)_{m \in \mathbb{Z}^+}$ converges. Then it converges to a solution of (1).*

Proof. Since the sequence converges, it is Cauchy. We choose $\epsilon = \frac{1}{2}$, and there exists $M \in \mathbb{Z}^+$ such that $|x_{m_1} - x_{m_2}| < \epsilon$ for $m_1, m_2 > M$. Since the sequence only consists of integers, the sequence becomes constant for sufficiently large m. Then the sequence converges to this constant, and is thus a solution. □

Proposition 2.10. *Suppose (1) has at least two solutions. We choose two solutions N_1 and N_2 where $N_1 < N_2$, and there exists no other solutions in the interval (N_1, N_2). If $N_1 \leq x_1 \leq N_2$, then $(x_m)_{m \in \mathbb{Z}^+}$ converges to N_1 or N_2.*

Proof. Let $g(x) = \sum_{i=1}^{n} \left\lfloor \frac{x}{p_i} \right\rfloor$. Since $\lfloor \cdot \rfloor$ is (not strictly) increasing, so is g. Thus, we can conclude that for $x \in [N_1, N_2]$:

$$N_1 \leq x \leq N_2 \Rightarrow g(N_1) \leq g(x) \leq g(N_2) \Rightarrow N_1 \leq g(x) \leq N_2$$

As such, the sequence is bounded by $[N_1, N_2]$. Since the sequence is monotone, it converges to some $L \in [N_1, N_2]$. Finally, since there exists no solutions in (N_1, N_2), and L is a solution, it follows that $L = N_1$ or $L = N_2$. □

These results will be extremely crucial in the next section, where we examine the case when $\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

3 The case $\sum_{i=1}^{n} \frac{1}{p_i} < 1$

We next consider when $\sum_{i=1}^{n} \frac{1}{p_i} < 1$. This case is more interesting than the alternate case of $\sum_{i=1}^{n} \frac{1}{p_i} > 1$, as we will prove that under this condition, we are guaranteed to have a non-trivial solution.

Lemma 3.1. *Let $N \in \mathbb{Z}^-$. Then $\sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor \leq -n$.*

Proof.

$$\sum_{i=1}^{n} \left\lfloor \frac{N}{p_i} \right\rfloor \leq \sum_{i=1}^{n} -1 \leq -n$$

□

Remark. As a corollary, we can conclude that if N is a negative non-trivial solution to (1), then $N \leq -n$.

Theorem 3.2. *If $\sum_{i=1}^{n} \frac{1}{p_i} < 1$, then (1) has at least one non-trivial solution.*

Proof. Observe that $\sum_{i=1}^{n} \left\lfloor \frac{-l}{p_i} \right\rfloor = \sum_{i=1}^{n} \frac{-l}{p_i} > -l$ as $\sum_{i=1}^{n} \frac{1}{p_i} < 1$. We consider the recurrence relation in (2) with $x_1 = -l$, and we have that $(x_m)_{m \in \mathbb{Z}^+}$ is a monotonically increasing sequence by Proposition 2.8. Since $x_m \leq -n$, $(x_m)_{m \in \mathbb{Z}^+}$ is bounded and thus converges to some $L \leq -n < 0$. Then L is a non-trivial solution to (1). □

Proposition 3.3. *If $\sum_{i=1}^{n} \frac{1}{p_i} < 1$, then $(x_m)_{m \in \mathbb{Z}^+}$ converges for all $x_1 \in \mathbb{Z}$.*

Proof. Let $s := \sum_{i=1}^n \frac{1}{p_i}$. If $x_1 > 0$, then observe that $x_2 = \sum_{i=1}^n \left\lfloor \frac{x_1}{p_i} \right\rfloor \leq \sum_{i=1}^n \frac{x_1}{p_i} < x_1$, and thus the sequence is monotonically decreasing. Since $x_m \geq 0$ for all m, the sequence converges.

Let S be the set of solutions for (1). If $\min S \leq x_1 \leq 0$, then it follows from the proof in Proposition 2.10 that $(x_m)_{m\in\mathbb{Z}^+}$ converges to some elements in S. If $x_1 < \min S$, then suppose for a contradiction that $(x_m)_{m\in\mathbb{Z}^+}$ diverges. This means that $(x_m)_{m\in\mathbb{Z}^+}$ cannot be increasing as it would be bounded otherwise. Since $(x_m)_{m\in\mathbb{Z}^+}$ is decreasing and diverges, there exists $k \in \mathbb{Z}^+$ such that $x_k < -\frac{n-s}{s-1}$. However, observe that:

$$\sum_{i=1}^n \left\lfloor \frac{x_k}{p_i} \right\rfloor - x_k \geq \sum_{i=1}^n \frac{x_k+1}{p_i} - n - x_k = (s-1)x_k - n + s > 0$$

And therefore we have that $x_{k+1} > x_k$, contradicting that $(x_m)_{m\in\mathbb{Z}^+}$ is decreasing. □

Remark. Since if $x_n, x_{n+1} > 0$, then $x_{n+1} < x_n$, the sequence must converge to 0. This is consistent with Theorem 2.2(25), where the equation does not have any positive solutions.

Remark. In particular, if $x_1 < \min S$, then x_m must be increasing as otherwise it converges to some $L < \min S$, but $L \in S$.

Remark. Since for $y \in \mathbb{Z}^+$, $\left\lfloor \frac{\lfloor x \rfloor}{y} \right\rfloor = \left\lfloor \frac{x}{y} \right\rfloor$, it follows that $(x_m)_{m\in\mathbb{Z}^+}$ converges for all $x_1 \in \mathbb{R}$.

We now consider a sequence $(p_n)_{n\in\mathbb{Z}^+}$, $p_i \in \mathbb{Z}_{\geq 2}$, such that $\sum_{i=1}^\infty \frac{1}{p_i} \leq 1$ (which implies that $\sum_{i=1}^n \frac{1}{p_i} < 1$ for all n).

Proposition 3.4. *Let S_n denote the set of solutions for $\sum_{i=1}^n \left\lfloor \frac{N}{p_i} \right\rfloor = N$. Then, $\forall i, j \in \mathbb{Z}^+$, $i \neq j$, $S_i \cap S_j = \{0\}$.*

Proof. It is obvious that $0 \in S_i \cap S_j$. Since $\sum_{i'=1}^n \frac{1}{p_{i'}} < 1$, if N is a solution to (1), then $N < 0$. Suppose for a contradiction that there exists $N \in \mathbb{Z}^-$ such that $N \in S_i \cap S_j$. WLOG, suppose $i < j$. Then since $\sum_{i'=1}^i \left\lfloor \frac{N}{p_{i'}} \right\rfloor = \sum_{i'=1}^j \left\lfloor \frac{N}{p_{i'}} \right\rfloor = N$, we have that $\sum_{i'=i+1}^j \left\lfloor \frac{N}{p_{i'}} \right\rfloor = 0$. However, $\left\lfloor \frac{N}{p_{i'}} \right\rfloor < 0$ for all i', a contradiction. □

4 Illustrations and Examples

We shall illustrate our results with several examples:

Example 4.1. $\left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{2} \right\rfloor = N$
By Theorem 2.2(1), exactly all even integers are the solutions as $\frac{1}{2} + \frac{1}{2} = 1$.

Example 4.2. $\lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{3} \rfloor + \lfloor \frac{N}{6} \rfloor = N$

Again by Theorem 2.2(1), exactly all multiples of $\operatorname{lcm}(2,3,6) = 6$ are the solutions.

Example 4.3. $\lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{3} \rfloor + \lfloor \frac{N}{4} \rfloor + \lfloor \frac{N}{5} \rfloor + \lfloor \frac{N}{6} \rfloor = N$

The set of solutions is $\{0, 4, 5\}$. As consistent with Theorem 2.2(2), there exists no negative solutions. In fact, more is true: For $\lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{3} \rfloor + \cdots + \lfloor \frac{N}{n} \rfloor$ where $n \geq 6$, the set of solutions is always $\{0, 4, 5\}$.

Example 4.4. $\lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{2} \rfloor = N$

This equation has no non-trivial solutions. This thus establishes a counterexample to Theorem 3.2 with the opposite premise, where $\sum_{i=1}^{n} \frac{1}{p_i} > 1$.

Example 4.5. $\lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{3} \rfloor + \lfloor \frac{N}{11} \rfloor + \lfloor \frac{N}{14} \rfloor = N$

This equation has exactly 231 solutions, of which by Theorem 2.2(3), none of them are positive.

Example 4.6. $\lfloor \frac{N}{k} \rfloor = N$, $k \geq 2$

Regardless of the value of k, this equation always has $\{-1, 0\}$ as its set of solutions.

Just as it was mentioned in 3.4, we can instead consider $(p_n)_{n \in \mathbb{Z}^+}$ as an integer sequence.

Example 4.7. $\lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{4} \rfloor + \cdots + \lfloor \frac{N}{2^n} \rfloor = N$

This equation has exactly 2^n solutions, of which none are positive. Below is a short list of the solutions for small values of n:

n	Solutions
1	-1, 0
2	-5, -3, -2, 0
3	-17, -13, -11, -10, -7, -6, -4, 0
4	-49, -41, -37, -35, -34, -29, -27, -26, -23, -22, -20, -15, -14, -12, -8, 0

As stated in Proposition 3.4, there are no common non-trivial solutions for distinct values of n.

One may also notice that the largest non-trivial solution for $1 \leq n \leq 4$ is -2^n. This is in fact true for any positive integer n, and the author invites the reader to try to provide a proof.

Example 4.8. $\lfloor \frac{N}{r} \rfloor + \lfloor \frac{N}{r} \rfloor + \cdots + \lfloor \frac{N}{r^n} \rfloor = N$, $r \geq 3$

Unlike Example 4.7 which the number of solutions grows exponentially as n increases, the number of solutions for this equation as n increases grows very slowly (for fixed r), and does not increase monotonically. Note that by Theorem 3.2, at least one non-negative solution must exist for each pair of (r, n).

5 Conclusion and Discussion

The author believes that solving the problem allows us to understand the distribution of multiples of various integers much better, especially if $p_1, \ldots, p_n$ are prime numbers, and invites readers to attempt to establish more significant results to this problem.

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