1. Let Q+ denote the set of all positive rational numbers. Determine all functions f: Q+ −→ Q+ satisfying

f(x2f(y)

2)= f(x)

2f(y)

for all x, y ∈ Q+.

2. Let n ≥ 3 be an integer. Prove that there exists a set S of 2n positive integers satisfying the following property: For every m =2, 3,...,n, the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

3. Let ABC be a triangle with AB = AC, and let M be the midpoint of BC. Let P be a point such that PB < PC and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and ∠PXM = ∠PYM. Prove that the quadrilateral AP XY is cyclic.

4. Let n> 1 be a positive integer. Each cell of an n × n table contains an integer. Suppose the following conditions are satisfied:

(i) Each number in the table is congruent to 1 modulo n;

(ii) The sum of numbers in any row, as well as the sum of numbers in any column is congruent to n modulo n2.

Let Ri be the product of the numbers in the i

th row, and Cj be the product of
numbers in the jth column. Prove that the sums R1 +···+Rn and C1 +···+Cn are congruent modulo n4.

Time allowed: 4 hours

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Day 0

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5. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of n + 1 squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone should stay within the board). Sisyphus’s aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

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turns. (As usual, ⌈x⌉ stands for the least integer not smaller than x.)

6. Let a0,a1,a2,... be a sequence of real numbers such that a0 =0,a1 = 1, and for every n ≥ 2, there exists 1 ≤ k ≤ n satisfying

an = an−1 + ··· + an−k

k .

Find the maximal possible value of |a2018 − a2017|.

7. Let n be a natural number. A sequence is k-complete if it contains all residues modulo nk. Let Q(x) be a polynomial with integer coefficients. For k ≥ 2, define Qk(x)= Q(Qk−1(x)), where Q1(x)= Q(x). Show that if

0,Q(0),Q2(0),Q3(0), ···

is 2018-complete, then it is k-complete for all positive integers k.

[Proposed by Ma Zhao Yu]

Time allowed: 4.5 hours

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8. A point T is chosen inside a triangle ABC. Let A1,B1 and C1 be the reflec-tions of T in BC, CA, and AB respectively. Let Ω be the circumcircle of the triangle A1B1C1. The lines A1T,B1T, and C1T meet Ω again at A2,B2, and C2, respectively. Prove that the lines AA2,BB2, and CC2 are concurrent on Ω.

9. Define the sequence a0,a1,a2,... by an =2n +2⌊n/2⌋. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of two or more distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

10. Let k be a positive integer. The organising committee of a tennis tournament is to schedule the matches for 2k players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day
a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players’ stays. Determine this minimum cost.

Time allowed: 4.5 hours

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Day 2

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Solutions to NTST 2019

1. Let Q+ denote the set of all positive rational numbers. Determine all functions f: Q+ −→ Q+ satisfying

f(x2f(y)

2)= f(x)

2f(y)

for all x, y ∈ Q+.

Solution. The answer is f(x) = 1 for all x ∈ Q+. We shall prove that this is the only answer. Consider the function equation

f(x2f(y)

2)= f(x)

2f(y). (1.1)

Take any a, b ∈ Q+. By substituting x = f(a),y = b and x = f(b),y = a into (1.1), we get f(f(a))2f(b)= f(f(a)2f(b)2)= f(f(b))2f(a), which yields f(f(a))2
f(a) = f(f(b))2

f(b) for all a, b ∈ Q+. In other words, there exists a constant
C ∈ Q+ such that f(f(a))2 = Cf(a), or

!f(f(a))

C

"2

= f(a)
C for all a ∈ Q+. (1.2)

Denote by fn(x)= f(f(...(f # $% & n

(x))...)) the nth iteration of f. The equality (1.2)

yields

f(a)
C =

!f 2(a)

C

"2

=

!f 3(a)

C

"4

= · · · =

!fn+1(a)

C

"2n
for all positive integer n. So, f(a)/C is the 2n-th power of a rational number for all positive integer n. This is impossible unless f(a)/C = 1, since otherwise the exponent of some prime in the prime decomposition of f(a)/C is not divisible by sufficiently large powers of 2. Therefore, f(a)= C for all a ∈ Q+.

Finally, after substituting f ≡ C into (1.1), we get C = C3; whence C = 1. So f(x) = 1 is the unique function satisfying (1.1).

Remark. If we let d = f(1), then by letting x = y = 1 and x = d2,y = 1 in (1.1), we get f(d2)= d3 and f(d6)= f(d2)2 · d = d7. By substituting x = 1 and y = d2 into (1.1), we obtain f(d6)= d2 · d3 = d5. Therefore, d7 = f(d6)= d5, whence d = 1. After that, the rest of the solution simplifies a bit, since we already have C = f(f(1))2

f(1) = 1. Hence equation (1.2) is simply f(f(a))2 = f(a), which yields
f(a) = 1 in a similar manner.

There exists nonconstant function f: R+ −→ R+ satisfying (1.1) for all real
x, y > 0. For example, f(x)= √x.

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2. Let n ≥ 3 be an integer. Prove that there exists a set S of 2n positive integers satisfying the following property: For every m =2, 3,...,n, the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

Solution. Let S = {F1,F2,...,F2n−1,F1 + F2 + ··· + F2n−4}, where {Fi} is the Fibonacci sequence with F1 =1,F2 =2,Fi+1 = Fi + Fi−1.

Then S has 2n elements and the sum Σ of all elements in S is Σ = F1 +F2 +···+

F2n−1+(F1+F2+···+F2n−4) = 2(F1+F2+···+F2n−4)+F2n−3+F2n−2+F2n−1 = 2[(F1 + F2 + ··· + F2n−4)+ F2n−1].

Thus 1

2Σ = (F1 + F2 + ··· + F2n−4)+ F2n−1 =(F1 + F2 + ··· + F2n−4)+(F2n−2 + F2n−3)=(F1 + F2 + ··· + F2n−4)+(F2n−2 + F2n−4 + F2n−5)= ··· etc. Therefore, we may take

Am = {F1 + F2 + ··· + F2n−4,F2n−2,F2n−4,...,F2n−2(m−2),F2n−2m+3}, consisting of m distinct elements from S with sum 1

2Σ.

3. Let ABC be a triangle with AB = AC, and let M be the midpoint of BC. Let P be a point such that PB < PC and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and ∠PXM = ∠PYM. Prove that the quadrilateral AP XY is cyclic.

Solution. Since AB = AC, AM is the perpendicular bisector of BC, hence ∠P AM = ∠AMC = 90◦.

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Now let Z be the common point of AM and the perpendicular through Y to PC (notice that Z lies on to the ray AM beyond M.) We have ∠P AZ = ∠PYZ = 90◦. Thus the points P, A, Y, Z are concyclic.

Since ∠CMZ = ∠CY Z = 90◦, the quadrilateral CY ZM is cyclic, hence ∠CZM = ∠CYM. By the condition in the statement, ∠CYM = ∠BXM, and, by sym-metry in ZM, ∠CZM = ∠BZM. Therefore, ∠BXM = ∠BZM. It follows that the points B, X, Z,M are concyclic, hence ∠BXZ = 180◦ − ∠BMZ = 90◦.

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Finally, we have ∠PXZ = ∠PYZ = ∠P AZ = 90◦, hence the five points P, A, X, Y, Z are concyclic. In particular, the quadrilateral AP XY is cyclic, as required.

4. Let n> 1 be a positive integer. Each cell of an n × n table contains an integer. Suppose the following conditions are satisfied:

(i) Each number in the table is congruent to 1 modulo n;

(ii) The sum of numbers in any row, as well as the sum of numbers in any column is congruent to n modulo n2.

Let Ri be the product of the numbers in the i

th row, and Cj be the product of
numbers in the jth column. Prove that the sums R1 +···+Rn and C1 +···+Cn are congruent modulo n4.

Solution. Let Ai,j be the entry in the i

th row and the jth column; let P be the
product of all n2 entries. For convenience, denote ai,j = Ai,j − 1 and ri = Ri − 1.
We show that

!n

i=1

Ri ≡ (n − 1) + P (mod n4). (4.1)

Due to the symmetry of the problem conditions, the sum of all the Cj is also congruent to (n − 1) + P modulo n4, whence the conclusion.

By condition (i), the number n divides ai,j for all i and j. So, every product of at least two of the ai,j is divisible by n2, hence

Ri = "n

j=1

(1 + ai,j)=1+!n

j=1

ai,j +!
1≤ji<j2≤n

ai,j1 a2,j2 + ···

≡ 1+!n

j=1

ai,j ≡ 1 − n +!n

j=1

Ai,j (mod n2)

for every index i. Using condition (ii), we obtain Ri ≡ 1 (mod n2), and so n2 | ri.

Therefore, every product of at least two of the ri is divisible by n4. Repeating the same argument, we obtain

P = "n

i=1

Ri = "n

i=1

(1 + ri) ≡ 1+!n

i=1

ri (mod n4),

whence

!n

i=1

Ri = n +!n

i=1

ri ≡ n +(P − 1) (mod n4),

as desired.

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5. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of n + 1 squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone should stay within the board). Sisyphus’s aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

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turns. (As usual, ⌈x⌉ stands for the least integer not smaller than x.) Solution. The stones are indistinguishable, and all have the same origin and the same final position. So, at any turn we can prescribe which stone from the chosen square to move. We do it in the following manner. Number the stones from 1 to n. At any turn, after choosing a square, Sisyphus moves the stones with the largest number from this square.

This way, when stone k is moved from some square, that square contains not more than k stones (since all their numbers are at most k). Therefore, stone k is moved by at most k squares at each turn. Since the total shift of the stone is exactly n, at least ⌈n/k⌉ moves of the stone k should have been made, for every k =1, 2,...,n. By summing up over all k =1, 2,...,n, we get the required estimate.

6. Let a0,a1,a2,... be a sequence of real numbers such that a0 =0,a1 = 1, and for every n ≥ 2, there exists 1 ≤ k ≤ n satisfying

an = an−1 + ··· + an−k

k .

Find the maximal possible value of |a2018 − a2017|.
Solution. The maximal value is 1

2018 and is achieved at

a1 = a2 = ··· = a2017 =1,a2018 = a2017 + ··· + a0

2018

= 1 − 1
2018.

We need to show that this value is optimal. For brevity, we use the notation

S(n, k)= an−1 + an−2 + ··· + an−k for nonnegative integers k ≤ n.

In particular, S(n, 0) = 0 and S(n, 1) = an−1. In these terms, for every integer n ≥ 2 there exists a positive integer k ≤ n such that an = S(n, k)/k.

For every integer n ≥ 1 we define

Mn = max
1≤k≤n

S(n, k)
k,mn = min

1≤k≤n

S(n, k)
k, and △n = Mn − mn ≥ 0.

By definition, an ∈ [mn,Mn] for all n ≥ 2; on the other hand, an−1 = S(n, 1)/1 ∈ [mn,Mn]. Therefore,

a2018 − a2017 ≤ M2018 − m2018 = △2018,

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and we are interested in an upper bound for △2018.
Also by definition, for any 0 <k ≤ n we have kmn ≤ S(n, k) ≤ kMn; notice that these inequalities are also valid for k = 0.
Claim. For every n> 2, we have △n ≤ n−1

n △n−1.

Proof. Choose positive integers k, ℓ ≤ n such that Mn = S(n, k)/k and mn = S(n,ℓ)

ℓ. We have S(n, k)= an−1 + S(n − 1,k − 1), so

k(Mn−an−1)= S(n, k)−kan−1 = S(n−1,k−1)−(k−1)an−1 ≤ (k−1)(Mn−1−an−1),

since S(n − 1,k − 1) ≤ (k − 1)Mn−1. Similarly, we get

ℓ(an−1 − mn)=(ℓ − 1)an−1 − S(n − 1, ℓ − 1) ≤ (ℓ − 1)(an−1 − mn−1).

Since mn−1 ≤ an−1 ≤ Mn−1 and k, ℓ ≤ n, the obtained inequalities yield

Mn − an−1 ≤

k − 1

k (Mn−1 − an−1) ≤

n − 1

n (Mn−1 − an−1),

an−1 − mn ≤

ℓ − 1

ℓ (an−1 − mn−1) ≤

n − 1

n (an−1 − mn−1).

Therefore, △n =(Mn − an−1)+(an−1 − mn)
≤

n − 1

n ((Mn−1 − an−1)+(an−1 − mn−1))
= n − 1
n

△n−1.

This completes the proof of the claim.
Back to the problem, first note that △2 = M2 − m2 =1 − 1

2 = 1

2. By the claim,

|a2018 − a2017| ≤△2018 ≤

1

2 ·

2

3 ··· 2017

2018 = 1

2018.

7. Let n be a natural number. A sequence is k-complete if it contains all residues modulo nk. Let Q(x) be a polynomial with integer coefficients. For k ≥ 2, define Qk(x)= Q(Qk−1(x)), where Q1(x)= Q(x). Show that if

0,Q(0),Q2(0),Q3(0)···

is 2018-complete, then it is k-complete for all positive integers k.
Solution. We first prove that the statement is true for n = p, where p is a prime.

It is trivial to note that for each k< 2018, the sequence is k-complete, purely periodic (mod pk) with period pk and that the first d> 0 such that pk | Qd(0) is the period.

Also pk∥Qpk

(0) for if pk+1 | Qpk

(0), the sequence mod pk+1 has period pk and
would not be k + 1-complete.

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We have the identity Qp2016 (x) − Qp2016 (y)=(x − y)R(x, y) for some polyno-mial R(x, y)with integer coefficients. Let α be an integer which satisfies α̸=1 and α ≡ R(0, 0) (mod p3). For each k> 1, by putting x = Q(k−1)p2016 (0) and y = 0, and using the fact that p2016 | Q(k−1)p2016 (0), we have

Qkp2016 (0) = Qp2016 (0) + R(Q(k−1)p2016 (0), 0) Q(k−1)p2016 (0)

≡ Qp2016 (0) + αQ(k−1)p2016 (0) (mod p2019).

Then, iterating, we have Qkp2016 (0) ≡ (1 + α + ··· + αk−1)Qp2016 (0) (mod p2019).

Substituting k = p, and because p2016∥Qp2016 (0), p2017∥Qp2017 (0),
we have

p∥1+ α + ··· + αp−1 = αp − 1

α − 1 ⇒ p | αp − 1 ⇒ α ≡ 1(mod p).

By the Lifting The Exponent Lemma (LTE), for odd p, then vp(αp2

− 1) =
vp(p2)+ vp(α − 1) = 2 + vp(α − 1). For p = 2, we have 2∥1+ α, so v2(α22

− 1) =
v2(22)+ v2(α − 1) + v2(α + 1) − 1 = 2+ v2(α − 1). Either way, we have

p2∥αp2

− 1

α−1 = 1+α+α2+···+αp2−1, so when substituting k = p2, p2018∥Qp2018 (0). Similarly, when k = p3, we have p2019 | Qp2019 (0) by LTE.

Taking modulo p2019, the sequence is periodic with period d where d | p2019. However, d! p2018, because p2018∥Qp2018 (0), so we have d = p2019.

Since no residue can appear twice in the period, this means that the sequence is 2019-complete, with period p2019 modulo p2019. Similarly, we can induct up to show that it is k-complete with period pk modulo pk for all k ≥ 2018. It is also clear that the sequence is k-complete for k< 2018. Hence, we are done when n is prime.

If n is composite, let n = pβ1

1 pβ2

2 ··· p

βj
j, and let r be any residue class mod-
ulo nk = p

kβ1
1 p

kβ2
2 ··· p

kβj
j. Let r ≡ ri (mod p

kβi
i). The sequence is kβi-complete with respect to pi. Hence, there is a term Qmi (0) ≡ ri (mod p

kβi
i). By Chinese
Remainder Theorem (CRT), ∃m such that m ≡ mi (mod p

kβi
i). Since the se-
quence has period p

kβi
i modulo p

kβi
i, Qm(0) ≡ Qmi (0) ≡ ri (mod p

kβi
i), and by
CRT, Qm(0) ≡ r (mod nk), so we have proven that all residue classes are covered in the sequence.

8. A point T is chosen inside a triangle ABC. Let A1,B1 and C1 be the reflec-tions of T in BC, CA, and AB respectively. Let Ω be the circumcircle of the triangle A1B1C1. The lines A1T,B1T, and C1T meet Ω again at A2,B2, and C2, respectively. Prove that the lines AA2,BB2, and CC2 are concurrent on Ω.

Solution. By!(ℓ, n) we always means the directed angle of the lines ℓ and n, taken modulo 180◦. Let CC2 meet Ω again at K (as usual, if CC2 is tangent to

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Ω, we set T = C2). We show that the line BB2 contains K; similarly, AA2 will also pass through K. For this purpose, it suffices to prove that

!(C2C, C2A1)=!(B2B,B2A1). (8.1)

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By the condition of the problem, CB and CA are perpendicular bisectors of TA1 and TB1, respectively. Hence, C is the cicumcentre of the triangle A1TB1. Therefore,!(CA1,CB)=!(CB,CT)=!(B1A1,B1T)=!(B1A1,B1B2). In the circle Ω, we have!(B1A1,B1B2)=!(C2A1,C2B2). Thus,

!(CA1,CB)=!(B1A1,B1B2)=!(C2A1,C2B2). (8.2)

Similarly, we can obtain

!(BA1,BC)=!(C1A1,C1C2)=!(B2A1,B2C2). (8.3)

The two obtained relations (8.2) and (8.3) yield that the triangles A1BC and

A1B2C2 are similar and oriented in the same sense, hence A1B2

A1B = A1C2

A1C and

!(A1B,A1C)=!(A1B2,A1C2). (8.4)

It follows from (8.4) that we have!(A1B,A1B2)=!(A1C, A1C2), so the trian-gles A1BB2 and A1CC2 are also similar and oriented in the same sense. This proves (8.1).

9. Define the sequence a0,a1,a2,... by an =2n +2⌊n/2⌋. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of two or more distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

Solution. Call a nonnegative integer representable if it equals the sum of several

(possibly 0 or 1) distinct terms of the sequence. We say that two nonnegative

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integers b and c are equivalent (written as b ∼ c) if they are either both repre-sentable or both non-representable.
On can easily compute

Sn−1:= a0 + ··· + an−1 =2n +2⌈n/2⌉ +2⌊n/2⌋ − 3.

Indeed, we have Sn − Sn−1 =2n +2⌊n/2⌋ = an so we can use induction. In particular, S2k−1 =22k +2k+1 − 3.
Note that, if n ≥ 3, then 2⌈n/2⌉ ≥ 22 > 3, so

Sn−1 =2n +2⌈n/2⌉ +2⌊n/2⌋ − 3 > 2n +2⌊n/2⌋ = an.

Also notice that Sn−1 − an =2⌈n/2⌉ − 3 <an.
The main tool of the solution is the following claim.

Claim 1. Assume that b is a positive integer such that Sn−1 − an <b<an for some n ≥ 3. Then b ∼ Sn−1 − b.
Proof. As seen above, we have Sn−1 >an. Denote c = Sn−1−b; then Sn−1−an < c<an, so the roles of b and c are symmetrical.
Assume that b is representable. The representation cannot contain ai with i ≥ n, since b<an. So b is the sum of some subset of {a0,a1,...,an−1}; then c is the sum of the complement. The converse is obtained by swapping b and c.
We also need the following version of this claim.

Claim 2. For any n ≥ 3, the number an can be represented as a sum of two or more distinct terms of the sequence if and only if Sn−1 − an =2⌈n/2⌉ − 3 is representable.
Proof. Denote c = Sn−1 − an <an. If an satisfies the required condition, then it is the sum of some subset of {a0,a1,...,an−1}; then c is the sum of the comple-ment. Conversely, if c is representable, then its representation consists only of the numbers from {a0,a1,...,an−1}, so an is the sum of the complement.

By claim 2, in order to prove the problem statement, it suffices to find infinitely many representable numbers of the form 2t − 3, as well as infinitely many non-representable ones.

Claim 3. For every t ≥ 3, we have 2t − 3 ∼ 24t−6 − 3, and 24t−6 − 3 > 2t − 3. Proof. The inequality follows from t ≥ 3. In order to prove the equivalence, we apply claim 1 twice in the following manner.
First, since S2t−3 − a2t−2 =2t−1 − 3 < 2t − 3 < 22t−2 +2t−1 = a2t−2, by claim 1 we have 2t − 3 ∼ S2t−3 − (2t − 3) = 22t−2.
Second, since S4t−7 −a4t−6 =22t−3 −3 < 22t−2 < 24t−6 +22t−3 = a4t−6, by claim
1 we have 22t−2 ∼ S4t−7 − 22t−2 =24t−6 − 3.
Therefore, 2t − 3 ∼ 22t−2 ∼ 24t−6 − 3, as required.

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Now it is easy to find the required numbers. Indeed, the number 23 − 3=5= a0 + a1 is representable, so claim 3 provides an infinite sequence of representable numbers:

23 − 3 ∼ 26 − 3 ∼ 218 − 3 ∼ ··· ∼ 2t − 3 ∼ 24t−6 − 3 ∼ ···.

On the other hand, the number 27 −3 = 125 is non-representable (since by claim
1 we have 125 ∼ S6 − 125 = 24 ∼ S4 − 24 = 17 ∼ S3 − 17 = 4 which is clearly non-representable). So claim 3 provides an infinite sequence of non-representable numbers:

27 − 3 ∼ 222 − 3 ∼ 282 − 3 ∼ ··· ∼ 2t − 3 ∼ 24t−6 − 3 ∼ ···.

10. Let k be a positive integer. The organising committee of a tennis tournament is to schedule the matches for 2k players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day
a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players’ stays. Determine this minimum cost.

Solution. The required minimum is k(4k2 + k − 1)/2. The solution is as follow. Let b1 ≤ b2 ≤ ··· ≤ b2k be the days the players arrive to the tournament, arranged in nondecreasing order; similarly, let e1 ≥ ··· ≥ e2k be the days they depart arranged in nonincreasing order (it may happen that a player arrives on day bi and departs on day ej, where i ̸= j). If a player arrives on day b and departs on day e, then his stay cost is e − b + 1. Therefore, the total stay cost is

C =!

2k

i=1

ei −!

2k

i=1

bi +2k =!

2k

i=1

(ei − bi + 1).

Bounding the total cost from below. Before day bi and after day ei, i − 1 players

are present. If these two groups of players are disjoint, they play at most 2"i−1

2

#
games. Therefore

ei − bi +1 ≥
$2k

2

%

− 2

$i − 1

2

%

.

However, this is not possible when i>k. In this case, each group has i − 1 players and the overlap is at least 2(i − 1) − 2k. So the total number of games played by them is at most 2"i−1

2

#

− "2i−2k−2

2

#

. Therefore

ei − bi +1 ≥
$2k

2

%

− 2

$i − 1

2

%

+

$2i − 2k − 2

2

%

= (2k − i + 1)2.

An optimal tournament. We now describe a schedule in which the lower bounds above are all achieved simultaneously. Split the players into two groups X = {x1,x2,...,xk} and Y = {y1,y2,...,yk}. Next, partition the schedule into three parts. During the first part, the players from X arrive one by one, and each

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newly arrived player immediately plays with everyone already present. During the third part (after all players from X have already departed) the players from Y depart one by one, each playing with everyone still present just before departing. This ensures that equality is achieved in the case i ≤ k.

In the middle part, everyone from X should play with everyone from Y. Let y1,y2,...,yk arrive in this order and xk,xk−1,...,x1 depart in this order. The idea is that between yt’s arrival and xt’s departure, yj, j = t, t +1,...,k plays against xt,xt+1,...,xk so that the total number of games is (k − t + 1)2 and therefore ek+t − bk+t +1=(k − t + 1)2 = (2k − (k + t) + 1)2 and thus equality is achieved for the case i>k. The schedule is as follows. After yj arrives, he immediately plays with all the xi,i > j and xk departs the day tk arrives. Thereafter each xi, i = k − 1,k − 2,..., 1 plays with all yj,i ≤ j, just before his departure.

Evaluation. Finally, the total cost for the optimal schedule:

k

!2k

2

"

− 2
#

k

i=1

!i − 1

2

"

+
#

k

i=1

i

2 = k
!2k

2

"

− 2

!k

3

"

+

k(2k + 1)(k + 1)
6 = 4k3 + k2 − k

2 .

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