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This article aims to firstly illustrate how Binomial coefficients coincide with Pascal’s triangle, and secondly show that the Fibonacci sequence is embedded inside Pascal’s triangle. Additionally, this article features a general form of Pascal’s Triangle and proved that a general form of a second order difference equation is also embedded inside the General Pascal’s Triangle.

To first understand how Pascal’s triangle works, it is important to note that the entire Pascal’s triangle can be generated with just the number “1”. Based on the following figure, there is a number “1” in the middle and infinite zeroes that extend past both sides of “1” (left and right). By adding two adjacent terms, another term is created in the following row (as represented by the numerous inverted triangles). This is how Pascal’s triangle is generated continuously.

0 0 0 0 0 0 1 0 0 0 0 0 0

0 0 0 0 0 1 1 0 0 0

0 0

0 0

0 0 1 2 1 0 0 0

0

0 0 0 1 3 3 1

0

0

0

0 0 1 4 6 4

1 0

0

0 1 5 10 10 5 1 0

Next, this article aims to prove how the numbers in Pascal’s triangle coincide with Binomial coefficients.

However, first and foremost, the formula �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟� = 𝑛𝑛𝑛𝑛!

𝑟𝑟𝑟𝑟!(𝑛𝑛𝑛𝑛−𝑟𝑟𝑟𝑟)!

is essential in proving this relationship. The proof for this

formula can be argued using combinatorics.

Consider a scenario where there are 𝑟𝑟𝑟𝑟 identical 𝐴𝐴𝐴𝐴 letters and (𝑛𝑛𝑛𝑛 − 𝑟𝑟𝑟𝑟) identical 𝐵𝐵𝐵𝐵 letters to be arranged in a straight line.

This problem can be solved by one of the 3 following methods:

Method 1: Out of the 𝑛𝑛𝑛𝑛 slots chosen, there are 𝑟𝑟𝑟𝑟 slots to place the 𝐴𝐴𝐴𝐴 letters. As for the rest of the (𝑛𝑛𝑛𝑛 − 𝑟𝑟𝑟𝑟) slots,

there is only 1 choice to allocate the rest of the 𝐵𝐵𝐵𝐵 letters. Hence, this scenario can be solved in �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟�
ways.

Method 2: By solving this directly, there are 𝑛𝑛𝑛𝑛!

𝑟𝑟𝑟𝑟!(𝑛𝑛𝑛𝑛−𝑟𝑟𝑟𝑟)!

ways to rearrange all the letters.

Method 3: Out of the slots 𝑛𝑛𝑛𝑛, (𝑛𝑛𝑛𝑛 − 𝑟𝑟𝑟𝑟) slots are chosen to place the 𝐵𝐵𝐵𝐵 letters. As for the rest of the 𝑟𝑟𝑟𝑟 slots, there

is only 1 choice to allocate the rest of the 𝐴𝐴𝐴𝐴 letters. Hence, this scenario can be solved in � 𝑛𝑛𝑛𝑛

𝑛𝑛𝑛𝑛 − 𝑟𝑟𝑟𝑟

�
ways.

Since all 3 methods are solving the same scenario, their answers are all equivalent.

Hence, �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟� = 𝑛𝑛𝑛𝑛!

𝑟𝑟𝑟𝑟!(𝑛𝑛𝑛𝑛−𝑟𝑟𝑟𝑟)!

= � 𝑛𝑛𝑛𝑛

𝑛𝑛𝑛𝑛 − 𝑟𝑟𝑟𝑟

� .

The General Pascal’s Triangle

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Next, �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟� = 𝑛𝑛𝑛𝑛!

𝑟𝑟𝑟𝑟!(𝑛𝑛𝑛𝑛−𝑟𝑟𝑟𝑟)!

needs to be used to explain 0! = 1 because the first number “1” in the first row of Pascal’s

triangle is �

0
0

� = ଴!

଴!(଴−଴)!

= 1.

By substituting 𝑟𝑟𝑟𝑟 = 𝑛𝑛𝑛𝑛, the result is �

𝑛𝑛𝑛𝑛

𝑛𝑛𝑛𝑛� =1. Hence, �

𝑛𝑛𝑛𝑛

𝑛𝑛𝑛𝑛� = 𝑛𝑛𝑛𝑛!

𝑛𝑛𝑛𝑛!(𝑛𝑛𝑛𝑛−𝑛𝑛𝑛𝑛)!

= 1 and 0! = 1

By observation, since �

𝑛𝑛𝑛𝑛

݇� = 0 for ݇ < 0 or݇ > 𝑛𝑛𝑛𝑛, where 𝑛𝑛𝑛𝑛אԺା ׫} 0 {and݇ אԺ, this can be used to explain the
infinite series of zeroes that extend past the left and right borders of Pascal’s Triangle.

Finally, the formula �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟� + � 𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟 +1� = �

𝑛𝑛𝑛𝑛 +1

𝑟𝑟𝑟𝑟 +1� needs to proven. Since proof using algebraic manipulation is commonly
known, this article has created a new method of proving by using a combinatorics argument.

The proof begins by choosing from 𝑛𝑛𝑛𝑛 ladies and 1 gentleman to form a committee of 𝑟𝑟𝑟𝑟 +1 people. The number of ways can be calculated using two methods.

Method 1: The direct method is simply �

𝑛𝑛𝑛𝑛 +1

𝑟𝑟𝑟𝑟 +1� ways.
Method 2: There are 2 cases to be considered, namely whether the gentleman is chosen or not.

Case 1: If the gentleman choose another 𝑟𝑟𝑟𝑟 ladies from the 𝑛𝑛𝑛𝑛 ladies, which results in �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟�
ways.

� 𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟 +1� ways.

Note that case 1 and 2 are mutually exclusive, hence, the total number of ways is �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟� + � 𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟 +1�.

Since both methods are solving the same question, hence, �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟� + � 𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟 +1� = �

𝑛𝑛𝑛𝑛 +1

𝑟𝑟𝑟𝑟 +1�.
This can be rearranged to express the following geometrically:

�

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟� � 𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟 +1�

�

𝑛𝑛𝑛𝑛 +1

𝑟𝑟𝑟𝑟 +1�
Based on the figure above, it is now clear that the numbers in Pascal’s triangle and binomial coefficients must coincide.

Next, this article shall explain and prove the observation that the Fibonacci sequence is also embedded inside Pascal’s triangle.

Recalling the Fibonacci sequence:

ݑ𝑛𝑛𝑛𝑛ାଶ = ݑ𝑛𝑛𝑛𝑛ାଵ + ݑ ; �𝑛𝑛𝑛�ݑଵ = ݑଶ = 1, �𝑛𝑛𝑛�אԺା

This can be extended to

ݒ𝑛𝑛𝑛𝑛ାଶ = ݒ𝑛𝑛𝑛𝑛ାଵ + ݒ ,0 = ଴ݒ ; �𝑛𝑛𝑛�ݒଵ = 1, �𝑛𝑛𝑛�אԺା ׫ }0{

As such, ݒ �𝑛𝑛𝑛�is defined as follows:

𝑛𝑛𝑛𝑛 0 for �𝑛𝑛𝑛� =0 ൜ = ݒ

ݑ𝑛𝑛𝑛𝑛 for �𝑛𝑛𝑛�אԺା

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The General Pascal’s Triangle

By relooking at Pascal’s triangle in the following manner:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

As such, it can be observed that ܶ𝑛𝑛𝑛𝑛 needs to be considered when 𝑛𝑛𝑛𝑛 = 2݉ and 𝑛𝑛𝑛𝑛 = 2݉ + 1 where ݉ ൒ 1 with ܶ଴ = 0 :

Case 1: When 𝑛𝑛𝑛𝑛 = 2݉, ܶ𝑛𝑛𝑛𝑛 =ܶଶ௠ = �

2݉ − 1

0 � + �

2݉ − 2

1 � + �

2݉ − 3

� + ڮ + � 2

2݉ −݉

݉ − 1 �

ܶ𝑛𝑛𝑛𝑛−ଵ =ܶଶ௠−ଵ = �

2݉ − 2

0 � + �

2݉ − 3

1 � + �

2݉ − 4

� + ڮ + � 2

݉ − 1
݉ − 1

�

ܶ𝑛𝑛𝑛𝑛−ଶ =ܶଶ௠−ଶ = �

2݉ − 3

0 � + �

2݉ − 4

1 � + �

2݉ − 5

� + ڮ + � 2

݉ − 1
݉ − 2

�

ܶ𝑛𝑛𝑛𝑛−ଵ +ܶ𝑛𝑛𝑛𝑛−ଶ = �

2݉ − 2

0 � + ቂ�

2݉ − 3

0 � + �

2݉ − 3

1 �ቃ + ቂ�

2݉ − 4

1 � + �

2݉ − 4

2 �ቃ + ڮ + ቂ�

݉ − 1
݉ − 2

� + �

݉ − 1
݉ − 1

�ቃ

=�

2݉ − 1

0 � + �

2݉ − 2

1 � + �

2݉ − 3

݉ � + ڮ + � 2

݉ − 1� =ܶଶ௠ =ܶ𝑛𝑛𝑛𝑛 (shown)

Note that the formula �

𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟� + � 𝑛𝑛𝑛𝑛

𝑟𝑟𝑟𝑟 + 1� = �

𝑛𝑛𝑛𝑛 + 1
𝑟𝑟𝑟𝑟 + 1

� was applied repeatedly.

Case 2: When 𝑛𝑛𝑛𝑛 = 2݉ + 1, ܶ𝑛𝑛𝑛𝑛 =ܶଶ௠ାଵ = �

2݉

0 � + �

2݉ − 1

1 � + �

2݉ − 2

� + ڮ + � 2

2݉ −݉

݉ �

ܶ𝑛𝑛𝑛𝑛−ଵ =ܶଶ௠ = �

2݉ − 1

0 � + �

2݉ − 2

1 � + �

2݉ − 3

݉ � + ڮ + � 2

݉ − 1�

ܶ𝑛𝑛𝑛𝑛−ଶ =ܶଶ௠−ଵ = �

2݉ − 2

0 � + �

2݉ − 3

1 � + �

2݉ − 4

� + ڮ + � 2

݉ − 1
݉ − 1

�

ܶ𝑛𝑛𝑛𝑛−ଵ +ܶ𝑛𝑛𝑛𝑛−ଶ = �

2݉ − 1

0 � + ቂ�

2݉ − 2

0 � + �

2݉ − 2

1 �ቃ + ቂ�

2݉ − 3

1 � + �

2݉ − 3

2 �ቃ + ڮ + ቂ�

݉ − 1
݉ − 1

� + �݉

݉ − 1�ቃ

= �

2݉

0 � + �

2݉ − 1

1 � + �

2݉ − 2

� + ڮ + � 2

݉

݉� =ܶଶ௠ାଵ =ܶ𝑛𝑛𝑛𝑛 (shown)
Hence, it has been proven successfully that the Fibonacci sequence is embedded inside Pascal’s triangle by using Binomial coefficients.

ܶ଴ = 0

ܶଵ = 1

ܶଶ = 1

ܶଷ = 1 + 1 = 2

ܶସ = 1 + 2 = 3 =ܶଶ +ܶଷ

ܶହ = 1 + 3 + 1 = 5 =ܶଷ +ܶସ

ܶ଺ = 1 + 4 + 3 = 8 =ܶସ +ܶହ

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The General Pascal’s Triangle

Next, this article will consider the second order difference equation ݑ�𝑛𝑛𝑛�ାଶ = ݑߙ�𝑛𝑛𝑛�ାଵ + ݑ = ,0଴ ; ݑ�𝑛𝑛𝑛� ݑߚଵ = 1, 𝑛𝑛𝑛𝑛אԺା ׫} 0 {such that both ߙ and ߚ are some constants.
As such, it will generate ݑ ,0 = ଴ݑଵ = 1, ݑଶ = ݑ ,ߙଷ = ߙଶ + ݑ ,ߚସ = ߙ)ߙଶ + ߙ = (ߙ)ߚ + (ߚଷ +2ߚߙ,
(ߚߙ ଷ +2ߙ)ߚ + (ଶߚ + ߚଶߙ ସ +3ߙ)ߙ = ଺ݑ ,ଶߚ + ߚଶߙ ସ +3ߙ = (ߚ + ଶߙ)ߚ + (ߚߙ ଷ +2ߙ)ߙ = ହݑ
i.e. ߙ଺ ݑହ = +4ߙଷ ߚߙ3+ ߚଶ, and so on…
The General Pascal’s Triangle
We define �𝑛𝑛𝑛� 0 ௞ܩ = for ݇ < 0 or ݇ > 𝑛𝑛𝑛𝑛 where 𝑛𝑛𝑛𝑛אԺା ׫} 0{,݇ אԺ with �𝑛𝑛𝑛� 0 ௞ܩ = for 𝑛𝑛𝑛𝑛אԺ− and ଴ 1. ଴ܩ=
We also define ܩ௞) ߚ −ଵ(

(ܩ ௞)ߙ + �𝑛𝑛𝑛�

𝑛𝑛𝑛𝑛 = �𝑛𝑛𝑛� ܩ ାଵ ௞ . This can be rearrange this to express the following geometrically:
ܩ௞�𝑛𝑛𝑛� �𝑛𝑛𝑛� ܩଵ−௞
𝑛𝑛𝑛𝑛 ܩ ାଵ ௞
As such, the following General Pascal’s Triangle can be successfully generated:
0 0 0

0

0 0

0 0 0 0 0 0 ܩ଴଴

0 0 0 0 0 ଴ଵܩ ଵଵܩ 0 0 0 0 0

0 0 0 0 ଶ଴ܩ ଶଵܩ ଶଶܩ 0 0 0 0

0 0 0 ଷ଴ܩ ଷଵܩ ଶଷܩ ଷଷܩ 0 0 0

0 0 ଴ସܩ ସଵܩ ଶସܩ ଷସܩ ସସܩ 0 0

0 ଴ହܩ ଵହܩ ଶହܩ ଷହܩ ସହܩ ହହܩ 0

By relooking at the General Pascal’s triangle in the following manner:

1

ߚ α

ଶߚ ߚߙ ଶ 2ߙ

ଷߚ ଶߚߙ3 ߚଶߙ ଷ 3ߙ

ସߚ ଷߚߙ ଶ 4ߚଶߙ6 ߚଷߙ ସ 4ߙ
ହߙ

ହߚ ସߚߙ ଷ 5ߚଶߙ ଶ 10ߚଷߙ10 ߚସߙ5

Similarly, one can observe that ܶ𝑛𝑛𝑛𝑛 needs to be considered when 𝑛𝑛𝑛𝑛 =2݉ and 𝑛𝑛𝑛𝑛 =2݉ +1, where ݉൒ 1 with ܶ଴ =0 :

ܶ଴ = 0
ܶଵ = 1
ߙ = ଶܶ
ଵܶߚ + ଶܶߙ = ߚ + ଶߙ = ଷܶ
ଶܶߚ + ଷܶߙ = ߚߙ ଷ + 2ߙ = ସܶ
ଷܶߚ + ସܶߙ = ଶߚ + ߚଶߙ ସ + 3ߙ = ହܶ
ସܶߚ + ହܶߙ = ଶߚߙ3 + ߚଷߙ ହ + 4ߙ = ଺ܶ

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The General Pascal’s Triangle

Case 1: When 𝑛𝑛𝑛𝑛 = 2݉, ܶ𝑛𝑛𝑛𝑛 =ܶଶ௠ = ଴

ଶ௠−ଵ ܩ + ଵ

ଶ௠−ଶ ܩ + ଶ

ଶ௠−ଷ ڮ + ܩ +௠ −ଵ

ܩ ௠−௠ଶ
ܶ𝑛𝑛𝑛𝑛−ଵ =ܶଶ௠−ଵ = ଴

ଶ௠−ଶ ܩ + ଵ

ଶ௠−ଷ ܩ + ଶ

ଶ௠−ସ ڮ + ܩ +௠ ௠−ଵܩ −ଵ
ܶ𝑛𝑛𝑛𝑛−ଶ =ܶଶ௠−ଶ = ଴

ଶ௠−ଷ ܩ + ଵ

ଶ௠−ସ ܩ + ଶ

ଶ௠−ହ ڮ + ܩ +௠ ௠−ଶܩ −ଵ
ߙܶ𝑛𝑛𝑛𝑛−ଵ + ߚܶ𝑛𝑛𝑛𝑛−ଶ
଴ (ߙ =

଴ (ߚ[+) ܩ ଶ௠−ଶ

ଶ௠−ଷ ߙ + (ܩ)ଵ

ଶ௠−ଷ ߚ] + [(ܩ)ଵ

ଶ௠−ସ ߙ + (ܩ)ଶ

]) ܩଵଵ−−௠௠)ߙ +) ܩଶଵ−−௠௠)ߚ[+ ڮ +]) ܩ ଶ௠−ସ

= ଴

ଶ௠−ଵ ܩ + ଵ

ଶ௠−ଶ ܩ + ଶ

ଶ௠−ଷ ڮ + ܩ +௠ ଶ௠ −ଵܩ = ܶଶ௠ =ܶ𝑛𝑛𝑛𝑛 (shown)
Note that we apply the formula ܩ௞(ߚ −ଵ(

)ܩ ௞ (ߙ + �𝑛𝑛𝑛�

𝑛𝑛𝑛𝑛 = ௞

�𝑛𝑛𝑛� ܩ ାଵ repeatedly many times.
Case 2: When 𝑛𝑛𝑛𝑛 = 2݉ + 1, ܶ𝑛𝑛𝑛𝑛 =ܶଶ௠ାଵ = ଴

ଶ௠ ܩ + ଵ

ଶ௠−ଵ ܩ + ଶ

௠ + ڮ + ܩ ଶ௠−ଶ

ଶ௠−௠ ܩ = ܶଶ௠ାଵ
ܶ𝑛𝑛𝑛𝑛−ଵ =ܶଶ௠ = ଴

ଶ௠−ଵ ܩ + ଵ

ଶ௠−ଶ ܩ + ଶ

ଶ௠−ଷ ڮ + ܩ +௠ −ଵ

ܩ ௠
ܶ𝑛𝑛𝑛𝑛−ଶ =ܶଶ௠−ଵ = ଴

ଶ௠−ଶ ܩ + ଵ

ଶ௠−ଷ ܩ + ଶ

ଶ௠−ସ ڮ + ܩ +௠ −ଵ

ܩଵ−௠
ߙܶ𝑛𝑛𝑛𝑛−ଵ + ߚܶ𝑛𝑛𝑛𝑛−ଶ
଴ (ߙ =

଴ (ߚ[+) ܩ ଶ௠−ଵ

ଶ௠−ଶ ߙ + (ܩ)ଵ

ଶ௠−ଵ ߚ] + [(ܩ)ଵ

ଶ௠−ଷ ߙ + (ܩ)ଶ

ଵ−௠)ߚ[+ ڮ +]) ܩ ଶ௠−ଷ

௠ (ߙ +) ܩଵ−௠

]) ܩ ௠−௠ଶ

= ଴

ଶ௠ ܩ + ଵ

ଶ௠−ଵ ܩ + ଶ

௠ + ڮ + ܩ ଶ௠−ଶ

ଶ௠−௠ ܩ =ܶଶ௠ାଵ =ܶ𝑛𝑛𝑛𝑛 (shown)

Hence, it has been proven successfully that the sequence of a general 2nd order difference equation, with initial conditions ݑ = ,0଴ ݑଵ = 1 and ݑ�𝑛𝑛𝑛�ାଶ = ݑߙ�𝑛𝑛𝑛�ାଵ + א �𝑛𝑛𝑛�,�𝑛𝑛𝑛� ݑߚ Ժା ׫} 0 {is embedded inside the General Pascal’s Triangle. In the General Pascal’s triangle, it is obvious that the terms that appear in every horizontal row are the expansion of
(ߚ + ߙ (�𝑛𝑛𝑛�whereby 𝑛𝑛𝑛𝑛 א Ժା ׫} 0{. It is also interesting to note that ߙ and ߚ are assumed to be constants which are not
restricted to only real numbers. In fact, ߙ and ߚ can also be complex numbers.
The trinomial coefficients can be represented in a 3-dimensional tetrahedron diagram. Furthermore, it may be possible
to show geometrically how a general 3rd order difference equation with specific conditions is embedded inside the 3-dimensional tetrahedron diagram. Perhaps, this shall be explored in future publications.

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