IMO 2018
59th International
Mathematical Olympiad
Cluj-Napoca Romania

English (eng), day 1

Monday, July 9, 2018

Problem 1. Let Γ be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).

Problem 2. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_{n+2}$, such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \ldots, n$.

Problem 3. An *anti-Pascal triangle* is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \cdots + 2018$?

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

59th International Mathematical Olympiad
Cluj-Napoca Romania

English (eng), day 2

Tuesday, July 10, 2018

Problem 4. A *site* is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Problem 5. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Problem 6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Solutions

1. Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

Solution by Lucas Boo Tse Yang. If $AB = AC$, then the triangle ABC is isosceles and it is obvious that DE is parallel to FG. We may assume without loss of generality that $AB > AC$. Let ℓ be the tangent at A. Construct the point J such that J, A are on the same side of the line BF and $\triangle FBJ \equiv \triangle FDA$. Similarly construct the point K such that K, A are on the same side of the line CG and $\triangle GCK \equiv \triangle GEA$. Since $\angle BJF = \angle BAF$ and $\angle CKG = \angle CAG$, the points J, K lie on Γ. Then $BJ = DA = EA = CK$ implies that $BCKJ$ is an isosceles trapezium with JK parallel to BC. Also $FA = FJ$ implies $\angle AGF = \angle FGJ$ and $GA = GK$ implies $\angle AFG = \angle GJK$.

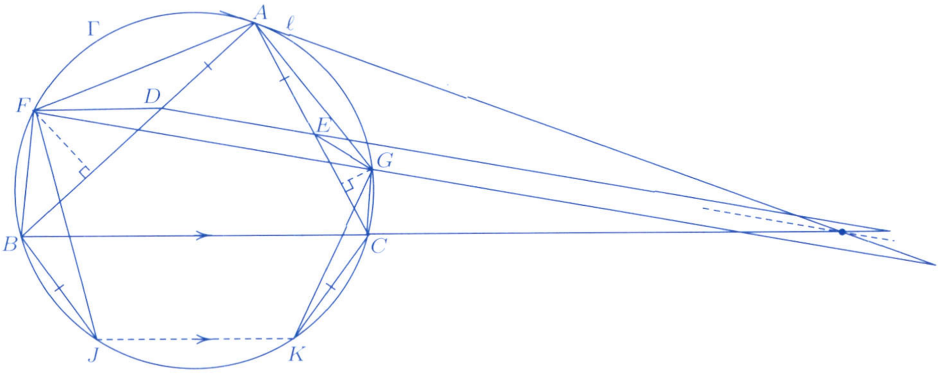

Figure 1.1

The angle between ℓ and $FG = \angle AGF - \angle AFG = \angle FGJ - \angle GJK =$ the angle between FG and $JK =$ the angle between FG and BC.

The angle between ℓ and $DE = \angle AED - \angle ABC = \angle ADE - \angle ABC =$ the angle between DE and BC.

This implies that both FG and DE are parallel to the bisector of ℓ and BC.

2. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \ldots, n$.

Solution by Cheng Puhua. We shall prove that n is a multiple of 3. For convenience, extend the sequence $a_1, a_2, \ldots, a_n$ to an infinite periodic sequence with period n. Then we shall prove that the shortest period of the sequence is 3. Note that the equation $x^2 + 1 = x$ has no real solution, so the numbers $a_1, a_2, \ldots, a_n$ cannot be all equal, hence the shortest period cannot be 1.

By the recurrence relation for i and $i+1$, we have $(a_{i+2}-1)a_{i+2} = a_i a_{i+1} a_{i+2} = a_i(a_{i+3}-1)$ so that $a_{i+2}^2 - a_i a_{i+3} = a_{i+2} - a_i$. By summing over $i = 1, 2, \ldots, n$, we get

$$\sum_{i=1}^{n} (a_i - a_{i+3})^2 = 0.$$

Thus $a_i = a_{i+3}$ for every index i, so the sequence $a_1, a_2, \ldots$ is periodic with period 3. Note that when n is a multiple of 3, $(a_1, a_2, \ldots) = (-1, -1, 2, -1, -1, 2, \ldots)$ is a solution.

3. An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + 3 + \cdots + 2018$?

Partial Solution by Lee Ker Yang and Official Solution. No, such an anti-Pascal triangle does not exist. Suppose it exists. Let T be an anti-Pascal triangle with n rows, containing every integer from 1 to $1 + 2 + \cdots + n$, and let a_1 be the topmost number in T as in figure 2.1. The two numbers below a_1 are some a_2 and $b_2 = a_1 + a_2$, the two numbers below b_2 are some a_3 and $b_3 = a_1 + a_2 + a_3$, and so on and so forth all the way down to the bottom row, where some a_n and $b_n = a_1 + a_2 + \cdots + a_n$ are two neighbours below $b_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$. Since the a_k are pairwise distinct positive integers whose sum does not exceed the largest number in T, which is $1 + 2 + \cdots + n$, it follows that they form a permutation of $1, 2, \ldots, n$.

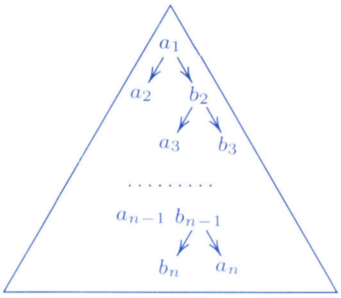

Figure 3.1

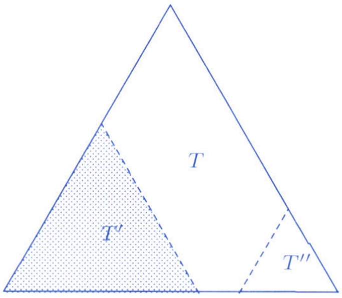

Figure 3.2: There are ℓ numbers in the bottom row of T'.

Consider the two equilateral subtriangles of T whose bottom rows contain the numbers to the left, respectively right, of the pair a_n, b_n as in Figure 3.2. One of these subtriangles could be empty. At least one of these subtriangles, say T', has side length $\ell \ge \lceil (n-2)/2 \rceil$. Since T' is also an anti-Pascal triangle, it contains ℓ pairwise distinct positive integers $a'_1, a'_2, \ldots, a'_\ell$, where a'_1 is at the apex, and a'_k and $b'_k = a'_1 + a'_2 + \cdots + a'_k$ are two neighours below b'_{k-1} for each $k = 2, 3, \ldots, \ell$. Since the a_k all lie outside T', and they form a permutation of $1, 2, \ldots, n$, the a'_k are all greater n. Consequently,

$$b'_\ell \ge (n+1)+(n+2)+\cdots+(n+\ell) = \frac{\ell(2n+\ell+1)}{2} \ge \frac{1}{2}\cdot\frac{n-2}{2}\left(2n+\frac{n-2}{2}+1\right) = \frac{5n(n-2)}{8},$$

which is greater than $1+2+\cdots+n = \frac{n(n+1)}{2}$ for $n = 2018$ resulting in a contradiction.

Remark. In the literature, an anti-Pascal triangle is called an exact difference triangle. It has been proved in [1] that an anti-Pascal triangle with n rows formed by the numbers from 1 to $1+2+\cdots+n$ exists if and only if $n \le 5$.

[1] Chang, Hu, Lih and Shieh, *Exact Difference Triangles*, Bulletin of the Institute of Mathematics, Academia Sinica, Vol. 5, pp 191-197 (1977). The paper is available at `http://w3.math.sinica.edu.tw/bulletin/bulletin_old/d51/5120.pdf`.

4. A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

   Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

   Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Solution by Shi Cheng. The answer is $K = 100$. Colour the sites of the grid black and white in a checkerboard fashion. There are 200 black sites and 200 white sites. As the distance between any 2 black sites is not equal to $\sqrt{5}$, Amy can place a red stone on any of these black sites. Since Amy and Ben take turn to place a stone on the grid and there are 200 black sites, Amy can place at least 100 red stones on the black sites. Thus $K \ge 100$.

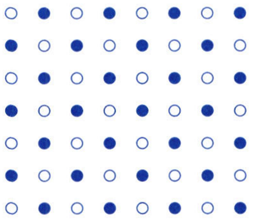

Figure 4.1: Colour the sites of the grid black and white in a checkerboard fashion.

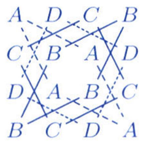

Figure 4.2: Partition the 4 × 4 grid into 4 groups.

Next we shall show that Ben can prevent Amy from placing more than 100 red stones on the grid. Divide the 20×20 grid into 25 sub-grids of size 4×4. In each subgrid of size 4 × 4, group the sites into 4 groups each having 4 sites as in Figure 3.2. In each group, the 4 sites are labelled by the same letter, and they form the vertices of either a square or a rhombus. Note that the length of a side of such a square or a rhombus is $\sqrt{5}$. Also the distance between 2 sites within the same group which are diagonally opposite is not equal to $\sqrt{5}$. If Amy places a red stone on a site of the 4 × 4 grid which belongs to one of the groups with labels say A, then Ben can place a blue stone on the diagonal site within the same group. This prevents Amy from placing a stone on the other 2 sites in that group. As such Amy can place at most 4 red stones on the 4 × 4 grid. Therefore, if Ben adopts this strategy, then Amy can place at most 100 red stones on the 20 × 20 grid. This proves that $K = 100$.

5. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Solution by Tan Junyao, Joel and Official Solution. Let $s_n = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$, where $n \geq N$. Then

$$s_{n+1} - s_n = \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} - \frac{a_n}{a_1} \tag{5.1}$$

is an integer. Multiplying it by a_1, we see that $a_1 a_n / a_{n+1}$ is an integer, so that $a_{n+1} \mid a_1 a_n$ for all $n \geq N$. This implies that $a_n \mid a_1^{n-N} a_N$. Thus all the prime factors of a_n are among those of $a_1 a_N$. It suffices to prove that the exponent of each of them in the prime factorization of a_n is eventually constant. This implies all the a_n are eventually equal.

Let p be a prime factor of $a_1 a_N$. Denote by $v_p(q)$ the exponent of p in the prime factorization of a nonzero rational number q.

Case 1. There exists an index $n \geq N$ such that $v_p(a_n) \geq v_p(a_1)$.

If $v_p(a_{n+1}) < v_p(a_1)$, then $v_p(a_n/a_{n+1}) \geq 0$ and $v_p(a_n/a_1) \geq 0$, but $v_p(a_{n+1}/a_1) < 0$; hence $s_{n+1} - s_n$ in (5.1) cannot be an integer. This contradiction shows that $v_p(a_{n+1}) \geq v_p(a_1)$.

On the other hand, if $v_p(a_{n+1}) > v_p(a_n)$, then $v_p(a_n/a_{n+1}) < 0$, while $v_p(a_{n+1}/a_1) \geq 0$ and $v_p(a_n/a_1) \geq 0$; again contradicting the fact that $s_{n+1} - s_n$ in (5.1) is an integer. Therefore, $v_p(a_n) \geq v_p(a_{n+1}) \geq v_p(a_1)$.

This argument can be applied successively to indices $n+1, n+2, \ldots$, showing that $v_p(a_n), v_p(a_{n+1}), v_p(a_{n+2}), \ldots$ is non-increasing and bounded below by $v_p(a_1)$, hence eventually constant.

Case 2. $v_p(a_n) < v_p(a_1)$ for all $n \geq N$.

If we have $v_p(a_{n+1}) < v_p(a_n)$ for some $n \geq N$, then $v_p(a_{n+1}/a_1) < v_p(a_n/a_1) < 0 < v_p(a_n/a_{n+1})$, which contradicts the fact that $s_{n+1} - s_n$ in (5.1) is an integer. Therefore, $v_p(a_{n+1}) \geq v_p(a_n)$ for all $n \geq N$. Consequently, the sequence $v_p(a_N), v_p(a_{N+1}), v_p(a_{N+1}), \ldots,$ is non-decreasing and bounded by $v_p(a_1)$ from above, and hence it is also eventually constant.

6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

Solution by Ng Yu Peng. In order to take care of different configurations, we use oriented angles. The notation $\measuredangle ABC$ denotes the angle measured in the anticlockwise sense from the ray BA to the ray BC.

Let the circumcircles of the triangle AXB and CXD intersect at X and P. Since $\measuredangle XPB + \measuredangle DPX = 180° - \measuredangle XAB + \measuredangle XCD = 180°$, the point P lies on BD. We have the following.

(i) $\measuredangle CPA = \measuredangle CPX + \measuredangle XPA = \measuredangle CDX + 180° - \measuredangle ABX = (\measuredangle CDA - \measuredangle XDA) + 180° - (\measuredangle ABC - \measuredangle XBC) = 180° + \measuredangle CDA - \measuredangle ABC$.

(ii) $\measuredangle BXA + \measuredangle DXC = 180° \Leftrightarrow \measuredangle BPA + \measuredangle DPC = 180° \Leftrightarrow \measuredangle DPC = 180° - \measuredangle BPA = \measuredangle APD$.

The point P on BC such that $\measuredangle CPA = 180° + \measuredangle CDA - \measuredangle ABC$ is unique, so it suffices to show that such a point P satisfies the condition that the line BD bisects $\measuredangle CPA$.

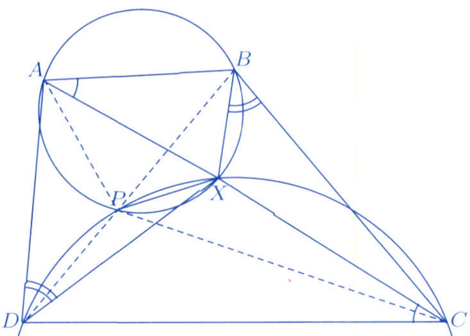

Figure 6.1

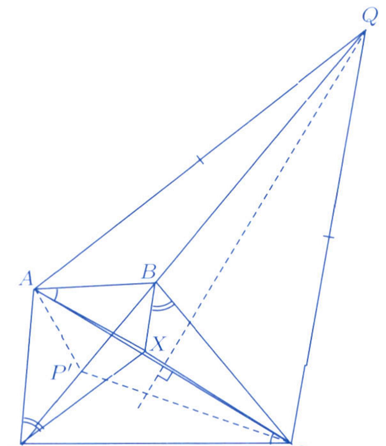

Figure 6.2

As $AB \cdot CD = BC \cdot DA$, we first suppose $AB \neq DA$ so that $BC \neq CD$. For instance, we may suppose $AB < DA$ and $BC < CD$. This implies $\measuredangle BDA < \measuredangle ABD$ and $\measuredangle CDB < \measuredangle DBC$.

If the perpendicular bisector of AC is parallel to BD, then BD is perpendicular to AC. Let the internal and external bisectors of $\measuredangle ABC$ meet the line AC at H and K respectively. The circle with diameter HK is the Apollonius circle of the two points A and C. The relation $AB/BC = AD/DC$ implies that both B and D lie on this circle. This implies that AC is the perpendicular bisector of BD and hence $AB = DA$ and $BC = CD$, which contradict our assumption that $AB \neq DA$ and $BC \neq CD$. Therefore, the perpendicular bisector AC is not parallel to BD.

Let BD intersect the perpendicular bisector of AC at Q. See Figure 6.2. Without loss of generality, we may assume Q and B are on the same side of AC. Let the circumcircle of the triangle ACQ intersect BD at P'. Since $AQ = CQ$, we have the line BD bisects $\measuredangle CP'A$.

To show $P' = P$, we show that $\measuredangle CP'A = 180° + \measuredangle CDA - \measuredangle ABC$. Since $\measuredangle CP'A = 180° - \measuredangle AQC$, it is equivalent to show $\measuredangle AQC = -\measuredangle CDA + \measuredangle ABC$.

Let Q' be the point on the extension of DB such that $\measuredangle BAQ' = \measuredangle BDA$. Then the triangles ABQ' and DAQ' are similar. It follows that $\frac{Q'D}{Q'B} = \frac{Q'D}{AB}\frac{AB}{Q'B} = \frac{Q'D}{AB}\frac{DA}{Q'A} = \frac{Q'D}{Q'A}\frac{DA}{AB} = \frac{DA^2}{AB^2}$.
Similarly, if Q'' is the point on the extension of DB such that $\measuredangle Q''CB = \measuredangle CDB$, the the triangles BCQ'' and CDQ'' are similar. It follows that $\frac{Q''D}{Q''B} = \frac{CD^2}{BC^2}$.

Since $AB \cdot CD = BC \cdot DA$, we have $\frac{Q'D}{Q'B} = \frac{Q''D}{Q''B}$. This implies $Q' = Q''$. Denote this common point of Q' and Q'' by Q^*. Since $Q^*A^2 = Q^*B \cdot Q^*D = Q^*C^2$, the point Q^* lies on the perpendicular bisector of AC. This means $Q^* = Q$. This prove that $\measuredangle BAQ = \measuredangle BDA$ and $\measuredangle QCB = \measuredangle CDB$.

Therefore $\measuredangle AQC = \measuredangle ABC - \measuredangle BAQ - \measuredangle QCB = \measuredangle ABC - \measuredangle BDA - \measuredangle CDB = \measuredangle ABC - \measuredangle CDA$.

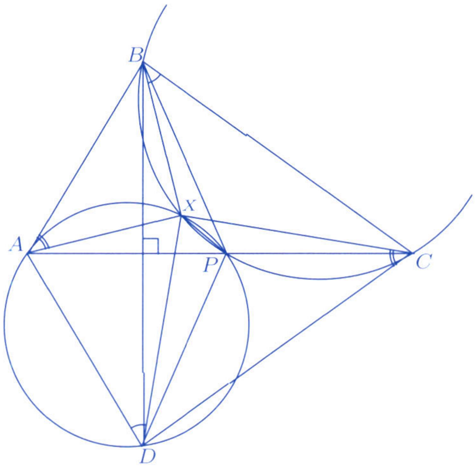

Figure 6.3

Lastly if $AB = DA$ so that $BC = CD$, then $ABCD$ is a kite. See Figure 6.3. If we construct the point P as the second intersection of the circumcircles of the triangles BXC and DXA, then P lies on the AC. As in the above case, it suffices to prove AC bisects $\measuredangle BPD$ which is obviously true.