On Double Series

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Using geometric series and p-series, we present double series whose sums are partial sums of Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, and Alternating Harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$. Special cases of the following theorems are $\sum_{p=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^p} = 1, \sum_{n=2}^{\infty} \sum_{p=2}^{\infty} (-1)^n \frac{1}{n^p} = 2 \ln 2 - 1.$

Definitions, theorems, terminologies and notations used can be found in introductory Real Analysis textbooks.

A well known Theorem of Fubini states that for positive integers r, t and for real numbers a_{mn}, where m, n are integers such that $m \geq t$ and $n \geq r$, if the infinite series $\sum_{m=t}^{\infty} \sum_{n=r}^{\infty} |a_{mn}|$ is convergent, then the infinite series $\sum_{n=r}^{\infty} \sum_{m=t}^{\infty} a_{mn}$ is convergent and $\sum_{m=t}^{\infty} \sum_{n=r}^{\infty} a_{mn} = \sum_{n=r}^{\infty} \sum_{m=t}^{\infty} a_{mn}$.

Theorem 1. *Let k be an integer and $1 \leq k$. Then*

$$\sum_{n=k+1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{k} \left(\frac{k}{n} \right)^m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k},$$

$$\sum_{m=2}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{k} \left(\frac{k}{n} \right)^m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}.$$

Proof. Let k, n be integers and $1 \leq k < n$. Then the geometric series $\sum_{m=2}^{\infty} \frac{1}{k} \left(\frac{k}{n} \right)^m$ converges to $\frac{k}{n(n-k)}$. For integers $p \geq 2k + 1$,

$$\begin{aligned} \sum_{n=k+1}^{p} \frac{k}{n(n-k)} &= \sum_{n=k+1}^{p} \left[\frac{1}{n-k} - \frac{1}{n} \right] = \sum_{n=1}^{p-k} \frac{1}{n} - \sum_{n=k+1}^{p} \frac{1}{n} \\ &= \sum_{n=1}^{k} \frac{1}{n} + \sum_{n=k+1}^{p-k} \frac{1}{n} - \sum_{n=k+1}^{p} \frac{1}{n} = \sum_{n=1}^{k} \frac{1}{n} - \sum_{n=p-k+1}^{p} \frac{1}{n}, \end{aligned}$$

$\sum_{n=p-k+1}^{p} \frac{1}{n} \leq \frac{k}{p-k+1}$ and $\lim_{p\to\infty} \frac{k}{p-k+1} = 0$. Hence

$$\sum_{n=k+1}^{\infty} \frac{k}{n(n-k)} = \lim_{p\to\infty} \sum_{n=k+1}^{p} \frac{k}{n(n-k)} = \sum_{n=1}^{k} \frac{1}{n}.$$

That is,

$$\sum_{n=k+1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{k} \left(\frac{k}{n}\right)^m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k},$$

thus, by the Fubini's Theorem above,

$$\sum_{m=2}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{k} \left(\frac{k}{n}\right)^m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}.$$

■

Corollary 2. $\sum_{p=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^p} = 1 = \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$.

Proof. Let $k = 1$ in Theorem 1. ■

Remark 1. *A consequence of Theorem 1 is*

$$\lim_{k\to\infty} \frac{\sum_{n=k+1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{k} \left(\frac{k}{n}\right)^m}{\ln k} = 1.$$

Proof. Since $\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\, dx < \frac{1}{n}$ $\forall n \in \mathbb{N}$, for $k > 1$,

$$\sum_{n=2}^{k} \frac{1}{n} = \sum_{n=1}^{k-1} \frac{1}{n+1} < \sum_{n=1}^{k-1} \int_n^{n+1} \frac{1}{x}\, dx = \int_1^k \frac{1}{x}\, dx = \ln k < \sum_{n=1}^{k-1} \frac{1}{n}.$$

Hence

$$\ln k + \frac{1}{k} < \sum_{n=k+1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{k} \left(\frac{k}{n}\right)^m = \sum_{n=1}^{k} \frac{1}{n} < 1 + \ln k$$

and

$$1 = \lim_{k\to\infty} \frac{\ln k + \frac{1}{k}}{\ln k} \leq \lim_{k\to\infty} \frac{\sum_{n=k+1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{k} \left(\frac{k}{n}\right)^m}{\ln k} \leq \lim_{k\to\infty} \frac{1 + \ln k}{\ln k} = 1.$$

■

Theorem 3. *Let k be an integer and $1 \leq k$. Then*

$$\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}(-1)^n\frac{1}{k}\left(\frac{k}{n}\right)^m = \sum_{n=1}^{k}(-1)^n\frac{1}{n}$$

if k is even, and

$$\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}(-1)^n\frac{1}{k}\left(\frac{k}{n}\right)^m = \sum_{n=1}^{k}(-1)^{n+1}\frac{1}{n} + 2\sum_{n=k+1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

if k is odd.

Proof. It follows from Theorem 1 that $\sum_{m=2}^{\infty}\frac{1}{k}\left(\frac{k}{n}\right)^m$ converges to $\frac{1}{n-k} - \frac{1}{n}$. Hence $\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}(-1)^n\frac{1}{k}\left(\frac{k}{n}\right)^m = \sum_{n=k+1}^{\infty}(-1)^n\frac{1}{(n-k)} - \sum_{n=k+1}^{\infty}(-1)^n\frac{1}{n} = \sum_{n=1}^{\infty}(-1)^{n+k}\frac{1}{n} - \sum_{n=k+1}^{\infty}(-1)^n\frac{1}{n}$. By noting that$(-1)^{n+k} = (-1)^n$ if k is even, and $(-1)^{n+k} = (-1)^{n+1}$ if k is odd, we get the required result. ■

Theorem 4. *Let k be an integer and $1 \leq k$. Then*

$$\sum_{m=2}^{\infty}\sum_{n=k+1}^{\infty}(-1)^n\frac{1}{k}\left(\frac{k}{n}\right)^m = \sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}(-1)^n\frac{1}{k}\left(\frac{k}{n}\right)^m.$$

Proof. This Theorem follows from the Fubini's Theorem. ■

Corollary 5.

$$\lim_{k\to\infty}\left|\sum_{m=2}^{\infty}\sum_{n=k+1}^{\infty}(-1)^n\frac{1}{k}\left(\frac{k}{n}\right)^m\right| = \lim_{k\to\infty}\left|\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}(-1)^n\frac{1}{k}\left(\frac{k}{n}\right)^m\right| = \ln 2.$$

Proof. By applying Theorems 3, 4 and the fact that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$, we get the required result. ■

Corollary 6. $\sum_{n=2}^{\infty}\sum_{p=2}^{\infty}(-1)^n\frac{1}{n^p} = 2\ln 2 - 1 = \sum_{n=2}^{\infty}(-1)^n\frac{1}{n(n-1)}.$

Proof. Let $k = 1$ in Theorem 3. ■

Remark 2. *A special case of Theorem 3 in [2] is that* $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{n+m} = \ln 2 - \frac{1}{2}$. *It is interesting to note that the sum of the series in Corollary 6 is twice the sum of the series* $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{n+m}$.

Theorem 7. *Let k be an integer and $1 \le k$. Then*

(i) $\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{k}\left(-\frac{k}{n}\right)^m = \sum_{n=k+1}^{2k}\frac{1}{n}$ *and*

$$\lim_{k\to\infty}\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{k}\left(-\frac{k}{n}\right)^m = \ln 2.$$

(ii) $\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}(-1)^n\frac{1}{k}\left(-\frac{k}{n}\right)^m = \sum_{n=k+1}^{2k}(-1)^n\frac{1}{n}$ *if k is even,*

$$\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}(-1)^n\frac{1}{k}\left(-\frac{k}{n}\right)^m = \sum_{n=k+1}^{2k}(-1)^n\frac{1}{n} + 2\sum_{n=2k+1}^{\infty}(-1)^n\frac{1}{n}$$

if k is odd, and $\lim_{k\to\infty}\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}(-1)^n\frac{1}{k}\left(-\frac{k}{n}\right)^m = 0.$

Proof. (i) As shown in the proof of Theorem 1 ,

$$\sum_{n=k+1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{k}\left(-\frac{k}{n}\right)^m = \sum_{n=k+1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+k}\right] = \sum_{n=k+1}^{2k}\frac{1}{n},$$

$\frac{1}{2k} + \ln\frac{2k}{k+1} < \sum_{n=k+1}^{2k}\frac{1}{n} < \frac{1}{k+1} + \ln\frac{2k}{k+1}$ and $\lim_{k\to\infty}\sum_{n=k+1}^{2k}\frac{1}{n} = \ln 2.$

(ii) It is easy to see that $\sum_{n=k+1}^{\infty}(-1)^n\left[\frac{1}{n}-\frac{1}{n+k}\right] = \sum_{n=k+1}^{2k}(-1)^n\frac{1}{n}$

if k is even and $\sum_{n=k+1}^{\infty}(-1)^n\left[\frac{1}{n}-\frac{1}{n+k}\right] = \sum_{n=k+1}^{2k}(-1)^n\frac{1}{n}+2\sum_{n=2k+1}^{\infty}(-1)^n\frac{1}{n}$

if k is odd. Since the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent,

$$\lim_{k\to\infty}\sum_{n=k+1}^{2k}(-1)^{n}\frac{1}{n} = 0 = \lim_{k\to\infty}\sum_{n=2k+1}^{\infty}(-1)^{n}\frac{1}{n}.$$

■

Corollary 8.

$$\sum_{n=2}^{\infty}\sum_{p=2}^{\infty}\left(-\frac{1}{n}\right)^{p} = \frac{1}{2} \textit{ and } \sum_{n=2}^{\infty}\sum_{p=2}^{\infty}(-1)^{n}\left(-\frac{1}{n}\right)^{p} = \frac{3}{2} - 2\ln 2.$$

Proof. By letting $k = 1$ in Theorem 7 and using the fact that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2,$$

we get the result. ■

Theorem 9. *Let k be an integer and $2 \le k$. Then*

$$(i)\ \sum_{m=2}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{kn}\right)^{m} = \sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\left(\frac{1}{kn}\right)^{m}$$
$$= \int_0^1 \frac{t^{k-2}}{1+t+t^2+\cdots+t^{k-1}}dt,$$

$$(ii)\ \sum_{m=2}^{\infty}\sum_{n=1}^{\infty}\left(-\frac{1}{kn}\right)^{m} = \sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\left(-\frac{1}{kn}\right)^{m}$$
$$= \int_0^1 \frac{t^{k-1}}{1+t+t^2+\cdots+t^{k-1}}dt.$$

Proof. Let k, n be integers, $1 \le n$ and $2 \le k$. Then the geometric series $\sum_{m=2}^{\infty}\left(\frac{1}{kn}\right)^{m}$ coverges to $\frac{1}{k^2 n\left(n-\frac{1}{k}\right)}$. Then we use equation (5) in [1], that is,

$$S(a,b) = \frac{1}{a-b}\int_0^1 \frac{t^b - t^a}{1-t}\,dt,$$

where $S(a,b) = \sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)}, a \neq b$ and neither a nor b is a negative integer, and find $\sum_{n=1}^{\infty} \frac{1}{k^2 n \left(n - \frac{1}{k}\right)} = \frac{1}{k} \int_0^1 \frac{1 - t^{\frac{1}{k}}}{t^{\frac{1}{k}}(1-t)} dt$. By making a substitution $u = t^{\frac{1}{k}}$, it can be easily seen that

$$\frac{1}{k} \int_0^1 \frac{1 - t^{\frac{1}{k}}}{t^{\frac{1}{k}}(1-t)} dt = \int_0^1 \frac{u^{k-2}}{1 + u + u^2 + \cdots + u^{k-1}} du.$$

Thus $\sum_{n=1}^{\infty} \sum_{m=2}^{\infty} \left(\frac{1}{kn}\right)^m = \int_0^1 \frac{t^{k-2}}{1 + t + t^2 + \cdots + t^{k-1}} dt$. Hence by Fubini's theorem (i) follows. Similarly, (ii) can be proved. ■

Remark 3. *The following is a generalization of Theorem 1.*
Let k, n be integers, r be a real number greater than -1 and $1 \leq k$. Then

$$\sum_{n=k+1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{k} \left(\frac{k}{n+r}\right)^m = \sum_{n=1}^{k} \frac{1}{n+r}.$$

References

[1] Costas J. Efthimiou, (1999) Finding Exact Values for Infinite Sums. *Mathematics Magazine* **72**(1):45-51.

[2] Ovidiu Furdui, Tiberiu Trif, (2011) On the Summation of Certain Iterated Series. *Journal of Integer Sequences* **14**(6):Article 11.6.1.

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