Complexity of Pólya's Positivstellensatz for polynomials of low degree

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ABSTRACT. Pólya's Positivstellensatz on the 1-simplex says that if $P(x)$ is a real polynomial such that $P(x) > 0$ whenever $x \geq 0$, then all the coefficients of $(1+x)^m P(x)$ are positive whenever m is large. Powers-Reznick gave a complexity estimate for Pólya's Positivstellensatz. Namely, they proved that, for such $P(x)$ of degree d, all the coefficients of $(1+x)^m P(x)$ are positive whenever $m > \frac{1}{2}(d^2-d)\rho(P) - d$, where $\rho(P)$ is an invariant of $P(x)$. For $d = 3$ and $d = 4$ specifically, we improve Powers-Reznick's bound by showing $m > \frac{3}{2}\rho(P) - 1$ for $d = 3$ and $m > \frac{4232}{2505}\rho(P) - 1$ for $d = 4$.

1. Introduction and main result

A positivstellensatz certifies the strict positivity of a polynomial $f \in \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$ on a semialgebraic set $K \subseteq R^n$ by representing f as an algebraic expression. This algebraic expression of f witnesses the strict positivity of f on K. Pólya proved a positivstellensatz for real homogeneous polynomials on the n-simplex [3] (reproduced in [2, pp. 57-60]). For the 1-simplex an equivalent formulation is that if $P(x)$ be a polynomial such that $P(x) > 0$ whenever $x \geq 0$, then all the coefficients of $(1+x)^m P(x)$ are positive whenever m is large (see also [4, corollary 5]).

Powers-Reznick obtained an upper bound for the least m such that all the coefficients of $(1+x)^m P(x)$ are positive. Given that $\rho(P) = \frac{L(P)}{\lambda(P)}$, for a polynomial $P(x) = \sum_{j=0}^{d} a_j x^j$ of degree d, they showed that all the coefficients of $(1+x)^m P(x)$ are positive whenever

$$m > \frac{1}{2}(d^2-d)\rho(P) - d,$$

where

$$L(P) := \max_{j=0,\ldots,d} \frac{1}{\binom{d}{j}}|a_j| \quad \text{and} \quad \lambda(P) := \inf_{x\in[0,\infty)} \frac{P(x)}{(1+x)^d}, \tag{1}$$

Lemma 1. *Suppose $P(x)$ is a polynomial of degree d such that $P(x) > 0$ whenever $x \geq 0$. Then $\lambda(P)$ is a well-defined positive real number.*

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Proof. The leading coefficient of $P(x)$ is equal to $P(0)$ and hence is positive. Thus $P(x)$ and $(1+x)^d$ are polynomials with positive leading coefficient of the same degree, hence $\lim_{x\to\infty}\frac{P(x)}{(1+x)^d} > 0$. Thus $\lambda(P) := \inf_{x\in[0,\infty)}\frac{P(x)}{(1+x)^d}$ is well-defined. Furthermore, since $P(x)$ and $(1+x)^d$ takes only positive values for $x \geq 0$, thus $\lambda(P) := \inf_{x\in[0,\infty)}\frac{P(x)}{(1+x)^d}$ is positive. □

Next, we shall show

$$L(P) \geq \lambda(P). \tag{2}$$

Note that

$$L(P) \geq \max_{j=0,\dots,d}\frac{1}{\binom{d}{j}}a_j \quad \text{and} \quad \sup_{x\in[0,\infty)}\frac{\sum_{j=0}^d a_j x^j}{(1+x)^d} \geq \lambda(P)$$

Hence it suffices to show that $\max_{j=0,\dots,d}\frac{1}{\binom{d}{j}}a_j \geq \sup_{x\in[0,\infty)}\frac{\sum_{j=0}^d a_j x^j}{(1+x)^d}$. Let $A = \max_{j=0,\dots,d}\frac{1}{\binom{d}{j}}a_j$, we know that $A \geq \frac{1}{\binom{d}{j}}a_j$. $\frac{\sum_{j=0}^d a_j x^j}{(1+x)^d} \leq \frac{\sum_{j=0}^d \binom{d}{j}Ax^j}{(1+x)^d} = \frac{(1+x)^d}{(1+x)^d}A$. Hence $\frac{\sum_{j=0}^d a_j x^j}{(1+x)^d} \leq \max_{j=0,\dots,d}\frac{1}{\binom{d}{j}}a_j$. This proves (2).

Lemma 2. *Let $P(x)$ be a polynomial of degree d. If $L(P) = \lambda(P)$, then $P(x) = b(1+x)^d$ for some b.*

Proof. Since

$$L(P) \geq \max_{j=0,\dots,d}\frac{1}{\binom{d}{j}}a_j \geq \sup_{x\in[0,\infty)}\frac{P(x)}{(1+x)^d} \geq \inf_{x\in[0,\infty)}\frac{P(x)}{(1+x)^d} =: \lambda(P),$$

if $L(P) = \lambda(P)$, then $\sup_{x\in[0,\infty)}\frac{P(x)}{(1+x)^d} = \inf_{x\in[0,\infty)}\frac{P(x)}{(1+x)^d}$. This means $\frac{P(x)}{(1+x)^d}$ has the same supremum and infimum over $[0,\infty)$, and hence $\frac{P(x)}{(1+x)^d}$ must be a constant. □

In this report we improve Powers-Reznick's degree bound for small d.

Theorem 3. *Let $P(x)$ be a polynomial of degree d such that $P(x) > 0$ whenever $x \geq 0$. For $d = 1, \dots, 4$, all the coefficients of $(1+x)^m P(x)$ are positive whenever*

$$m > C_d\frac{L(P)}{\lambda(P)} - 1,$$

where C_d is given in the table below

To prove that $(1+x)^m P(x)$ have positive coefficients, we use the equivalent condition:

$$[x^{c(m+d)}](1+x)^m P(x) > 0 \text{ whenever } c(m+d) \text{ is a non-negative integer for } 0 \leq c \leq 1 \tag{3}$$

Pólya chose $\frac{m!(m+d)^d}{(c(m+d))!}(1-c)^d P(\frac{c}{1-c})$ to approximate $[x^{c(m+d)}](1+x)^m P(x)$. We chose our approximator to be $\binom{m+d}{c(m+d)}(1-c)^d P(\frac{c}{1-c})$ instead.

d	1	2	3	4
C_d	0	1	$\frac{3}{2}$	$\frac{4232}{2505}$
$\frac{d^2-d}{2}$	0	1	3	6

TABLE 1. Improvement in degree bounds

2. Proof of theorem 3

Lemma 4. *Let $P(x)=\sum_{j=0}^{d} a_j x^j$ be a polynomial of degree d. For each $m \in \mathbb{Z}_{\geq 0}$ and each $0 \leq c \leq 1$ such that $c(m+d) \in \mathbb{Z}_{\geq 0}$,*

$$[x^{c(m+d)}](1+x)^m P(x) - \binom{m+d}{c(m+d)}(1-c)^d P(\frac{c}{1-c})$$
$$= \binom{m+d}{c(m+d)} \sum_{j=0}^{d} a_j (f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0)),$$

where

$$f_c^{(j)}(x) := \frac{((c)(c-x)\cdots(c-(j-1)x))((1-c)(1-c-x)\cdots(1-c-(d-j-1)x)}{(1)(1-x)(1-2x)\cdots(1-(d-1)x)}. \tag{4}$$

Proof. Since $[x^{c(m+d)}](1+x)^m P(x) = \sum_{j=0}^{d} a_j \binom{m}{c(m+d)-j}$ and $\binom{m}{c(m+d)-j} = \binom{m+d}{c(m+d)} f_c^{(j)}(\frac{1}{m+d}))$, for $0 \leq j \leq d$, hence

$$[x^{c(m+d)}](1+x)^m P(x) = \binom{m+d}{c(m+d)} \sum_{j=0}^{d} a_j f_c^{(j)}(\frac{1}{m+d}). \tag{5}$$

Combining (5) with $(1-c)^d P(\frac{c}{1-c}) = \sum_{j=0}^{d} a_j c^j (1-c)^{d-j} = f_c^{(j)}(0)$, we will prove lemma 4. □

From (4),

$$f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0) = \frac{c(1-c)}{(m+1)(m+2)\cdots(m+d-1)} h_c^{(j)}(m), \tag{6}$$

for some polynomial $h_c^{(j)}(m)$ of degree $d-2$ in m and of degree $d-2$ in c. By symmetry

$$h_c^{(j)}(m) = h_{1-c}^{d-j}(m) \tag{7}$$

Lemma 5. *For $d=3$ or 4, let $f_c^{(j)}(m)$ be as in (4). For each $m \in \mathbb{Z}_{\geq 0}$ and $0 \leq c \leq \frac{1}{2}$,*

$$\sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0)| \leq \frac{C_d}{m+1},$$

where $C_3 = \frac{3}{2}$ and $C_4 = \frac{4232}{2505}$ as given in table 1.

Proof of lemma 5 for $d = 3$. We will first determine the sign of $f_c^{(j)}(\frac{1}{m+3}) - f_c^{(j)}(0)$. For $d = 3$, we calculate $h_c^{(j)}(m)$ explicitly as follows:

$$h_c^{(j)}(m) = \begin{cases} (-3+3c)m + (-5+7c) & \text{if } j = 0 \\ (2-3c)m + (4-7c) & \text{if } j = 1 \end{cases} \tag{8}$$

Note that $h_c^{(2)}(m) = h_{1-c}^{(1)}(m)$ and $h_c^{(3)}(m) = h_{1-c}^{(0)}(m)$ from (7). Since $h_c^{(j)}(m)$ is linear in m, we can determine the sign as $0 \le c \le 1$.

j	$0 \le c \le \frac{2}{7}$	$\frac{2}{7} \le c \le \frac{1}{3}$	$\frac{1}{3} \le c \le \frac{3}{7}$	$\frac{3}{7} \le c \le \frac{4}{7}$	$\frac{4}{7} \le c \le \frac{2}{3}$	$\frac{2}{3} \le c \le \frac{5}{7}$	$\frac{5}{7} \le c \le 1$
0	≤ 0	≤ 0	≤ 0	≤ 0	≤ 0	≤ 0	?
1	≥ 0	≥ 0	≥ 0	≥ 0	?	≤ 0	≤ 0
2	≤ 0	≤ 0	?	≥ 0	≥ 0	≥ 0	≥ 0
3	?	≤ 0	≤ 0	≤ 0	≤ 0	≤ 0	≤ 0

TABLE 2. The entry of the j-th row and the column $a \le c \le b$ is the sign of $h_c^{(j)}(m)$ for $a \le c \le b$ and $m \ge 0$. When there is a ?, it means that the sign depends on m.

For example, $j = 1$ when $0 \le c \le \frac{4}{7}$, the linear function $h_c^{(1)}(m)$ has positive leading coefficient since $2 - 3c \ge 2 - 3(\frac{4}{7}) = \frac{2}{7} > 0$ hence

$$h_c^{(1)}(m) \ge h_c^{(1)}(0) = 4 - 7c \ge 0, \quad \text{for } 0 \le c \le \frac{4}{7}$$

The possible signs of $(h_c^{(0)}(m), h_c^{(1)}(m), h_c^{(2)}(m), h_c^{(3)}(m))$ are given by

$$(h_c^{(0)}(m), h_c^{(1)}(m), h_c^{(2)}(m), h_c^{(3)}(m)) \begin{cases} (\le 0, \ge 0, \le 0, \ge 0) & \text{only if } 0 \le c \le \frac{2}{7} \\ (\le 0, \ge 0, \le 0, \le 0) & \text{only if } 0 \le c \le \frac{3}{7} \\ (\le 0, \ge 0, \ge 0, \le 0) & \text{only if } \frac{1}{3} \le c \le \frac{1}{2} \end{cases}$$

We return to the estimation of the sum $\sum_{j=0}^{3} \binom{3}{j} |f_c^{(j)}(\frac{1}{m+3}) - f_c^{(j)}(0)|$.

Case 1: $(h_c^{(0)}(m), h_c^{(1)}(m), h_c^{(2)}(m), h_c^{(3)}(m)) = (\le 0, \ge 0, \le 0, \ge 0)$ and $0 \le c \le \frac{2}{7}$. Note that for this case, using (8) with its respective signs, we will obtain $\sum_{j=0}^{3} \binom{3}{j} |h_c^{(j)}(m)| = (12 - 24c)m + (28 - 56c)$. Hence, using (6), we will get

$$\sum_{j=0}^{3} \binom{3}{j} |f_c^{(j)}(m) - f_c^{(j)}(0)| = (12 + \frac{4}{m+2}) \frac{c(1-c)(1-2c)}{m+1} \le \frac{7}{3\sqrt{3}} \frac{1}{m+1}, \tag{9}$$

where the last inequality follows from $0 < \frac{4}{m+2} \le 2$ and $c(1-c)(1-2c) \le \frac{1}{6\sqrt{3}}$ since $0 \le c \le \frac{2}{7}$ and $m \ge 0$.

Case 2: $(h_c^{(0)}(m), h_c^{(1)}(m), h_c^{(2)}(m), h_c^{(3)}(m)) = (\le 0, \ge 0, \le 0, \le 0)$ and $0 \le c \le \frac{3}{7}$. Similarly, for this case, using (8) with its respective signs, we will obtain $\sum_{j=0}^{3}\binom{3}{j}|h_c^{(j)}(m)| = (6-18c)m + (24-42c)$. Hence, using (6), we will get

$$\text{(10)} \quad \sum_{j=0}^{3}\binom{3}{j}|f_c^{(j)}(m) - f_c^{(j)}(0)| = 6(1+\frac{2-c}{m+2})\frac{c(1-c)(1-3c)}{m+1} \le \frac{8(7\sqrt{7}-10)}{81}\frac{1}{m+1},$$

where the last inequality follows from $0 < \frac{2-c}{m+2} \le 1$ and $c(1-c)(1-3c) \le \frac{2(7\sqrt{7}-10)}{243}$ since $0 \le c \le \frac{3}{7}$ and $m \ge 0$.

Case 3: $(h_c^{(0)}(m), h_c^{(1)}(m), h_c^{(2)}(m), h_c^{(3)}(m)) = (\le 0, \ge 0, \ge 0, \le 0)$ and $\frac{1}{3} \le c \le \frac{1}{2}$. Also for this case, using (8) with its respective signs, we will obtain $\sum_{j=0}^{3}\binom{3}{j}|h_c^{(j)}(m)| = 6m+6$. Hence, using (6), we will get

$$\text{(11)} \quad \sum_{j=0}^{3}\binom{3}{j}|f_c^{(j)}(m) - f_c^{(j)}(0)| = \frac{c(1-c)}{(m+1)(m+2)}(6m+6) \le \frac{3}{2}\frac{1}{m+1}$$

where the last inequality follows from $c(1-c) \le \frac{1}{4}$ since $\frac{1}{3} \le c \le \frac{1}{2}$ and $m \ge 0$.

Hence, comparing (9), (10), (11), we can conclude that

$$\sum_{j=0}^{3}\binom{3}{j}|f_c^{(j)}(m) - f_c^{(j)}(0)| \le \max\left\{\frac{7}{3\sqrt{3}}, \frac{8(7\sqrt{7}-10)}{81}, \frac{3}{2}\right\}\frac{1}{m+1} = \frac{3}{2}\frac{1}{m+1}$$

for $0 \le c \le \frac{1}{2}$ □

Proof of lemma 5 for $d=4$. We will first determine the sign of $f_c^{(j)}(\frac{1}{m+4}) - f_c^{(j)}(0)$. For $d=4$, we calculate $h_c^{(j)}(m)$ explicitly as follows:

(12)

$$h_c^{(j)}(m) = \begin{cases} m^2(-6c^2+12c-6)+m(-37c^2+63c-26)+(-58c^2+78c-26) & \text{if } j=0 \\ m^2(6c^2-9c+3)+m(37c^2-50c+15)+(58c^2-68c+18) & \text{if } j=1 \\ m^2(-6c^2+6c-1)+m(-37c^2+37c-7)+(-58c^2+58c-12) & \text{if } j=2 \end{cases}$$

Note that $h_c^{(3)}(m) = h_{1-c}^{(1)}(m)$ and $h_c^{(4)}(m) = h_{1-c}^{(0)}(m)$.

Since $h_c^{(j)}(m)$ is quadratic in m, we can find the turning point of the quadratic given by $m_0 = -\frac{b}{2a}$ for any quadratic am^2+bm+c. Due to symmetry, we would only show the case for

$j = 1, 2, 3.$

$$m_0 = \begin{cases} \frac{26-37c}{12(c-1)} & \text{if } j = 0 \\ -\frac{37c^2-50c+15}{12c^2-18c+6} & \text{if } j = 1 \\ -\frac{37c^2-37c+7}{12c^2-12c+2} & \text{if } j = 2 \end{cases}$$

Using the turning point above we can obtain,

$$h_c^{(j)}(m_0) = \begin{cases} \frac{1}{24}(-23c^2 - 52c + 52) & \text{if } j = 0 \\ -\frac{-23c^4+20c^3+34c^2-36c+9}{24c^2-36c+12} & \text{if } j = 1 \\ \frac{-23c^4+46c^3-25c^2+2c+1}{24c^2-24c+4} & \text{if } j = 2 \end{cases}$$

Now, for $m > 1$, we will show $h_c^{(j)}(m) \leq h_c^{(j)}(-\frac{b}{2a})$ and for $m < 1$, we will show that $h_c^{(j)}(m) \leq h_c^{(j)}(1)$.

Given that $m > 1$, we can solve m_0 for c as follow

$$m_0(c) = 1 \quad \text{if and only if} \begin{cases} c = \frac{38}{49} & \text{if } j = 0 \\ c = \frac{34}{49} \pm \frac{\sqrt{127}}{49} & \text{if } j = 1 \\ c = \frac{1}{2} \pm \frac{\sqrt{13}}{14} & \text{if } j = 2 \end{cases}$$

For the case $j = 0$, the leading coefficient of $h_c^{(0)}(m)$ is negative for $0 \leq c \leq \frac{1}{2}$. Here, we only have 1 range ($0 \leq c \leq \frac{1}{2}$) to deal with. Since the leading coefficient is negative, hence $h_c^{(0)}(m)$ is is decreasing in m for $m \geq m_0$. In this case $m_0 \leq 1$, hence $h_c^{(0)}(m) \geq h_c^{(0)}(1)$ where $h_c^{(0)}(1)$ can be solved to be negative for $0 \leq c \leq \frac{1}{2}$.

Next, for $j = 1$, the leading coefficient of $h_c^{(1)}(m)$ is positive for $0 \leq c \leq \frac{1}{2}$.

Case B_1: $0 \leq c \leq \frac{34-\sqrt{127}}{49}$. Since the leading coefficient is positive, hence $h_c^{(1)}(m)$ is increasing in m for $m \geq m_0$. In this case, $m_0 \leq 1$, hence $h_c^{(1)}(1) \leq h_c^{(1)}(m)$ where $h_c^{(1)}(1)$ can be solved to be positive for $0 \leq c \leq \frac{127-\sqrt{1585}}{202}$.

Case B_2: $\frac{34-\sqrt{127}}{49} \leq c < \frac{1}{2}$. Since the leading coefficient is positive, and in this case, $m_0 \geq 1$, hence $h_c^{(1)}(m_0) \leq h_c^{(1)}(m)$ where $h_c^{(1)}(m_0)$ can be solved to be negative for $\frac{34-\sqrt{127}}{49} \leq c < \frac{1}{2}$.

Lastly, for $j = 2$, the leading coefficient of $h_c^{(2)}(m)$ is negative for $0 \leq c \leq \frac{1}{6}(3 - \sqrt{3})$ and positive for $\frac{1}{6}(3 - \sqrt{3}) \leq c \leq \frac{1}{2}$.

Case C_1: $0 \leq c < \frac{1}{6}(3 - \sqrt{3})$. Since the leading coefficient is negative, hence $h_c^{(2)}(m)$ is decreasing in m for $m \geq m_0$. In this case, $m_0 \leq 1$, hence $h_c^{(2)}(m) \leq h_c^{(2)}(1)$ where $h_c^{(2)}(1)$ can be solved to be negative for $0 \leq c \leq \frac{1}{6}(3 - \sqrt{3})$.

Case C_2: $\frac{1}{6}(3-\sqrt{3}) < c \le \frac{1}{2} - \frac{\sqrt{13}}{14}$. Since the leading coefficient is positive, hence $i_c^{(2)}(m)$ is increasing in m for $m \ge m_0$. In this case, $m_0 \ge 1$, hence $h_c^{(2)}(1) \le h_c^{(2)}(m)$ where $h_c^{(2)}(1)$ can be solved to be negative for $\frac{1}{6}(3-\sqrt{3}) \le c \le \frac{7-\sqrt{13}}{14}$

Case C_3: $\frac{7-\sqrt{13}}{14} \le c \le \frac{1}{2}$. Since the leading coefficient is positive and in this case, $m_0 \le 1$, hence $h_c^{(2)}(m_0) \le h_c^{(2)}(m)$ where $h_c^{(2)}(m_0)$ can be solved to be negative for $\frac{7-\sqrt{13}}{14} \le c \le \frac{1}{2}$.

j	$0 \le c < \frac{3-\sqrt{3}}{6}$	$\frac{3-\sqrt{3}}{6} < c \le \frac{127-\sqrt{1585}}{202}$	$\frac{127-\sqrt{1585}}{202} \le c < \frac{1}{2}$	$c = \frac{1}{2}$
0	≤ 0	≤ 0	≤ 0	≤ 0
1	≥ 0	≥ 0	?	≤ 0
2	≤ 0	?	?	≥ 0

TABLE 3. The entry of the j-th row and the column(range of c) is the sign of $h_c^{(j)}(m)$ for that range. When there is a ?, it means that the sign depends on m.

Hence the possible signs of $(h_c^{(0)}(m), h_c^{(1)}(m), h_c^{(2)}(m), h_c^{(3)}(m), h_c^{(4)}(m))$ are given by

$$\begin{cases} (\le 0, \ge 0, \le 0, \ge 0, \le 0) & \text{only if } 0 \le c < \frac{1}{2} \\ (\le 0, \ge 0, \ge 0, \ge 0, \le 0) & \text{only if } \frac{3-\sqrt{3}}{6} \le c < \frac{1}{2} \\ (\le 0, \le 0, \le 0, \le 0, \le 0) & \text{only if } \frac{127-\sqrt{1585}}{202} \le c < \frac{1}{2} \\ (\le 0, \le 0, \ge 0, \le 0, \le 0) & \text{only if } \frac{127-\sqrt{1585}}{202} \le c \le \frac{1}{2} \end{cases}$$

(13)
$$\sum_{j=0}^{4} \binom{4}{j} |f_c^{(j)}(m) - f_c^{(j)}(0)| = \frac{c(1-c)(\phi(m)c^2 - \phi(m)c + \psi(m))}{(m+1)(m+2)(m+3)} \le \frac{(\psi(m))^2}{4\phi(m)(m+1)(m+2)(m+3)},$$

Let $g(c) := c(1-c)(\phi(m)c^2 - \phi(m)c + \psi(m))$, where $\phi(m)\psi(m) \ge 0$ and $\phi(m)(\phi(m) - 2\psi(m)) \ge 0$. The above inequality follows since $g(c) \le g(\frac{\phi(m) - \sqrt{\phi(m)(\phi(m)-2\psi(m))}}{2\phi(m)}) = \frac{\psi(m)^2}{4(\phi(m)}$. Case 1: $(\le 0, \ge 0, \le 0, \ge 0, \le 0)$. In this case, $\phi(m) = 96m^2 + 592m + 982$ and $\psi(m) = 24m^2 + 136m + 208$. From (13),

(14)
$$\sum_{j=0}^{4} \binom{4}{j} |f_c^{(j)}(m) - f_c^{(j)}(0)| = \frac{8(3m^2+17m+26)^2}{((m+2)(m+3)(48m^2+296m+491)} \cdot \frac{1}{m+1} \le \frac{4232}{2505} \frac{1}{m+1}$$

This inequality holds since $\frac{8(3m^2+17m+26)^2}{((m+2)(m+3)(48m^2+296m+491)}$ is decreasing in m for $m \ge 1$ so that $\frac{8(3m^2+17m+26)^2}{((m+2)(m+3)(48m^2+296m+491)} \le \frac{8(3\cdot 1^2+17\cdot 1+26)^2}{((1+2)(1+3)(48\cdot 1^2+296\cdot 1+491)} = \frac{4232}{2505}$.

Case 2: $(\leq 0, \geq 0, \geq 0, \geq 0, \leq 0)$. In this case, $\phi(m) = 24m^2 + 128m + 232$ and $\psi(m) = 12m^2 + 52m + 64$. From (13),

$$\sum_{j=0}^{4} \binom{4}{j} |f_c^{(j)}(m) - f_c^{(j)}(0)| \leq \frac{8}{9}\frac{1}{m+1}. \tag{15}$$

Case 3: $(\leq 0, \leq 0, \leq 0, \leq 0, \leq 0)$ In this case,

$$\sum_{j=0}^{4} \binom{4}{j} |f_c^{(j)}(m) - f_c^{(j)}(0)| = 0 \tag{16}$$

Case 4: $(\leq 0, \leq 0, \geq 0, \leq 0, \leq 0)$. In this case,

(17)

$$\begin{aligned}
&\sum_{j=0}^{4} \binom{4}{j} |f_c^{(j)}(m) - f_c^{(j)}(0)| \\
&= \frac{c(1-c)(c^2(-72m^2 - 444m - 696) + c(72m^2 + 444m + 696) - (12m^2 + 84m + 144))}{(m+1)(m+2)(m+3)} \\
&= \frac{12c(1-c)(-c^2(6 + \frac{7}{m+2} + \frac{1}{(m+2)(m+3)}) + c(6 + \frac{7}{m+2} + \frac{1}{(m+2)(m+3)}) - (1 + \frac{2}{m+2}))}{m+1} \\
&< \frac{3}{2}\frac{1}{m+1}
\end{aligned}$$

where the last inequality holds since $(-c^2(6 + \frac{7}{m+2} + \frac{1}{(m+2)(m+3)}) + c(6 + \frac{7}{m+2} + \frac{1}{(m+2)(m+3)}) - (1 + \frac{2}{m+2})) \leq \frac{2m+5}{4m+12} < \frac{1}{2}$ and $c(1-c) \leq \frac{1}{4}$ at $c = \frac{1}{2}$, the inequality is true.

Hence, comparing (14), (15), (16),(17), we can conclude that

$$\sum_{j=0}^{4} \binom{4}{j} |f_c^{(j)}(m) - f_c^{(j)}(0)| \leq \max\left\{\frac{4232}{2505}, \frac{8}{9}, 0, \frac{3}{2}\right\} \frac{1}{m+1} = \frac{4232}{2505}\frac{1}{m+1}$$

for $0 \leq c \leq \frac{1}{2}$ □

Lemma 6. *For $0 \leq c \leq 1$ and each $d \in \mathbb{Z}_{\geq 0}$*

$$\sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0)| = \sum_{j=0}^{d} \binom{d}{j} |f_{1-c}^{(j)}(\frac{1}{m+d}) - f_{1-c}^{(j)}(0)|$$

Proof. For $j = 0, \ldots, d$, by (7), we have $f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0) = f_{1-c}^{(d-j)}(\frac{1}{m+d}) - f_{1-c}^{(d-j)}(0)$. Hence

$$\begin{aligned}
\sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0)| &= \sum_{j=0}^{d} \binom{d}{j} |f_{1-c}^{(d-j)}(\frac{1}{m+d}) - f_{1-c}^{(d-j)}(0)| \\
&= \sum_{j=0}^{d} \binom{d}{j} |f_{1-c}^{(j)}(\frac{1}{m+d}) - f_{1-c}^{(j)}(0)|,
\end{aligned}$$

where the last equality follows by replacing the index j with $d - j$. □

Corollary 7. *For $d=3$ or 4, let $f_c^{(j)}(m)$ be as in (4). For each $m \in \mathbb{Z}_{\geq 0}$ and $0 \leq c \leq 1$,*

$$\sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0)| \leq \frac{C_d}{m+1},$$

where $C_3 = \frac{3}{2}$ and $C_4 = \frac{4232}{2505}$ as given in table 1.

Proof. From lemma 5, for $c \leq \frac{1}{2}$ we have $\sum_{j=0}^{3} \binom{3}{j} |f_c^{(j)}(m) - f_c^{(j)}(0)| \leq \frac{3}{2}\frac{1}{m+1}$ Using lemma 6, we can show that for $1-c \leq \frac{1}{2}$, $\sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0)| = \sum_{j=0}^{3} \binom{3}{j} |f_{1-c}^{(j)}(m) - f_{1-c}^{(j)}(0)| \leq \frac{3}{2}\frac{1}{m+1}$, where the last inequality follows from lemma 5 (with c replaced by $1-c$). □

Proof of theorem 3. Let $P(x)$ be a polynomial of degree d. By lemma 4 and the definition of $L(P)$ in (1),

$$\begin{aligned}
&|[x^{c(m+d)}](1+x)^m P(x) - \binom{m+d}{c(m+d)} (1-c)^d P(\frac{c}{1-c})| \\
&\leq \binom{m+d}{c(m+d)} L(P) \sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0)| \\
&\leq \binom{m+d}{c(m+d)} L(P) \frac{C_d}{m+1},
\end{aligned}$$

where the last line follows from corollary 7. Hence,

$$\begin{aligned}
[x^{c(m+d)}](1+x)^m P(x) &\geq \binom{m+d}{c(m+d)} \left((1-c)^d P(\frac{c}{1-c}) - L(P)\frac{C_d}{m+1} \right) \\
&\geq \binom{m+d}{c(m+d)} \left(\lambda(P) - L(P)\frac{C_d}{m+1} \right)
\end{aligned}$$

Thus $[x^{c(m+d)}](1+x)^m P(x)$ is positive whenever $m > C_d \frac{L(P)}{\lambda(P)} - 1$. Therefore, by the equivalent condition given in (3), all the coefficients of $(1+x)^m P(x)$ are positive for such m. □

3. Future work

We will run a different method to prove lemma 5 for general d. Let $f_c^{(j)}(m)$ be as in (4). For each positive integer d, we wish to find a constant $C_d > 0$ such that, for each $m \geq 1$ and $0 \leq c \leq \frac{1}{2}$,

$$\sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(\frac{1}{m+d}) - f_c^{(j)}(0)| \leq \frac{C_d}{m+1}, \tag{18}$$

Proof. Since $f_c^{(j)}(x)$ is a rational function whose numerator is a polynomial of degree at most $(d-1)$ and whose denominator is a polynomial of degree $d-1$, by the theory of partial fractions,

$$f_c^{(j)}(x) = \gamma_c^{(j)} + \sum_{r=1}^{d-1} \frac{\alpha_c^{(j)}(r)}{1-rx}, \tag{19}$$

where

$$\alpha_c^{(j)}(r) = \frac{(-1)^{d-r-1}d}{r^2\binom{d}{j}} \binom{cr}{j}\binom{(1-c)r}{d-j}\binom{d-1}{r}. \quad \text{for } r = 1,\dots,d-1. \tag{20}$$

The value of the constant term $\gamma_c^{(j)}$ will not concern us here. Since $f_c^{(j)}(x) - f_c^{(j)}(0) = \sum_{r=1}^{d-1} \alpha_c^{(j)}(r)(\frac{1}{1-rx} - 1) = x\sum_{r=1}^{d-1} \alpha_c^{(j)}(r)(\frac{r}{1-rx})$, hence for $x \geq 0$,

$$|f_c^{(j)}(x) - f_c^{(j)}(0)| \leq x \sum_{r=1}^{d-1} |\alpha_c^{(j)}(r)|(\frac{r}{1-rx})$$

Hence

$$\sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(x) - f_c^{(j)}(0)| \leq c(1-c)x \sum_{r=1}^{d-1} \left|\frac{r}{1-rx}\right| Q_c(r) \quad \text{where } Q_c(r) := \sum_{j=0}^{d} \binom{d}{j} \frac{|\alpha_c^{(j)}(r)|}{c(1-c)}.$$

Since $Q_c(r) \geq 0$ and $\frac{r}{1-rx}$ is increasing in x for $x \leq \frac{1}{d} < \frac{1}{r}$, for $r = 1,\dots,d-1$, hence, for $0 \leq x \leq \frac{1}{d}$,

$$\sum_{j=0}^{d} \binom{d}{j} |f_c^{(j)}(x) - f_c^{(j)}(0)| \leq c(1-c)x \sum_{r=1}^{d-1} \frac{dr}{d-r} Q_c(r). \tag{21}$$

For $d = 5$, it can be shown numerically that, for $0 \leq c \leq \frac{1}{2}$,

$$c(1-c) \sum_{r=1}^{5-1} \frac{5r}{5-r} Q_c(r) < 16.5$$

Hence, together with (21), we can take $C_5 = 16.5$ in (18). □

Degree bounds for Pólya's positvstellensatz has various applications to the complexity of archimedean positivstellensatze. For example, Schweighofer showed that a degree bound for Pólya's positvstellensatz implies a corrosponding degree bound for Schmüdgen's positivstellensatz [5], and Nie-Schweighofer showed that a degree bound for Pólya's positvstellensatz implies a corrosponding degree bound for Putinar's positivstellensatz [6]. There are also applications of degree bounds for Pólya positvstellensatz to estimate the rate of convergence of hierarchy of lower bounds that converge to infimum of fixed degree polynomials on the simplex [1].

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