Singapore International Mathematical Olympiad 2018

National Team Selection Test

Day 0

1. Given a line ℓ outside a circle ω centred at O, draw OP perpendicular to ℓ at P. Let Q be an arbitrary point on ℓ distinct from P. Lines QA and QB touch ω at A, B respectively. Draw PM perpendicular to QA at M and PN perpendicular to QB at N. Let the line MN intersect OP at K. Prove that the position of K does not depend on the choice of Q.

2. Seven cards, numbered $1, 2, 3, 4, 5, 6$ and 7, are initially arranged to obtain the number 7654321. In each move, Sheldon is allowed to remove one card and place it exactly **two** cards away to its left or to its right to obtain a new number.

   What is the smallest number that can be obtained if Sheldon is allowed to perform any number of moves?

3. Consider all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$(x+y)f(x) \le x^2 + f(xy) + a,$$

   where $a \in \mathbb{N}$. Determine the minimum and maximum values of $f(613) + f(2018)$ amongst all such functions.

4. Consider an $a \times b$ grid with its rows numbered 1 to a from top to bottom and its columns numbered 1 to b from left to right. Each cell of the grid is colored black or white. For each row i, the black cells are precisely the rightmost r_i cells of the row; and for each column j, the white cells are precisely the bottommost c_j cells of the column. Consider the $a+b$ numbers, $i + r_i$ for each $1 \le i \le a$ and $j + c_j$ for each $1 \le j \le b$. Show that if $a+b$ is even, then these $a+b$ numbers can be paired into $\frac{a+b}{2}$ disjoint pairs, each with the same sum.

Time allowed: 4 hours

Singapore International Mathematical Olympiad 2018
National Team Selection Test
Day 1

5. Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

   - In the first line, Gugu writes down every number of the form $a - b$, where a and b are two (not necessarily distinct) numbers on his napkin.
   - In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the *first line*.
   - In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the *first line*.

   Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

6. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

7. Find the smallest positive integer n, or show no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \qquad \text{and} \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

   are integers.

Time allowed: 4.5 hours

Singapore International Mathematical Olympiad 2018

National Team Selection Test

Day 2

8. Let O be the circumcentre of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q respectively. The altitudes meet at H. Prove that the circumcentre of triangle PQH lies on a median of the triangle ABC.

9. Determine all integers $n \geq 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

10. An integer $n \geq 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \ldots, y_n$ of these numbers we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \cdots + y_{n-1} y_n \geq -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leq i < j \leq n} x_i x_j \geq K$$

holds for every Shiny n-tuple $(x_1, x_2, \ldots, x_n)$.

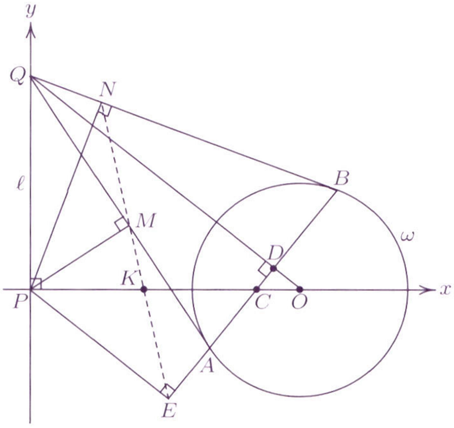
Time allowed: 4.5 hours

Solutions

Day 0

1. Given a line ℓ outside a circle ω centred at O, draw OP perpendicular to ℓ at P. Let Q be an arbitrary point on ℓ distinct from P. Lines QA and QB touch ω at A, B respectively. Draw PM perpendicular to QA at M and PN perpendicular to QB at N. Let the line MN intersect OP at K. Prove that the position of K does not depend on the choice of Q.

Solution 1.



Let OP and OQ intersect AB at C and D respectively. It is easy to see that C, D, Q, P are concyclic and hence $OC \cdot OP = OD \cdot OQ = OA^2$. This implies C does not depend on the choice of Q. In fact C is the inverse point of P with respect to ω. We claim that K is the midpoint of CP.

It is easy to see that A, O, B, Q, P are concyclic. In particular, P lies on the circumcircle of the triangle ABQ. Draw PE perpendicular to AB at E. We must have E, M, N collinear because it is the Simson line of P with respect to the triangle ABQ. Now it suffices to show that $\angle KEC = \angle KCE$. First we have

$$\angle KCE = \angle OCD = \angle PQO = \angle PQN - \tfrac{1}{2}\angle AQB = 90^\circ - \angle QPN - \tfrac{1}{2}\angle AQB.$$

Since M, N, Q, P are concyclic, $\angle QPN = \angle QMN = \angle AME$.

We also have $90^\circ - \frac{1}{2}\angle AQB = 90^\circ - \angle AQO = \angle BAQ$.

Consequently, $\angle KCE = \angle BAQ - \angle AME = \angle KEC$. Hence K is the midpoint of CP which does not depend on the choice of Q.

Solution 2. Let P be the origin, PO the x-axis and ℓ the y-axis. Let $O = (a + r, 0)$, where $a > 0$ and r is the radius of ω, and let $Q = (0, q)$. The equation of ω is

$$(x - a - r)^2 + y^2 = r^2. \qquad (1.1)$$

For any point (x_0, y_0), the equation of its polar with respect to ω is $x_0x + y_0y - (a + r)x + a^2 + 2ar = 0$.

Thus the polar AB of $Q = (0, q)$ has the equation

$$qy - (a + r)x + a^2 + 2ar = 0. \qquad (1.2)$$

Therefore the coordinates of A and B are solutions of (1.1) and (1.2). Substituting (1.2) into (1.1) and simplifying, we have

$$(q^2+(a+r)^2)x^2-2(a+r)(a^2+2ar+q^2)x+(a^2+2ar)(a^2+2ar+q^2)=0. \quad (1.3)$$

Let $A=(x_1,y_1)$ and $B=(x_2,y_2)$. Then x_1,x_2 are the roots of (1.3), and

$$x_1+x_2=\frac{2(a+r)(a^2+2ar+q^2)}{q^2+(a+r)^2}. \quad (1.4)$$

It is easy to see that P lies on the circumcircle of ABQ. Thus M,N,E are collinear and is the Simon line of the point P. Direct calculation gives

$M=(\frac{qx_1(q-y_1)}{a^2+2ar+q^2},\frac{qx_1^2}{a^2+2ar+q^2})$, $N=(\frac{qx_2(q-y_2)}{a^2+2ar+q^2},\frac{qx_2^2}{a^2+2ar+q^2})$, and
$E=(\frac{a(a+r)(a+2r)}{(a+r)^2+q^2},-\frac{qa(a+2r)}{(a+r)^2+q^2})$.

Using (1.2) and (1.4), the slope of the line MN is calculated to be $\frac{2q(a+r)}{q^2-(a+r)^2}$.

Thus the equation of the line PQ is $y+\frac{qa(a+2r)}{(a+r)^2+q^2}=\frac{2q(a+r)}{q^2-(a+r)^2}\left(x-\frac{a(a+r)(a+2r)}{(a+r)^2+q^2}\right)$.

Letting $y=0$ and solving for x, we get $x=\frac{a(a+2r)}{2(a+r)}$. Hence $K=(\frac{a(a+2r)}{2(a+r)},0)$ which is independent of q.

2. Seven cards, numbered $1,2,3,4,5,6$ and 7, are initially arranged to obtain the number 7654321. In each move, Sheldon is allowed to remove one card and place it exactly **two** cards away to its left or to its right to obtain a new number.

   What is the smallest number that can be obtained if Sheldon is allowed to perform any number of moves?

   Solution. The smallest value is 1234576.

   This can be easily obtained, for example, by performing the following 10 moves:

   7654321 → 7654**1**32 → 76**1**5432 → **1**765432 → 1765**2**43 → 17**2**6543 → 1726**3**54 → 17235**6**4 → 1723**4**56 → 123**7**456 → 12345**7**6

   We now prove that it is impossible to obtain the value 1234567. For any integer n, define $f(n)$ to be the number of order pairs (a,b), where $a<b$, found in the integer n. For example, $f(7654321)=0, f(1726543)=10, f(4531267)=13$, $f(1234567)=21$. Note that in each move, the value of $f(n)$ will change by $+2, 0$ or -2. We illustrate this in the table below, where $a<b<c$ represents 3 of the numbers:

original n	new n	change in $f(n)$
...*abc*...	...*bca*...	-2
...*acb*...	...*cba*...	-2
...*bac*...	...*acb*...	0
...*bca*...	...*cab*...	0
...*cab*...	...*abc*...	$+2$
...*cba*...	...*bac*...	$+2$

Note that those pairs not containing 2 of the a, b and c do not affect the value of $f(n)$. Hence the parity of $f(n)$ will remain invariant, which means it is impossible to obtain the value 1234567 since $f(1234567) = 21$ and $f(7654321) = 0$.

3. Consider all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$(x + y)f(x) \leq x^2 + f(xy) + a,$$

where $a \in \mathbb{N}$. Determine the minimum and maximum values of $f(613) + f(2018)$ amongst all such functions.

Solution. Let $s \in \mathbb{N}$. Consider $(x, y) = (s, 1)$. Then we have

$$(s + 1)f(s) \leq s^2 + f(s) + a \Rightarrow f(s) \leq s + \frac{a}{s}.$$

Let $t \in \mathbb{N}$. Consider $(x, y) = (t, 2a)$. Then from the above, we have

$$(t + 2a)f(t) \leq t^2 + f(2at) + a \leq t^2 + 2at + \frac{1}{2t} + a.$$

Rearranging,

$$f(t) \leq t + \frac{1}{2t(t + 2a)} + \frac{a}{t + 2a} < t + \frac{1}{2} + \frac{1}{2} = t + 1.$$

This means that $f(t) \leq t$. Since $f(1) \leq 1$, we must have $f(1) = 1$. Now substitute $(x, y) = (1, t)$ and we have

$$(1 + t) \leq 1 + f(t) + a \Rightarrow t - a \leq f(t).$$

So we have $t - a \leq f(t) \leq t$. Conversely, for any function satisfying $t - a \leq f(t) \leq t$, we have

$$(x + y)f(x) \leq (x + y)x = x^2 + (xy - a) + a \leq x^2 + f(xy) + a$$

and hence all such functions satisfy the condition given. Since

$$\max(613 - a, 1) \leq f(613) \leq 613, \quad \max(2018 - a, 1) \leq f(2018) \leq 2018,$$

the minimum and maximum values of $f(613) + f(2018)$ are $\max(2631 - 2a, 2019 - a, 2)$ and 2631 respectively.

4. Consider an $a \times b$ grid with its rows numbered 1 to a from top to bottom and its columns numbered 1 to b from left to right. Each cell of the grid is colored black or white. For each row i, the black cells are precisely the rightmost r_i cells of the row; and for each column j, the white cells are precisely the bottommost c_j cells of the column. Consider the $a + b$ numbers, $i + r_i$ for each $1 \leq i \leq a$ and $j + c_j$ for each $1 \leq j \leq b$. Show that if $a + b$ is even, then these $a + b$ numbers can be paired into $\frac{a+b}{2}$ disjoint pairs, each with the same sum.

Solution. We will show that we can pair the numbers such that each pair has sum $a + b + 1$. We induct on the number of black cells. When there are no black cells, the $a+b$ numbers under consideration are precisely $1, 2, \ldots, a+b$, and the conclusion is immediate.

Suppose now there are some non-zero number of black cells in the grid G. Consider the black cell(s) with minimal column number j, and among all such cell(s), the cell C with maximal row number i. Note that the cells to the right and above C are all black, while the cells to the left and below C are all white, from which we can deduce that $j+r_i = b+1$ and $i+c_j = a$; and thus, $(i+r_i)+(j+c_j) = a+b+1$ for this pair of values (i, j).

Now, consider the grid G' formed by changing the black cell C to a white cell, and apply the inductive hypothesis to G'. It means that for each s with $1 \le s \le a+b$, there are an equal number of values s and $a+b+1-s$ among the $a+b$ numbers under consideration. Suppose now $(i+r_i)$ in the original grid G has value t, and accordingly, $(j+c_j)$ in G has value $a+b+1-t$. Then $(i+r_i)$ in G' has value $t-1$, and $(j+c_j)$ in G' has value $(a+b+1-t)+1 = a+b+1-(t-1)$, with all other values unchanged. Thus we see that in G also, for each s with $1 \le s \le a+b$, there are an equal number of values s and $a+b+1-s$ among the $a+b$ numbers under consideration, which implies the desired conclusion. Hence, inductively, we are done.

Day 1

5. Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

   - In the first line, Gugu writes down every number of the form $a-b$, where a and b are two (not necessarily distinct) numbers on his napkin.
   - In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the *first line.*
   - In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the *first line.*

   Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

   Solution. The answers are $-2, 0, 2$.

   Call a number q *good* if every number in the second line appears in the third line unconditionally. We first show that the numbers $-2, 0, 2$ are good. The third line necessarily contains 0, so 0 is good. For any two numbers a, b in the first line, write $a = x-y$ and $b = u-v$, where x, y, u, v are (not necessarily distinct) numbers on the napkin. We may write

   $$2ab = 2(x-y)(u-v) = (x-v)^2 + (y-u)^2 - (x-u)^2 - (y-v)^2,$$

   which shows that 2 is good. By negating both sides of the above equation, we also see that -2 is good.

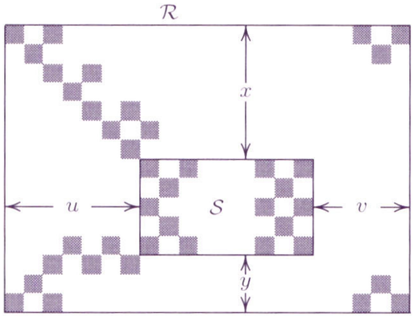
   We now show that $-2, 0, 2$ are the only good numbers. Assume for sake of contradiction that q is a good number, where $q \notin \{-2, 0, 2\}$. We now consider some particular choices of numbers on Gugu's napkin to arrive a contradiction.

Assume that the napkin contains the integers $1, 2, \ldots, 10$. Then, the first line contains the integers $-9, -8, \ldots, 9$. The second line then contains q and $81q$, so the third line must also contain both of them. But the third line only contains integers, so q must be an integer. Furthermore, the third line contains no numbers greater than $162 = 9^2 + 9^2 - 0^2 - 0^2$ or less than -162, so we must have $-162 \leq 81q \leq 162$. This shows that the only possibilities for q are ± 1.

Now assume that $q = \pm 1$. Let the napkin contain $0, 1, 4, 8, 12, 16, 20, 24, 28, 32$. The first line contains ± 1 and ± 4, so the second line contains ± 4. However, for every number a in the first line, $a \not\equiv 2 \pmod 4$, so we may conclude that $a^2 \equiv 0, 1 \pmod 8$. Consequently, every number in the third line must be congruent to $-2, -1, 0, 1, 2 \pmod 8$; in particular, ± 4 cannot be in the third line, which is a contradiction.

6. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Solution. Let the width and height of $\mathcal{R}$ be odd numbers a and b. Divide $\mathcal{R}$ into ab unit squares and color them black and white in a checkered pattern. Since the side lengths of a and b are odd, the corner squares of $\mathcal{R}$ will all have the same color, say black.



Call a rectangle (either $\mathcal{R}$ or a small rectangle) *black* if its corners are all black; call it *white* if the corners are all white, and call it *mixed* if it has both black and white corners. In particular, $\mathcal{R}$ is a black rectangle.

We will use the following trivial observations.

- Every mixed rectangle contains the same number of black and white squares;
- Every black rectangle contains one more black square than white square;
- Every white rectangle contains one more white square than black square.

The rectangle $\mathcal{R}$ is black, so it contains more black unit squares than white unit squares. Therefore, among the small rectangles, at least one is black. Let $\mathcal{S}$ be such a small black rectangle, and let its distances from the sides of $\mathcal{R}$ be x, y, u and v as in the figure. The top-left corner of $\mathcal{R}$ and the top-left corner of $\mathcal{S}$ have the same color, which happen if and only if x and u have the same parity. Similarly, the other three black corners of $\mathcal{S}$ indicate that x and v have the same parity, y and u have the same parity, i.e. x, y, u and v are all odd or all even.

7. Find the smallest positive integer n, or show no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \qquad \text{and} \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

Solution. The smallest such natural n is $n = 3$.

For $n = 2$, let $a_1 = \frac{a}{b}, a_2 = \frac{c}{d}$ where $\gcd(a, b) = \gcd(c, d) = 1$. Then we must have $bd|ad + bc, ac|ad + bc$. Taking mod d in the first condition, we get that $d|bc$ but since c, d are coprime we have that $d|b$. Taking mod b, we get $b|ad$ and since a, b are coprime we get $b|d$ which tells us that $b = d$. Similarly we get $a = c$ and hence we have $a_1 = a_2$. Clearly only $a_1 = a_2 = 1$ or $\frac{1}{2}$ works and we only have finitely many such duplets.

Now for $n = 3$, we will look for triplets of the form $(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c})$ where a, b, c are naturals. Fixing $a = 1$, it suffices to find infinitely many pairs of (b, c) such that $\frac{1}{b} + \frac{1}{c} + \frac{b}{c} + \frac{c}{b}$ is a natural. We shall show that there are infinitely many solutions to $\frac{1}{b} + \frac{1}{c} + \frac{b}{c} + \frac{c}{b} = 3$ by Vieta Jumping. If we let $c \geq b$, the equation rearranges into $b^2 - b(3c - 1) + c^2 + c = 0$. Viewing it as a quadratic in b, we get that there exists another b' satisfying $\frac{1}{b'} + \frac{1}{c} + \frac{b'}{c} + \frac{c}{b'} = 3$ and that this b' satisfies $b + b' = 3c - 1, bb' = c^2 + c$. Since b is a natural, we get that $b' = 3c - 1 - b$ is also a natural. Furthermore, $b' = \frac{c^2+c}{b} \geq \frac{c^2+c}{c} > c$. Hence our triplet $(1, b, c)$ will transform into another triplet $(1, c, b')$ where $b' > c$ and we can then jump again but this time with c as the subject of the quadratic. Starting with $b = 3, c = 6$, this algorithm will generate infinitely many distinct triplets of desired naturals (a, b, c) which will then give us infinitely many distinct triplets of rationals satisfying the problem conditions.

An explicit recursion can also be given in the following form. Define the sequence (x_n) as follows: $x_1 = 3, x_2 = 6$ and $x_{n+2} = 3x_{n+1} - x_n - 1$ for $n \geq 2$. Then the triplet

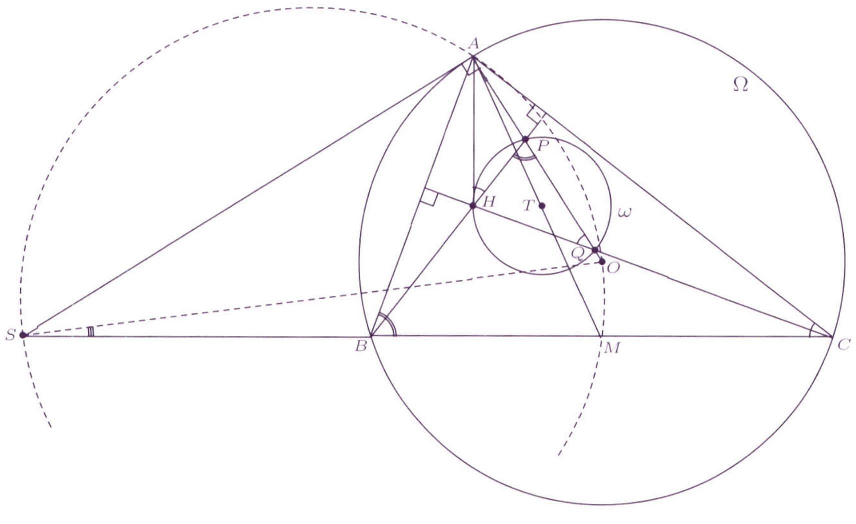
$$\left(\frac{1}{1 + x_m + x_{m+1}}, \frac{x_m}{1 + x_m + x_{m+1}}, \frac{x_{m+1}}{1 + x_m + x_{m+1}}\right)$$

satisfies the problem conditions for all natural m. One can also express x_n in terms of the Fibonacci sequence, $F_1 = 1, F_2 = 2, F_{n+2} = F_{n+1} + F_n$, as $x_n = F_{2n} + 1$.

Day 2

8. Let O be the circumcentre of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q respectively. The altitudes meet at H. Prove that the circumcentre of triangle PQH lies on a median of the triangle ABC.

Solution 1. Suppose without loss of generality that $AB < AC$. We have $\angle PQH = 90° - \angle QAB = 90° - \angle OAB = \frac{1}{2}\angle AOB = \angle ACB$, and similarly $\angle QPH = \angle ABC$. Thus triangles ABC and HPQ are similar. Let Ω and ω be the circumcircles of ABC and HPQ, respectively. Since $\angle AHP = 90° - \angle HAC = \angle ACB = \angle HQP$, line AH is tangent to ω.



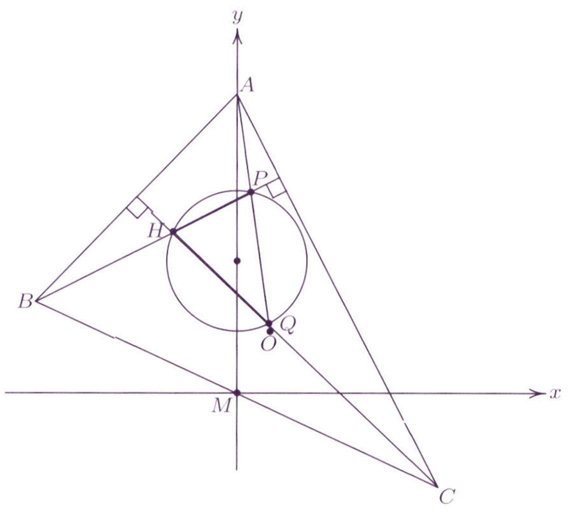
Let T be the centre of ω and let lines AT and BC meet at M. We will take advantage of the similarity between ABC and HPQ and the fact that AH is tangent to ω at H, with A on the line PQ. Consider the corresponding tangent AS to Ω, with S on BC. Then S and A correspond to each other in $\triangle ABC \sim \triangle HPQ$, and therefore $\angle OSM = \angle OAT = \angle OAM$. Hence quadrilateral $SAOM$ is cyclic, and since the tangent line AS is perpendicular to AO, $\angle OMS = 180° - \angle OAS = 90°$. This means that M is the orthogonal projection of O onto BC, which is its midpoint. So T lies on the median AM of triangle ABC.

Remark. Take ABC as the reference triangle and use barycentric coordinates. We have $P = (c^2S_C^2 : b^2S_AS_B : c^2S_CS_A)$ and $Q = (b^2S_B^2 : b^2S_AS_B : c^2S_CS_A)$. The equation of the circumcircle ω of the triangle HPQ is found to be

$$a^2yz + b^2zx + c^2xy - (x+y+z)\left(\frac{a^2S_A^2}{S^2}x + \frac{c^2S_BS_C}{S^2}y + \frac{b^2S_BS_C}{S^2}z\right) = 0,$$

where S is twice the area of the triangle ABC. The centre of ω is found to be $(2S_BS_C : a^2S_A : a^2S_A)$, which lies on the median AM.

Solution 2.



In a coordinate system, let $A = (0, a)$, $B = (b, c)$ and $C = (-b, -c)$. Then the midpoint of BC is the origin and the y-axis is the median through A. Also the slope of BC is c/b, the slope of AB is $(c-a)/b$, the slope of AC is $(c+a)/b$, the slope of BH is $-b(a+c)$, and the slope of CH is $b/(a-c)$. Direct calculations give

$H = (-\frac{c(b^2+c^2-a^2)}{ab}, \frac{b^2+c^2}{a})$, and $O = (\frac{c(b^2+c^2-a^2)}{2ab}, -\frac{b^2+c^2-a^2}{2a})$.

The equation of the line AO is $y = -\frac{b(a^2+b^2+c^2)}{c(b^2+c^2-a^2)}x + a$, and the equation of the line BP is $y = -\frac{bx}{a+c} + \frac{b^2+ca+c^2}{a+c}$. Solving these two equations, we get $P = (-\frac{c(b^2+c^2-a^2)^2}{ab((a+c)^2+b^2)}, \frac{b^2+c^2}{a} - \frac{2c(b^2+c^2-a^2)}{(a+c)^2+b^2})$. Thus, the midpoint of HP is $\frac{1}{2}(H+P) = (-\frac{c(b^2+c^2-a^2)}{ab((a+c)^2+b^2)}, \frac{b^2+c^2}{a} - \frac{c(b^2+c^2-a^2)}{(a+c)^2+b^2})$. Then the equation of the perpendicular bisector of HP is

$$y - \frac{b^2+c^2}{a} + \frac{c(b^2+c^2-a^2)}{(a+c)^2+b^2} = \left(x + \frac{c(b^2+c^2-a^2)}{ab((a+c)^2+b^2)}\right) \cdot \frac{a+c}{b}.$$

It follows that the intersection of the perpendicular bisector of HP with the y-axis is the point $(0, \frac{(b^2+c^2)^2-a^2c^2}{ab^2})$.

The computation of the coordinates of the point Q and the intersection of the perpendicular bisector of HQ with the y-axis can easily be done by changing b and c to $-b$ and $-c$. Since the expression $\frac{(b^2+c^2)^2-a^2c^2}{ab^2}$ involves only the squares of b and c, it will be the same expression for the intersection of the perpendicular bisector of HQ with the y-axis. In other words, the circumcentre of the triangle HPQ lies on the y-axis which is the median through A.

9. Determine all integers $n \geq 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

Solution. These integers are exactly the prime numbers.

Let us first show that, if $n = ab$, with $a, b \geq 2$, then the property in the statement of the problem does not hold. Indeed, let $a_k = a$ for $1 \leq k \leq n-1$ and $a_n = 0$. The sum $a_1 + a_2 + \cdots + a_n = a(n-1)$ is not divisible by n. Let i with $1 \leq i \leq n$ be an arbitrary index. Taking $j = b$ if $1 \leq i \leq n-b$, and $j = b+1$ if $n-b < i \leq n$, we have

$$a_i + a_{i+1} + \cdots + a_{i+j-1} = ab = n \equiv 0 \pmod{n}.$$

It follows that the given example is indeed a counterexample to the property of the statement.

Now let n be a prime number. Suppose by contradiction that the property in the statement of the problem does not hold. Then there are integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n such that for each i, $1 \leq i \leq n$, for which the number $a_i + a_{i+1} + \cdots + a_{i+j-1}$ is divisible by n. Notice that, in any such case, we should have $1 \leq j \leq n-1$, since $a_1 + a_2 + \cdots + a_n$ is not divisible by n. So we may construct recursively a finite sequence of integers $0 = i_0 < i_1 < i_2 < \cdots < i_n$ with $i_{s+1} - i_s \le n-1$ for $0 \le s \le n-1$ such that, for $0 \le s \le n-1$,

$$a_{i_s+1} + a_{i_s+2} + \cdots + a_{i_{s+1}} \equiv 0 \pmod{n},$$

where we take indices modulo n. Indeed, for $0 \le s < n$, we apply the previous observation to $i = i_s + 1$ in order to define $i_{s+1} = i_s + j$.

In the sequence of $n+1$ indices $i_0, i_1, i_2, \ldots, i_n$, by pigeonhole principle, we have two distinct elements which are congruent modulo n. So there are indices r, s with $0 \le r < s \le n$ such that $i_s \equiv i_r \pmod{n}$ and

$$a_{i_r+1} + a_{i_r+2} + \cdots + a_{i_s} = \sum_{j=r}^{s-1}(a_{i_j+1} + a_{i_j+2} + \cdots + a_{i_{j+1}}) \equiv 0 \pmod{n}.$$

Since $i_s \equiv i_r \pmod{n}$, we have $i_s - i_r = kn$ for some positive integer k, and since $i_{j+1} - i_j \le n-1$ for $0 \le j \le n-1$, we have $i_s - i_r \le (n-1)n$, so $k \le n-1$. But in this case

$$a_{i_r+1} + a_{i_r+2} + \cdots + a_{i_s} = k(a_1 + a_2 + \cdots + a_n)$$

cannot be a multiple of n, since n is a prime and neither k nor $a_1 + a_2 + \cdots + a_n$ is a multiple of n, a contradiction.

10. An integer $n \ge 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \ldots, y_n$ of these numbers we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1y_2 + y_2y_3 + y_3y_4 + \cdots + y_{n-1}y_n \ge -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \le i < j \le n} x_i x_j \ge K$$

holds for every Shiny n-tuple $(x_1, x_2, \ldots, x_n)$.

Solution. The answer is $K = -(n-1)/2$.

First of all, we show that we may not take a larger constant K. Let t be a positive number, and take $x_2 = x_3 = \cdots = t$ and $x_1 = -1/(2t)$. Then, every product $x_i x_j$ $(i \ne j)$ is equal to either t^2 or $-1/2$. Hence, for every permutation y_i of x_i, we have

$$y_1y_2 + y_2y_3 + y_3y_4 + \cdots + y_{n-1}y_n \ge (n-3)t^2 - 1 \ge -1.$$

This justifies that the n-tuple $(x_1, x_2, \ldots, x_n)$ is Shiny. Now, we have

$$\sum_{i<j} x_i x_j = -\frac{n-1}{2} + \frac{(n-1)(n-2)}{2}t^2.$$

Thus, as t approaches 0 from above, $\sum_{i<j} x_i x_j$ gets arbitrarily close to $-(n-1)/2$. This shows that we may not take K any larger than $-(n-1)/2$. It remains to show that $\sum_{i<j} x_i x_j \ge -\frac{n-1}{2}$ for any Shiny choice of the x_i.

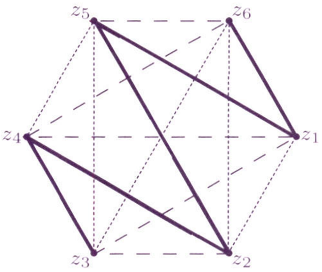


Figure 6.1: For $n = 6$, take 2 Hamiltonian paths and $L = z_3z_4 + z_4z_2 + z_2z_5 + z_5z_1 + z_1z_6$ is another Hamiltonian path.

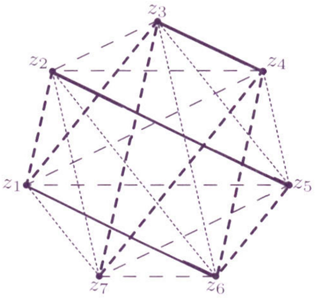


Figure 6.2: For $n = 7$, there are 3 Hamiltonian paths and $L = z_1z_6 + z_2z_5 + z_3z_4$ forms a matching.

From now on, assume that $(x_1, x_2, \ldots, x_n)$ is a Shiny n-tuple. Without loss of generality, we may assume that $x_1 \le x_2 \le \cdots \le x_k < 0 \le x_{k+1} \le \cdots \le x_n$. It is also easy to see that the negative n-tuple $(-x_1, \ldots, -x_n)$ is also shiny.

Consider the complete graph K_n with vertices labelled as $x_1, \ldots, x_n$ and with each edge x_ix_j identified with the product x_ix_j. Each permutation $(z_1, z_2, \ldots, z_n)$ of the n-tuple corresponds to a Hamiltonian path and the sum $z_1z_2 + \cdots + z_{n-1}z_n$ is the sum of the edges of the path. Thus the sum of the edges of each Hamiltonian path is ≥ -1 and the sum $\sum x_ix_j$ is equal to the sum of all the edges.

We first consider the case where n is odd. It is well known that the graph K_n can be partitioned into $\frac{n-1}{2}$ Hamiltonian paths and a set of $\frac{n-1}{2}$ independent edges. We may assume that k is even, otherwise consider the negative sequence. Label the vertices so that the independent edges are $x_1x_2, x_3x_4, \ldots, x_{n-2}x_{n-1}$. Then the sum of the edges is equal to the sum of the edges in the Hamiltonian paths plus the sum of the independent edges and is therefore $\ge -\frac{n-1}{2} + 0$ as desired.

Now consider the case where n is even. It is well known that K_n can be partitioned into $n/2$ Hamiltonian paths. Label the vertices so that $x_1x_2 \ldots x_n$ is one of the paths. It suffices to show that $L = x_1x_2 + \ldots + x_{n-1}x_n \ge -\frac{1}{2}$. Without loss of generality, we may assume that $n > k + 1$. Also note that x_kx_{k+1} is the only nonpositive summand in L. Suppose on the contrary that $L < -1/2$, then $x_kx_{k+1} < -1/2$. Then since $x_n \ge x_{k+1} \ge 0$ and $x_1 \le x_k < 0$, $x_nx_1 \le x_kx_{k+1} < -1/2$. (Note that k can be equal to 1.) Thus for the permutation $(x_n, x_1, x_2, \ldots, x_{n-1})$, the corresponding sum

$$x_nx_1 + x_1x_2 + \cdots + x_{n-2}x_{n-1} = L + (x_nx_1 - x_{n-1}x_n) < -1/2 + x_nx_1 < -1,$$

a contradiction.