A Refinement of Young's Inequality

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Abstract

For each positive integer $n \geqslant 2$, we prove the following result:

$$\sum_{k=1}^{n} \frac{\cos kx}{k} > \sum_{k=1}^{2\lfloor \frac{n}{2} \rfloor+1} \frac{(-1)^k}{k} \geqslant -\frac{5}{6} \quad (n = 2, 3, 4, \ldots; 0 < x < \pi),$$

which sharpens a previous result due to Brown and Koumandos. The lower bound $\sum_{k=1}^{2\lfloor \frac{n}{2} \rfloor+1} \frac{(-1)^k}{k}$ is sharp.

1 Introduction

In 1913, Young [3] proved that

$$1 + \sum_{k=1}^{n} \frac{\cos kx}{k} > 0 \ (n \in \mathbb{N}; 0 < x < \pi). \tag{1}$$

Since then, many mathematicians have been studying such trigonometric inequalities. For a survey of these results, please refer to the paper by Koumandos [2]. In particular, Brown and Koumandos [1] proved the following result:

$$\frac{5}{6} + \sum_{k=1}^{n} \frac{\cos kx}{k} > 0 \ (n = 2, 3, 4, \ldots; 0 < x < \pi). \tag{2}$$

However, since Brown and Koumandos devised a new method to establish (2), it is natural to ask whether Young's approach can be used to generalise (2).

In this report, we set

$$C_n(x) = \sum_{r=1}^{n} \frac{\cos kx}{k} \quad (n \in \mathbb{N}; x \in [0, \pi]) \tag{3}$$

and prove the following result.

Theorem 1.1.

$$C_n(x) > \sum_{k=1}^{2\lfloor \frac{n}{2} \rfloor+1} \frac{(-1)^k}{k} \geqslant -\frac{5}{6} \ (n = 2, 3, 4, \ldots; 0 < x < \pi),$$

where the lower bound $\sum_{k=1}^{2\lfloor \frac{n}{2} \rfloor+1} \frac{(-1)^k}{k}$ is sharp.

This report is structured as follows. In Section 2, we modify Young's approach to prove that

$$\min_{x\in[0,\pi]} C_{2N-1}(x) = C_{2N-1}(\pi) \text{ and } \min_{x\in[0,\pi]} C_{2N}(x) = C_{2N}\left(\frac{2N\pi}{2N+1}\right). \tag{4}$$

Then in Section 3, we use the sequence of absolute minimum values to extend the above result of Brown and Koumandos.

2 A new proof of Young's result concerning $\min_{x\in[0,\pi]} C_n(x)$

The goal of this section is to give an alternative proof of the following result due to Young [3].

Theorem 2.1. *If $N \in \mathbb{N}$, then*

$$\min_{x\in[0,\pi]} C_{2N-1}(x) = C_{2N-1}(\pi) \text{ and } \min_{x\in[0,\pi]} C_{2N}(x) = C_{2N}\left(\frac{2N\pi}{2N+1}\right).$$

We need some lemmata.

Lemma 2.2. *If $n \in \mathbb{N}$ and if $x \in (0,\pi)$, then*

$$\frac{d}{dx}\left(C_n(x)\right) = -\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}\csc\frac{x}{2}.$$

Proof. For each $x \in (0,\pi)$, we have

$$\begin{aligned}
\frac{d}{dx}\left(C_n(x)\right) &= -\sum_{r=1}^{n}\sin(rx) \\
&= -\frac{1}{2\sin\frac{x}{2}}\sum_{r=1}^{n} 2\sin(rx)\sin\frac{x}{2} \\
&= -\frac{1}{2}\csc\frac{x}{2}\sum_{r=1}^{n}\left(\cos\left(r-\frac{1}{2}\right)x - \cos\left(r+\frac{1}{2}\right)x\right) \\
&= -\frac{1}{2}\csc\frac{x}{2}\left(\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x\right) \\
&= -\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}\csc\frac{x}{2}.
\end{aligned}$$

□

Lemma 2.3. *Let N be a positive integer greater than 2. If $\sin\frac{Nx}{2} = 0$ and if $x \in (0,\pi)$, then*

$$x = \frac{2k\pi}{N} \text{ for some } k \in \left\{1, ..., \left\lceil\frac{N}{2}\right\rceil - 1\right\}.$$

Proof. Since $0 < \frac{Nx}{2} < \frac{N\pi}{2}$ for $0 < x < \pi$, we see that

$$\sin\frac{Nx}{2} = 0 \text{ and } x \in (0,\pi) \implies \frac{Nx}{2} = k\pi \text{ for } k \in \left\{1, ..., \left\lceil\frac{N}{2}\right\rceil - 1\right\}.$$

The result follows.

□

The following two lemmata follow from Lemma 2.3.

Lemma 2.4. If n is any integer greater than 1, then

$$\frac{d}{dx}\left(C_n(x)\right)\Big|_{x=\frac{2k\pi}{n+1}} = 0 \text{ and } \frac{d^2}{dx^2}\left(C_n(x)\right)\Big|_{x=\frac{2k\pi}{n+1}} > 0 \text{ for } k = 1, ..., \left\lceil \frac{n+1}{2} \right\rceil - 1.$$

Proof. For each $k = 1, \ldots, \lceil \frac{n+1}{2} \rceil - 1$, we have

$$\begin{aligned}
\frac{d}{dx}\left(C_n(x)\right)\Big|_{x=\frac{2k\pi}{n+1}} &= \left(-\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}\csc\frac{x}{2}\right)\Big|_{x=\frac{2k\pi}{n+1}} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2}{dx^2}\left(C_n(x)\right)\Big|_{x=\frac{2k\pi}{n+1}} &= \Bigg[\left(-\frac{(n+1)}{2}\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}\csc\frac{x}{2}\right) \\
&\quad - \left(\frac{n}{2}\sin\frac{(n+1)x}{2}\cos\frac{nx}{2}\csc\frac{x}{2}\right) \\
&\quad + \left(\frac{1}{2}\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}\csc\frac{x}{2}\cot\frac{x}{2}\right)\Bigg]\Bigg|_{x=\frac{2k\pi}{n+1}} \\
&= -\frac{(n+1)}{2}\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}\csc\frac{x}{2}\Big|_{x=\frac{2k\pi}{n+1}}. \\
&= -\frac{(n+1)}{2}\cos k\pi \sin\left(k\pi - \frac{k\pi}{n+1}\right)\csc\frac{k\pi}{n+1} \\
&= \frac{(n+1)}{2} \\
&> 0.
\end{aligned}$$

□

Lemma 2.5. *If n is any integer greater than 2, then*

$$\frac{d}{dx}\left(C_n(x)\right)\Big|_{x=\frac{2k\pi}{n}} = 0 \text{ and } \frac{d^2}{dx^2}\left(C_n(x)\right)\Big|_{x=\frac{2k\pi}{n}} < 0 \text{ for } k = 1, ..., \left\lceil \frac{n}{2} \right\rceil - 1.$$

Proof. Following the proof of Lemma 2.4, for each $k = 1, \ldots, \lceil \frac{n}{2} \rceil - 1$, we have

$$\frac{d}{dx}\left(C_n(x)\right)\Big|_{x=\frac{2k\pi}{n}} = 0$$

and

$$\begin{aligned}
\frac{d^2}{dx^2}\left(C_n(x)\right)\Big|_{x=\frac{2k\pi}{n}} &= \left(-\frac{n}{2}\sin\frac{(n+1)x}{2}\cos\frac{nx}{2}\csc\frac{x}{2}\right)\Big|_{x=\frac{2k\pi}{n}} \\
&= -\frac{n}{2}.
\end{aligned}$$

□

For each integer $n \geqslant 3$, it follows from Lemmata 2.4, 2.5 and the Second Derivative Test that

(i) the graph of C_n has relative minimum points

$$\left(\frac{2k\pi}{n+1}, C_n\left(\frac{2k\pi}{n+1}\right)\right), \text{ where } k = 1, \ldots, \left\lceil\frac{n+1}{2}\right\rceil - 1,$$

(ii) the graph of C_n has relative maximum points

$$\left(\frac{2k\pi}{n}, C_n\left(\frac{2k\pi}{n}\right)\right), \text{ where } k = 1, \ldots, \left\lceil\frac{n}{2}\right\rceil - 1.$$

For each positive integer $n \geqslant 2$ we next show that the sequence of relative minimum values of C_n is decreasing.

Lemma 2.6. *If p and q are integers such that $\lfloor \frac{n}{2} \rfloor \geqslant p > q \geqslant 1$, then*

$$C_n\left(\frac{2q\pi}{n+1}\right) > C_n\left(\frac{2p\pi}{n+1}\right).$$

Proof. Using Lemma 2.2 we obtain

$$\begin{aligned}
& C_n\left(\frac{2p\pi}{n+1}\right) - C_n\left(\frac{2q\pi}{n+1}\right) \\
= & \int_{\frac{2q\pi}{n+1}}^{\frac{2p\pi}{n+1}} \frac{d}{dx}\left(C_n(x)\right)\, dx \\
= & -\int_{\frac{2q\pi}{n+1}}^{\frac{2p\pi}{n+1}} \csc\frac{x}{2} \sin\frac{(n+1)x}{2} \sin\frac{nx}{2}\, dx \\
= & -\int_{\frac{2q\pi}{n+1}}^{\frac{2p\pi}{n+1}} \csc\frac{x}{2} \sin\frac{(n+1)x}{2} \left(\sin\frac{(n+1)x}{2}\cos\frac{x}{2} - \cos\frac{(n+1)x}{2}\sin\frac{x}{2}\right) dx \\
= & -\int_{\frac{2q\pi}{n+1}}^{\frac{2p\pi}{n+1}} \cot\frac{x}{2} \sin^2\frac{(n+1)x}{2}\, dx + \int_{\frac{2p\pi}{n+1}}^{\frac{2q\pi}{n+1}} \sin\frac{(n+1)x}{2} \cos\frac{(n+1)x}{2}\, dx \\
= & -\int_{\frac{2q\pi}{n+1}}^{\frac{2p\pi}{n+1}} \cot\frac{x}{2} \sin^2\frac{(n+1)x}{2}\, dx + 0 \\
< & \ 0 \text{ because } \frac{x}{2} \in \left[\frac{q\pi}{n+1}, \frac{p\pi}{n+1}\right] \subset \left(0, \frac{\lfloor\frac{n}{2}\rfloor\pi}{n+1}\right] \subset \left(0, \frac{\pi}{2}\right).
\end{aligned}$$

☐

When n is an odd positive integer, we have the following result.

Theorem 2.7. *If $N \in \mathbb{N}$, then*

$$\min_{x\in[0,\pi]} C_{2N-1}(x) = C_{2N-1}(\pi).$$

Proof. Since $\min_{x\in[0,\pi]} C_1(x) = C_1(\pi)$ and Lemma 2.5 implies

$$\min_{x\in\left[0,\frac{N\pi}{N+1}\right]} C_{2N-1}(x) = C_{2N-1}\left(\frac{N\pi}{N+1}\right) \text{ for } N = 2, 3, \ldots,$$

we need to show that

$$C_{2N-1}\left(\frac{N\pi}{N+1}\right) > C_{2N-1}(\pi) \text{ for } N = 2, 3, \ldots.$$

Following the proof of Theorem 2.5,

$$\begin{aligned}
& C_{2N-1}(\pi) - C_{2N-1}\left(\frac{N\pi}{N+1}\right) \\
= & \int_{\frac{N\pi}{N+1}}^{\pi} \frac{d}{dx}\left(C_{2N-1}(x)\right)\, dx \\
= & -\int_{\frac{N\pi}{N+1}}^{\pi} \sin Nx \sin\left(\left(N-\frac{1}{2}\right)x\right)\csc\frac{x}{2}\, dx \\
= & -\int_{\frac{N\pi}{N+1}}^{\pi} \sin^2 Nx \cot\frac{x}{2}\, dx + \int_{\frac{N\pi}{N+1}}^{\pi} \sin Nx \cos Nx\, dx \\
= & -\int_{\frac{N\pi}{N+1}}^{\pi} \sin^2 Nx \cot\frac{x}{2}\, dx + 0 \\
< & \ 0.
\end{aligned}$$

☐

For the case when n is even, the proof of the previous theorem does not work. Since Young [3] stated the following result without proof, we give a justification of the result.

Theorem 2.8. *If $N \in \mathbb{N}$, then*

$$\min_{x\in[0,\pi]} C_{2N}(x) = C_{2N}\left(\frac{2N}{2N+1}\pi\right).$$

Proof. Since Lemma 2.5 yields

$$\min_{x\in[0,\frac{2N\pi}{2N+1}]} C_{2N}(x) = C_{2N}\left(\frac{2N\pi}{2N+1}\right),$$

it is sufficient to check that

$$C_{2N}(\pi) > C_{2N}\left(\frac{2N}{2N+1}\pi\right).$$

Since Lemma 2.2 implies that

$$\frac{d}{dx}\left(C_{2N}(x)\right) = -\sin\frac{(2N+1)x}{2}\sin Nx \csc\frac{x}{2} \quad (0<x<\pi),$$

it remains to check that

$$\sin\frac{(2N+1)x}{2}\sin Nx < 0 \text{ for all } x \in \left(\frac{2N}{2N+1}\pi\ ,\pi\right).$$

Since $x \in \left(\frac{2N\pi}{2N+1},\pi\right)$, we have

$$N\pi < \frac{2N+1}{2}x < \frac{2N+1}{2}\pi = \left(N+\frac{1}{2}\right)\pi \tag{5}$$

and

$$\left(N-\frac{1}{2}\right)\pi < \frac{2N^2\pi}{2N+1} < Nx < N\pi. \tag{6}$$

If N is even and if the angle θ is in the open interval $\left(\left(N-\frac{1}{2}\right)\pi, N\pi\right)$, then it lies in the fourth quadrant. Likewise, if θ is in the open interval $\left(N\pi, \left(N+\frac{1}{2}\right)\pi\right)$, then it lies the first quadrant. On the other hand, if N is odd and if the angle θ is in the open interval $\left(\left(N-\frac{1}{2}\right)\pi, N\pi\right)$, then it lies in the second quadrant. Likewise, if θ is in the open interval $\left(N\pi, \left(N+\frac{1}{2}\right)\pi\right)$, then it lies in the third quadrant. Combining both cases, we have

$$\sin\frac{(2N+1)x}{2}\sin Nx < 0 \text{ for all } x \in \left(\frac{2N}{2N+1}\pi\ , \pi\right).$$

☐

3 An extension of a result of Brown and Koumandos

According to to Theorems 2.7 and 2.8, we have

$$\min_{x\in[0,\pi]} C_n(x) = \begin{cases} C_{2N-1}(\pi) & \text{if } n = 2N-1 \text{ for some } N \in \mathbb{N}, \\ C_{2N}\left(\frac{2N}{2N+1}\pi\right) & \text{if } n = 2N \text{ for some } N \in \mathbb{N}. \end{cases}$$

Since direct computations show that

$$\min_{x\in[0,\pi]} C_{2n}(x) > \min_{x\in[0,\pi]} C_{2n+1}(x) \text{ for } N = 1,2,3,4,5,$$

we conjecture that

$$\min_{x\in[0,\pi]} C_{Nr}(x) > \min_{x\in[0,\pi]} C_{2N+1}(x) \text{ for all } N \in \mathbb{N}. \tag{7}$$

The first goal of this section is to establish the conjecture (7).

Lemma 3.1. *If $N \in \mathbb{N}$, then*

$$C_{2N}\left(\frac{2N\pi}{2N+1}\right) > C_{2N+1}(\pi).$$

Proof. For each $N \in \mathbb{N}$, we have

$$\begin{aligned} C_{2N}\left(\frac{2N\pi}{2N+1}\right) - C_{2N+1}(\pi) &= \sum_{k=1}^{2N}\frac{\cos k\left(\frac{2N}{2N+1}\pi\right)}{k} - \sum_{k=1}^{2N+1}\frac{\cos k\pi}{k} \\ &= \sum_{k=1}^{2N}\frac{\cos k\left(\frac{2N}{2N+1}\pi\right)}{k} - \sum_{k=1}^{2N}\frac{\cos k\pi}{k} + \frac{1}{2N+1}. \end{aligned}$$

Hence, the desired inequality can be reformulated as follows:

$$\sum_{k=1}^{2N}\frac{\cos k\pi}{k} - \sum_{k=1}^{2N}\frac{\cos k\left(\frac{2N}{2N+1}\pi\right)}{k} < \frac{1}{2N+1}. \tag{8}$$

In order to prove (8), we use Lemma 2.2 to rewrite the left-hand side of the inequality as follows:

$$\sum_{k=1}^{2N}\frac{\cos k\pi}{k} - \sum_{k=1}^{2N}\frac{\cos k\left(\frac{2N}{2N+1}\pi\right)}{k} = -\int_{\frac{2N\pi}{2N+1}}^{\pi}\sin\frac{(2N+1)x}{2}\csc\frac{x}{2}\sin Nx\ dx. \tag{9}$$

Using (8) and (9), it remains to prove that

$$\int_{\frac{2N\pi}{2N+1}}^{\pi} \sin\frac{(2N+1)x}{2}\csc\frac{x}{2}\sin Nx\,dx+\frac{1}{2N+1}>0. \tag{10}$$

Consider the definite integral appearing in (10). Since

$$\max_{x\in\left[\frac{N\pi}{2N+1},\pi\right]}\csc\frac{x}{2}=\csc\frac{N\pi}{2N+1}>0 \text{ and } \sin\left(N+\frac{1}{2}\right)x\sin Nx<0 \text{ for } x\in\left(\frac{2N\pi}{2N+1},\pi\right) \text{ (cf. proof of Theorem 2.8),}$$

we deduce that

$$\int_{\frac{2N\pi}{2N+1}}^{\pi} \sin\frac{(2N+1)x}{2}\csc\frac{x}{2}\sin Nx\,dx>\left(\csc\left(\frac{N\pi}{2N+1}\right)\right)\int_{\frac{2N\pi}{2N+1}}^{\pi} \sin\frac{(2N+1)x}{2}\sin Nx\,dx. \tag{11}$$

Now we evaluate the definite integral on the right-hand side of (11). Using the factor formula

$$-2\sin a\sin b=\cos(a+b)-\cos(a-b),$$

we have

$$\begin{aligned}
&\int_{\frac{2N\pi}{2N+1}}^{\pi} \sin\frac{(2N+1)x}{2}\sin Nx\,dx\\
=\ &\frac{1}{2}\int_{\frac{2N\pi}{2N+1}}^{\pi}\left(\cos\frac{x}{2}-\cos\frac{(4N+1)x}{2}\right)dx\\
=\ &\sin\frac{\pi}{2}-\frac{1}{4N+1}\sin\left((4N+1)\frac{\pi}{2}\right)+\frac{1}{4N+1}\sin\left((4N+1)\frac{N\pi}{2N+1}\right)-\sin\left(\frac{N\pi}{2N+1}\right)\\
=\ &1-\frac{1}{4N+1}+\frac{1}{4N+1}\sin\left(2N-\frac{\pi}{2}+\frac{\pi}{2(2N+1)}\right)-\sin\left(\frac{N\pi}{2N+1}\right)\\
=\ &1-\frac{1}{4N+1}-\frac{1}{4N+1}\cos\frac{\pi}{2(2N+1)}-\cos\frac{\pi}{2(2N+1)}.
\end{aligned} \tag{12}$$

Finally, we use (11), (12) and the inequality $\cos\theta<1-\frac{\theta^2}{2}+\frac{\theta^4}{4}$ to conclude that

$$\begin{aligned}
&\int_{\frac{2N\pi}{2N+1}}^{\pi} \sin\frac{(2N+1)x}{2}\csc\frac{x}{2}\sin Nx\,dx+\frac{1}{2N+1}\\
>\ &\left(1-\frac{1}{4N+1}\right)\sec\frac{\pi}{2(2N+1)}-\left(1-\frac{1}{4N+1}\right)+\frac{1}{2N+1}\\
>\ &\left(1-\frac{1}{4N+1}\right)\left(\frac{1}{1-\frac{\pi^2}{8(2N+1)^2}+\frac{\pi^4}{384(2N+1)^4}}\right)-\left(1-\frac{1}{4N+1}\right)+\frac{1}{2N+1}\\
>\ &\left(1-\frac{1}{4N+1}\right)\left(1+\frac{\pi^2}{8(2N+1)^2}-\frac{\pi^4}{384(2N+1)^4}\right)-\left(1-\frac{1}{4N+1}\right)+\frac{1}{2N+1}\\
=\ &\left(1-\frac{1}{4N+1}\right)\left(\frac{\pi^2}{8(2N+1)^2}-\frac{\pi^4}{384(2N+1)^4}\right)+\frac{1}{2N+1}\\
>\ &0.
\end{aligned}$$

□

We are now ready to prove the main result of this report.

Theorem 3.2. *If $x\in(0,\pi)$ and if $n\in\mathbb{N}\backslash\{1\}$, then*

$$\sum_{k=1}^{n}\frac{\cos kx}{k}>\sum_{k=1}^{2\lfloor\frac{n}{2}\rfloor+1}\frac{(-1)^k}{k}.$$

Proof. Let n be any positive integer such that $n \geqslant 2$. When n is odd, the result follows from Theorem 2.6. On the other hand, suppose n is a positive even integer. In this case, there exists a positive integer N such that $n = 2N$; moreover, Theorem 2.8 and Lemma 3.1 yield

$$\sum_{k=1}^{2N} \frac{\cos kx}{k} \geqslant C_{2N}\left(\frac{2N\pi}{2N+1}\right) > C_{2N+1}(\pi).$$

The proof is complete. □

As an application of our main result, we sharpen a result due to Brown and Koumandos [1].

Theorem 3.3. *Let $n \in \mathbb{N}\backslash\{1\}$. If $x \in (0, \pi)$, then*

$$\sum_{k=1}^{n} \frac{\cos kx}{k} > \sum_{k=1}^{2\lfloor \frac{n}{2} \rfloor +1} \frac{(-1)^k}{k} \geqslant \sum_{k=1}^{3} \frac{(-1)^k}{k} = -\frac{5}{6},$$

where the lower bound $\sum_{k=1}^{2\lfloor \frac{n}{2} \rfloor +1} \frac{(-1)^k}{k}$ is sharp.

4 Conclusion

We have combined the classical approach of Young with calculus to extend a result due to Brown and Koumandos [?], and the lower bound is sharp.

References

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