Sum of Cubes is Equal to Square of Sum

Andrew Yapp Wei Rong, Gnoh Cheng Yi

Abstract

It is a well known fact that the sum of the cubes of the first n consecutive integers is equal to the square of their sum. However, not much research has been done about other sets that have this property. In this study, we attempt to establish tighter bounds on terms in such sets and find certain interesting properties about them.

1 Introduction

CS-sets are defined to be sets where the sum of the cubes of the terms in the set is equal to the square of the sum of the terms in the set. Namely, $\sum_{k=1}^{n} a_k^3 = \left(\sum_{k=1}^{n} a_k\right)^2$ where a_k is an element in the set. CS-$n$sets are defined as CS-sets of size n. In this paper, we are dealing with CS-sets which consists of entirely positive integers. WLOG, we assume that the elements of a CS-set are in increasing order. In this paper, we shall let n be the number of elements in a CS-set. Also, a_k will denote the k th element in a CS-set.

2 Proposition and Proofs

2.1 Bounds

Proposition 1. Sum of all elements of a CS-$n$set is at most n^2.

Proof: By Holder's Inequality,

$$\sum_{k=1}^{n} a_k \leq \left(\sum_{k=1}^{n} a_k^3\right)^{\frac{1}{3}} \left(\sum_{k=1}^{n} (1)^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left[\left(\sum_{k=1}^{n} a_k\right)^2\right]^{\frac{1}{3}} (n)^{\frac{2}{3}}$$

$$\iff \left(\sum_{k=1}^{n} a_k\right)^{\frac{1}{3}} \leq n^{\frac{2}{3}} \iff \sum_{k=1}^{n} a_k \leq n^2$$

Note that the equality can be achieved when all elements of the CS-set are equal to n.

Proposition 2. Sum of reciprocals of the elements of a CS-set is at least 1.

Proof 1: By Cauchy-Schwarz Inequality,

$$\left(\sum_{k=1}^{n} a_k\right)^2 \leq \left(\sum_{k=1}^{n} (a_k^{-\frac{1}{2}})^2\right)\left(\sum_{k=1}^{n} (a_k^{\frac{3}{2}})^2\right) = \left(\sum_{k=1}^{n} a_k^{-1}\right)\left(\sum_{k=1}^{n} a_k^{3}\right) = \left(\sum_{k=1}^{n} a_k^{-1}\right)\left(\sum_{k=1}^{n} a_k\right)^2$$

$$\iff 1 \leq \sum_{k=1}^{n} a_k^{-1}$$

Proof 2: By Cauchy-Schwarz Inequality,

$$n^2 = \left(\sum_{k=1}^{n} 1\right)^2 \leq \left(\sum_{k=1}^{n} (a_k^{-\frac{1}{2}})^2\right)\left(\sum_{k=1}^{n} (a_k^{\frac{1}{2}})^2\right) = \left(\sum_{k=1}^{n} a_k^{-1}\right)\left(\sum_{k=1}^{n} a_k\right) \leq \left(\sum_{k=1}^{n} a_k^{-1}\right)\left(n^2\right)$$

$$\iff 1 \leq \sum_{k=1}^{n} a_k^{-1}$$

Note that the equality can be achieved when all elements of the CS-set are equal to n.

Proposition 3. The upper bound of a_1 is n.

Proof: Suppose $a_1 > n$. Then, $a_1^{-1} < \frac{1}{n}$. Since $a_1 \leq a_2 \leq a_3 \leq ... \leq a_n$, $a_1^{-1} \geq a_2^{-1} \geq a_3^{-1} \geq ... \geq a_n^{-1}$. Then,

$$\sum_{k=1}^{n} a_k^{-1} \leq \sum_{k=1}^{n} a_1^{-1} = n(a_1^{-1}) < n\left(\frac{1}{n}\right) = 1$$

which is absurd as we know that $1 \leq \sum_{k=1}^{n} a_k^{-1}$ from Proposition 2. So, $a_1 \leq n$ with equality iff $a_1 = a_2 = a_3 = ... = a_n = n$ due to the fact that a n-element-set in the form of $\{n, n, n, ..., n\}$ is a CS-$n$set.

Proposition 4. The upper bound for a_n is at most $(n^4 - n + 1)^{\frac{1}{3}}$.

Proof: Suppose that a_k are the elements of a CS-$n$set such that m is the largest entry, that is, $a_n = m$. Then

$$\underbrace{1^3 + 1^3 + 1^3 + ... + 1^3}_{(n-1)\ 1^3\text{s}} + m^3 \leq \sum_{k=1}^{n} a_k^3 = \left(\sum_{k=1}^{n} a_k\right)^2 \leq (n^2)^2 = n^4$$

$$\iff m \leq (n^4 - n + 1)^{\frac{1}{3}}$$

Proposition 5. The lower bound for a_n is at least $n^{\frac{1}{3}}$.

Proof: Suppose that a_k are the elements of a CS-$n$set such that m is the largest entry, that is, $a_n = m$. Then

$$\underbrace{m^3 + m^3 + m^3 + ... + m^3}_{n\ m^3\text{s}} \geq \sum_{k=1}^{n} a_k^3 = \left(\sum_{k=1}^{n} a_k\right)^2 \geq (\underbrace{1 + 1 + 1 + ... + 1}_{n\ 1\text{s}})^2$$

$$\iff nm^3 \geq n^2$$

$$\iff m \geq n^{\frac{1}{3}}$$

2.2 CS-sets with only one repeated term

Proposition 6. $\{1, 2, 2, 3, 4, 5, ..., n-3, n-2, n\}$ is a CS-$n$set.

Proof: A property of CS set is that if $2\left(2\left(\sum_{k=1}^{n} a_k\right) + 1\right) = (y-x)^2 + (x-1)^2 + (y-1)^2$ for some integers x, y, then it can be appended with x and y to form another CS-set. Furthermore, appending an integer z is equivalent to removing $-z$. In addition, it is known that the set $\{1, 2, 3, ..., n-1, n\}$ is a CS-$n$set.

$$2\left(2\left(\frac{n(n+1)}{2}\right) + 1\right) = 2n(n+1) + 2 = 2n^2 + 2n + 2 = n^2 + (n+1)^2 + 1^2$$

Let $y - 1 = 1$ and $x - 1 = -n$, then $y - x = n + 1$. Hence, $(x, y) = (1-n, 2)$ is a solution and we can remove $n-1$ and append 2 to the CS-set $\{1, 2, 3, 4, 5..., n-3, n-2, n-1, n\}$ to form another CS-set $\{1, 2, 2, 3, 4, 5..., n-3, n-2, n\}$.

Lemma 1. For all sets of the form $\{1, 2, ..., x, x, ..., m-1, m\}$ (1 to x and x to m are consecutive integers),

$$\sum_{k=1}^{n} a_k^3 < \left(\sum_{k=1}^{n} a_k\right)^2$$

for all positive integers x, n where $x \leq m$.

Proof:

$$\sum_{k=1}^{n} a_k^3 = x^3 + \sum_{i=1}^{m} i^3 = x^3 + \frac{(m)^2(m+1)^2}{4} = x^2 + x^2(x-1) + \frac{(m)^2(m+1)^2}{4}$$

$$< x^2 + xm(m+1) + \frac{(m)^2(m+1)^2}{4} = \left(x + \frac{(m)(m+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} a_k\right)^2.$$

Lemma 2. For all sets of the form $\{1, 2, ..., x, x, ..., y-1, y+1, ..., m-1, m\}$,

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2$$

for all positive integers y, x, m where $x \leq m-2$ and

$$x < y < \sqrt{m(m+1) - x(x-1) + \left(\frac{x+1}{2}\right)^2} - \frac{x+1}{2}.$$

Proof: For $\{1, 2, ..., x, x, ..., y-1, y+1, ..., m-1, m\}$,

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2$$

when

$$x^3 - y^3 + \sum_{i=1}^{m} i^3 > \left(x - y + \sum_{i=1}^{m} i\right)^2.$$

$$x^3 - y^3 + \sum_{i=1}^{m} i^3 > x^2 + y^2 + \left(\sum_{i=1}^{m} i\right)^2 - 2xy + 2x\left(\sum_{i=1}^{m} i\right) - 2y\left(\sum_{i=1}^{m} i\right)$$

$$x^3 - y^3 > x^2 + y^2 - 2xy + 2x\left(\sum_{i=1}^{m} i\right) - 2y\left(\sum_{i=1}^{m} i\right)$$

$$x^3 - y^3 - x^2 + 2xy - y^2 > 2(x-y)\left(\sum_{i=1}^{m} i\right)$$

$$(x-y)(x^2+xy+y^2) - (x-y)^2 > 2(x-y)\left(\sum_{i=1}^{m} i\right)$$

Since $x < y$,

$$(x^2+xy+y^2) - (x-y) < 2\left(\sum_{i=1}^{m} i\right)$$

$$x^2 + xy + y^2 - x + y < m(m+1)$$

$$y^2 + (x+1)y < m(m+1) - x(x-1)$$

$$\left(y + \frac{x+1}{2}\right)^2 < m(m+1) - x(x-1) + \left(\frac{x+1}{2}\right)^2$$

And since we know that $y > x > 0$,

$$y + \frac{x+1}{2} < \sqrt{m(m+1) - x(x-1) + \left(\frac{x+1}{2}\right)^2}$$

Hence, for all sets of the form $\{1, 2, ..., x, x, ..., y-1, y+1, ..., m-1, m\}$,

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2$$

for all positive integers y, x, m where $x \leq m-2$ and

$$x < y < \sqrt{m(m+1) - x(x-1) + \left(\frac{x+1}{2}\right)^2} - \frac{x+1}{2}.$$

Lemma 3. For all sets where all the terms are distinct, if

$$x^3 + \sum_{k=1}^{n} a_k^3 > \left(x + \sum_{k=1}^{n} a_k\right)^2,$$

for all $y < n$,

$$x^3 - y^3 + \sum_{k=1}^{n} a_k^3 > \left(x - y + \sum_{k=1}^{n} a_k\right)^2.$$

Proof: If

$$x^3 + \sum_{k=1}^{n} a_k^3 > \left(x + \sum_{k=1}^{n} a_k\right)^2,$$

and

$$x^3 - y^3 + \sum_{k=1}^{n} a_k^3 > \left(x - y + \sum_{k=1}^{n} a_k\right)^2$$

then,

$$y^3 < y\left(2\left(\sum_{k=1}^{n} a_k\right) + 2x - y\right)$$

Since $y > 0$,

$$y^2 < \left(2\left(\sum_{k=1}^{n} a_k\right) + 2x - y\right).$$

$$y(y+1) < 2\left(\sum_{k=1}^{n} a_k\right) + 2x$$

All $y \leq n$ satisfies this condition as since the elements in the set are distinct and $x > 0$,

$$y(y+1) \leq n(n+1) < 2\left(\sum_{k=1}^{n} a_k\right) + 2x$$

Proposition 7. There exists no CS-set of the form $\{1, 1, a_3, ..., a_n\}$ where $a_k > a_{k-1}$ for all $k > 2$.

Proof: In order to prove this, we shall divide the problem into two cases. Case 1: For all sets of the form $\{1, 1, 2, ..., m\}$,

$$\sum_{k=1}^{n} a_k^3 < \left(\sum_{k=1}^{n} a_k\right)^2.$$

Applying lemma 1 here, we know that this is true.

Case 2: For all other sets of the form $\{1, 1, a_3, ..., a_n\}$ where $a_k > a_{k-1}$ for all $k > 2$,

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2.$$

In order to prove this, we first prove that for $\{1, 1, 2, ..., y-1, y+1, ..., m-1, m\}$ where $y > 1$ and $m > y$,

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2.$$

Applying Lemma 2, we know that this is true since $m > y > x = 1$ and since m and y are integers, $x \leq m - 2$ and

$$x < y \leq m - 1 < \sqrt{m^2 + m + 1} - 1$$
$$= \sqrt{m(m+1) - x(x-1) + \left(\frac{x+1}{2}\right)^2} - \frac{x+1}{2}.$$

Now, to prove that this is true for other sets, we need to find an order of removing terms from the set $\{1, 1, 2, ..., y - 1, y + 1, ..., m - 1, m\}$ where $y > 1$ and $m > y$, such that the property of

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2$$

is maintained.

From lemma 3, we know that if we remove terms that are less than n, the property is maintained as desired. Now, in order to prove that any desired set has this property, we can just start from a set $\{1, 1, 2, ..., y-1, y+1, ..., m-1, m\}$, where m is the biggest number of the desired set and $y < m$ is the biggest number missing from the desired set. Then, we remove terms in descending order, as the ith term we remove would be less than $n - i$, which is the size of the current set.

Example:

To prove that for $\{1, 1, 3, 6\}$,

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2,$$

we start from $\{1, 1, 2, 3, 4, 6\}$ which has this property from lemma 2. Then, we remove $4 < n = 6$ to get $\{1, 1, 2, 3, 6\}$ which maintains the property from lemma 3. Then, we remove $2 < n = 5$ to get $\{1, 1, 3, 6\}$ which maintains the property from lemma 3. Thus, proving that for $\{1, 1, 3, 6\}$,

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2,$$

Hence, for all other sets of the form $\{1, 1, a_3, ..., a_n\}$ where $a_k > a_{k-1}$ for all $k > 2$,

$$\sum_{k=1}^{n} a_k^3 > \left(\sum_{k=1}^{n} a_k\right)^2$$

and we have proven that there exists no CS-set of the form $\{1, 1, a_3, ..., a_n\}$ where $a_k > a_{k-1}$ for all $k > 2$.

3 Conclusion

We focused on CS-sets that consists of entirely positive integers, and explored the boundaries of various properties of CS-sets, as well as some special CS sets. The collected results (for each positive integer n) are:

1. Sum of all elements of a CS-$n$set is at most n^2.
2. Sum of reciprocals of all elements of a CS-$n$set is at least 1.
3. a_1 is at most n, with equality iff all elements of the CS-$n$set are equal to n.
4. The upper bound of a_n is improved to $(n^4 - n + 1)^{\frac{1}{3}}$.
5. The lower bound of a_n is improved to $n^{\frac{1}{3}}$.
6. $\{1, 2, 2, 3, 4, 5, ..., n-3, n-2, n\}$ is a CS-$n$set.
7. There exists no CS-set in the form of $\{1, 1, a_3, a_4, a_5, ..., a_n\}$ where $a_k > a_{k-1}$ for all $k > 2$.

From data that we produced using code, we observe that the boundaries of a_n still have a huge room for improvement. We also observe that for n smaller than 16, there are no CS-sets with 3 or 4 as the only repeated term. This could point to the fact that there are no CS-sets with 3 or 4 as the only repeated term at all.

4 References

1. Edward Barbeau, Samer Seraj, *Sum of Cubes is Square of Sum*. University of Toronto, 2013.