Cardan's Formula in a Dierent Planet

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Suppose we are living in a planet, which has prohibited root-operations; then is it possible to find a root of a cubic equation in this planet? The well-known Cardan's formula for a root of cubic equation contains square-roots and cube-roots. This paper attempts to express the Cardan's formula as an infinite convergent series, which is devoid of square-roots and cube-roots. Clearly such an expression (if it exists) will be suitable for iterative methods using computers, if the accuracy of the root so obtained is of the same order as that obtained directly from the compact Cardan's formula. Let us consider the following depressed cubic equation,

$$x^3 + ax + b = 0, \tag{1}$$

where its coefficients, a and b, are real. The three roots of cubic equation (1) obtained from Cardan's method [1] are:

$$x_1 = U + V, \quad x_2 = Uw + Vw^2, \quad x_3 = Uw^2 + Vw, \tag{2}$$

where U, V, and w are given by:

$$U = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{1/3}, \quad V = \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{1/3}, \quad \text{and}$$
$$w = (-1 + \sqrt{3}\, i)/2. \tag{3}$$

From (2) and (3) the expression for a root x_1 is obtained as,

$$x_1 = \left(\frac{b}{2}\right)^{1/3} \left[\left(-1 + \sqrt{1 + \frac{4a^3}{27b^2}}\right)^{1/3} - \left(1 + \sqrt{1 + \frac{4a^3}{27b^2}}\right)^{1/3}\right]. \tag{4}$$

Observing (4) it is evident that x_1 is real when the terms within the square roots are positive; i.e., when a is positive, or, when $|4a^3| < 27b^2$. However when a is negative and $|4a^3| > 27b^2$, the terms within the square roots are negative, and so the terms within the cube roots become complex. In such case, it is not clear whether the root x_1 is real or complex; in order to find out this, let the term within the square root be denoted as $-c^2$ as indicated below,

$$1 + (4a^3/27b^2) = -c^2, \tag{5}$$

where c is a positive number; and using it in (4) results in,

$$x_1 = -(b/2)^{1/3} \left[(1 - ci)^{1/3} + (1 + ci)^{1/3}\right]. \tag{6}$$

Expanding (6) using binomial theorem and simplifying leads to,

$$x_1 = -2(b/2)^{1/3}\left(1+\frac{c^2}{9}-\frac{10c^4}{243}+\frac{154c^6}{6561}-\frac{935c^8}{59049}+...\right). \qquad (7)$$

From (7) it is clear that x_1 is real even for $|4a^3| > 27b^2$, $(a < 0)$. Thus we conclude that x_1 is real for all real a and b. Even though (7) is an infinite series, it is not the expression we are looking for; since it contains a cube root term and it does not guarentee that it converges.

We shall now attempt to obtain an expression for one real root (x_1) of cubic (1), which shall be in the form of convergent infinite series, and devoid of square roots and cube roots.

Case 1. Coefficient a positive

In this case, the term within the square root in (4) is greater than unity, hence the square root term itself is greater than unity, and let us denote it as,

$$\sqrt{1+(4a^3/27b^2)} = (1/p), \qquad (8)$$

where p is a positive number less than unity. Using (8) in (4) yields,

$$x_1 = (b/2p)^{1/3}\left[(1-p)^{1/3}-(1+p)^{1/3}\right]. \qquad (9)$$

Expanding the cube root terms in (9) using binomial theorem and simplifying, we get,

$$x_1 = (b/2p)^{1/3}(-2p/3)\left(1+\frac{5p^2}{27}+\frac{22p^4}{243}+\frac{374p^6}{6561}+\frac{21505p^8}{531441}+...\right). \qquad (10)$$

Now, the first two terms in (10) are manipulated as below.

$$\begin{aligned}(b/2p)^{1/3}(-2p/3) = (-4bp^2/27)^{1/3} &= [-4b^3p^2a^3/(27b^2a^3)]^{1/3}\\ &= (-b/a)(4a^3p^2/27b^2)^{1/3}\end{aligned} \qquad (11)$$

Using (8) we deduce, $(4a^3p^2/27b^2) = (1-p^2)$, and so (11) becomes

$$(b/2p)^{1/3}(-2p/3) = (-b/a)(1-p^2)^{1/3},$$

which when used in (10) yields,

$$x_1 = (-b/a)(1-p^2)^{1/3}\left(1+\frac{5p^2}{27}+\frac{22p^4}{243}+\frac{374p^6}{6561}+\frac{21505p^8}{531441}+...\right). \qquad (12)$$

Again expanding the cube root term in (11) as before, we obtain,

$$\begin{aligned}x_1 = (-b/a)&\left(1-\frac{p^2}{3}-\frac{p^4}{9}-\frac{5p^6}{81}-\frac{10p^8}{243}-...\right)\\ &\left(1+\frac{5p^2}{27}+\frac{22p^4}{243}+\frac{374p^6}{6561}+\frac{21505p^8}{531441}+...\right),\end{aligned} \qquad (13)$$

which contains p^2 and its powers, and p^2 is obtained by squaring (8); thus (13) satisfies our requirement that it is devoid of square roots or cube roots.

Case 2. Coefficient a negative and $|4a^3| < 27b^2$

In this case, the term within the square root in (4) is positive but less than unity, so the square root term is less than unity, and let it be denoted as q,

$$\sqrt{1 + (4a^3/27b^2)} = q. \tag{14}$$

Substituting (14) in (4) yields,

$$x_1 = -(b/2)^{1/3} \left[(1-q)^{1/3} + (1+q)^{1/3}\right]. \tag{15}$$

Expanding the terms in square bracket in (15) leads to:

$$x_1 = -2(b/2)^{1/3} \left(1 - \frac{q^2}{9} - \frac{10q^4}{243} - \frac{154q^6}{6561} - \frac{935q^8}{59049} - \ldots\right). \tag{16}$$

We now manipulate the term $-2(b/2)^{1/3}$ in (16) as shown below.

$$-2(b/2)^{1/3} = (-4b)^{1/3} = (-8b^3/2b^2)^{1/3} = 2b(-1/2b^2)^{1/3} \tag{17}$$

After some algebraic manipulations on (14) we obtain the relation,

$$(-1/2b^2) = [27(1-q^2)/8a^3],$$

which when substituted in (17) leads to,

$$-2(b/2)^{1/3} = (3b/a)(1-q^2)^{1/3}. \tag{18}$$

Use of (18) in (16) yields,

$$x_1 = (3b/a)\left(1-q^2\right)^{1/3} \left(1 - \frac{q^2}{9} - \frac{10q^4}{243} - \frac{154q^6}{6561} - \frac{935q^8}{59049} - \ldots\right). \tag{19}$$

Expanding the cube root term in (19) yields,

$$\begin{aligned} x_1 = (3b/a) & \left(1 - \frac{q^2}{3} - \frac{q^4}{9} - \frac{5q^6}{81} - \frac{10q^8}{243} - \ldots\right) \\ & \left(1 - \frac{q^2}{9} - \frac{10q^4}{243} - \frac{154q^6}{6561} - \frac{935q^8}{59049} - \ldots\right), \end{aligned} \tag{20}$$

which contains only even powers of q; and since q^2 is obtained by squaring (14), we avoid use of square root, satisfying our criteria.

Case 3. Coefficient a negative and $1 < |4a^3/27b^2| < 2$

In this case $0 < c < 1$ [see (5)], so the infinite series in (7) becomes convergent and the term $-2(b/2)^{1/3}$ appearing in (7) is expressed [by using derivations similar to that used in (17) and (18)] as below.

$$-2(b/2)^{1/3} = (3b/a)(1+c^2)^{1/3} \tag{21}$$

Use of (21) in (7) results in,

$$x_1 = (3b/a)(1+c^2)^{1/3}\left(1+\frac{c^2}{9}-\frac{10c^4}{243}+\frac{154c^6}{6561}-\frac{935c^8}{59049}+...\right). \tag{22}$$

Expansion of cube root term in (22) yields an expression for x_1,

$$x_1 = (3b/a)\left(1+\frac{c^2}{3}-\frac{c^4}{9}+\frac{5c^6}{81}-\frac{10c^8}{243}+...\right)$$
$$\left(1+\frac{c^2}{9}-\frac{10c^4}{243}+\frac{154c^6}{6561}-\frac{935c^8}{59049}+...\right), \tag{23}$$

which meets our requirement.

Case 4. Coefficient a negative and $|4a^3/27b^2| > 2$
In this case, note that c [used in (6)] is greater than unity; let $c = 1/d$, where $0 < d < 1$. Using d in (6) results in,

$$x_1 = -(b/2di)^{1/3}\left[(1+di)^{1/3}-(1-di)^{1/3}\right]. \tag{24}$$

Using the procedures applied in earlier cases, we obtain an expression for x_1 as,

$$x_1 = -(b/a)\left(1+\frac{d^2}{3}-\frac{d^4}{9}+\frac{5d^6}{81}-\frac{10d^8}{243}+...\right)$$
$$\left(1-\frac{5d^2}{27}+\frac{22d^4}{243}-\frac{374d^6}{6561}+\frac{21505d^8}{531441}+...\right), \tag{25}$$

which has no square root or cube root terms.

Discussion

Notice that the four cases mentioned above can be clubbed into two main categories; cases 1 and 4 can be combined as, $|1+(4a^3/27b^2)| > 1$, and cases 2 and 3 can be clubbed together as, $|1+(4a^3/27b^2)| < 1$ and $a < 0$.

Case A: For $|1+(4a^3/27b^2)| > 1$
When $a > 0$, the expression (13) is applicable where p is real ($0 < p < 1$); when $a < 0$ we notice that the expression (13) still can be used with p as imaginary number, say $p = di$, where d is real and $0 < d < 1$.

Case B: For $|1+(4a^3/27b^2)| < 1$ and $a < 0$
When $|(4a^3/27b^2)| < 1$, the expression (20) is used with q as real number ($0 < q < 1$), however when $1 < |(4a^3/27b^2)| < 2$, we can still use the expression (20) with q imaginary, say $q = ci$, where c is real and $0 < c < 1$.

The convergent infinite series expressions obtained [(13), (20), (23), and (25), for all combinations of real coefficients a and b] for one real root of cubic equation (1) are devoid of square roots and cube roots; and are suitable for iterative methods using computers. Since square roots and cube roots of real numbers are also approximations (except for the cases of perfect squares and cubes), the expressions obtained here for one real root have the same level of accuracy as that of the so called *exact* formula (4) involving square roots and cube roots. Let us solve one numerical example and compare the results with the *exact* formula.

Numerical example

Consider the cubic equation: $x^3-49x+120=0$. We determine, $(4a^3/27b^2) = -1.210380658$, implying (23) to be used for x_1. Thus use of (23) with only four terms (including unity) in the two convergent series yields $x_1 = -8.000735311$, and with five terms we obtain $x_1 = -7.999887276$, while the exact value of root is -8. Defining error as, $|x_{1ex} - x_1|$, where x_{1ex} is the exact value of the root, we note that with four terms the error is 0.000735311, while with five terms the error has reduced to 0.000112724. As the number of terms in (23) increases the accuracy of the result (which is inversely proportional to the error) improves.

With the number of terms in (23) [or (13), (20), (25), as the case may be] approaching infinity, the value of root x_1 approaches x_{1ex}.

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References

[1] Leonard Eugene Dickson, "First course in the theory of equations", Ebook # 29785 available at www.gutenberg.org; p. 51-52.