Singapore International Mathematical Olympiad 2017

National Team Selection Test
Day 0

1. Let $ABCD$ be a parallelogram where AC is a diagonal. It is known the $AB = AC$ and $\angle BAC < 60^\circ$. Extend BA to E such that $AE = 2AB$. Draw lines ℓ_1 through E and ℓ_2 through D so that $\ell_1 \perp AC$ and $\ell_2 \perp AB$. If ℓ_1 and ℓ_2 intersect at P, prove that $\angle PCB = \angle BAC$.

2. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2017$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

3. Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

   - $m = 1$ and $l = 2k$; or
   - $l | k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

4. A 101×101 grid has a real number written in each of its cells.

   - If the number in a cell is greater than the numbers in the cells to its immediate left and immediately below it, we say that the cell is a **good cell**. A cell in the bottom row or the leftmost column is not a good cell.
   - If the number in a cell is less than the numbers in the cells to its immediate right and immediately above it, we say that the cell is a **bad cell**. A cell in the top row or the rightmost column is not a bad cell.

   Note that a cell can be both a good cell and a bad cell. If G is the number of good cells and B is the number of bad cells, what is the maximum value of $G - B$?

Time allowed: 4 hours

Singapore International Mathematical Olympiad 2017

National Team Selection Test
Day 01

5. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

6. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

7. Let a be a positive integer which is not a square number. Denote by A the set of all positive integers k such that

$$k = \frac{x^2 - a}{x^2 - y^2} \tag{7.1}$$

for some integers x and y with $x > \sqrt{a}$. Denote by B the set of all positive integers k such that (7.1) is satisfied for some integers x and y with $0 \leq x < \sqrt{a}$. Prove that $A = B$.

Time allowed: 4.5 hours

Singapore International Mathematical Olympiad 2017

National Team Selection Test
Day 02

8. Let ABC be a triangle with circumcircle Γ and incentre I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to side BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

9. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integers $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

10. Let $n \geq 3$ be an integer. Find the maximum number of diagonals of a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

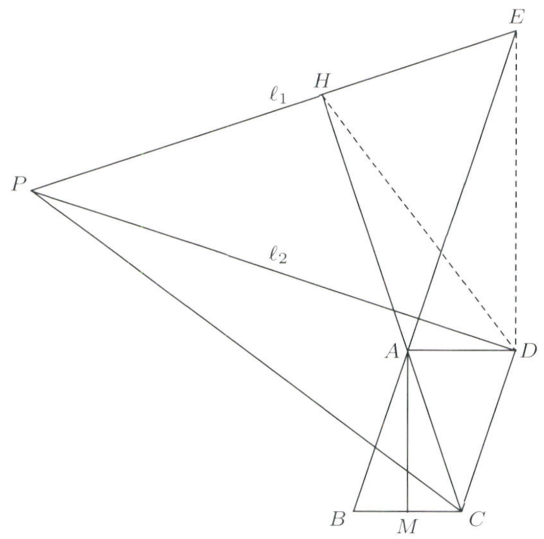
Time allowed: 4.5 hours

Solutions to NTST

Day 0.

1. Let $ABCD$ be a parallelogram where AC is a diagonal. It is known the $AB = AC$ and $\angle BAC < 60^\circ$. Extend BA to E such that $AE = 2AB$. Draw lines ℓ_1 through E and ℓ_2 through D so that $\ell_1 \perp AC$ and $\ell_2 \perp AB$. If ℓ_1 and ℓ_2 intersect at P, prove that $\angle PCB = \angle BAC$.

   Solution. Let ℓ_1 intersect the line AC at H. Let $\angle BAC = \alpha$.



   Let M be the midpoint of BC. Since $AE = 2AB, AD = 2BM$ and $AD \parallel BC$, it follows that $\triangle ABM \sim \triangle EAD$.

   Hence $\angle ADE = 90^\circ$ and A, D, E, H are concyclic. Now $\angle ADH = \angle AEH = 90^\circ - \angle EAH = 90^\circ - \alpha$.

   It is easy to see that $\angle ADP = \angle AED = \frac{1}{2}\alpha$, $\angle PDH = \angle ADH - \angle ADP = 90^\circ - \frac{3}{2}\alpha$.

   Clearly P, C, D, H are concyclic. Now $\angle PCH = \angle PDH = 90^\circ - \frac{3}{2}\alpha$.

   It follows that $\angle BCP = \angle ACB - \angle PCH = (90^\circ - \frac{1}{2}\alpha) - (90^\circ - \frac{3}{2}\alpha) = \alpha$. This completes the proof.

2. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2017$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)). \tag{2.1}$$

   Solution. The answers are $P(x) = c$, where $1 \leq c \leq 9$ is an integer; or $P(x) = x$.

   We consider 3 cases according to the degree of P.

   Case 1. $P(x)$ is a constant polynomial.

   Let $P(x) = c$ where c is an integer constant. Then (2.1) becomes $S(c) = c$. This holds if and only if $1 \leq c \leq 9$.

Case 2. $\deg P(x) = 1$.

We have the following observation. For any positive integers m, n, we have

$$S(m+n) \leq S(m) + S(n), \tag{2.2}$$

and equality holds if and only if there is no carry in the addition $m + n$.

Let $P(x) = ax + b$ for some integers a, b, where $a \neq 0$. As $P(n)$ is positive for large n, we must have $a \geq 1$. The condition (2.1) becomes $S(an + b) = aS(n) + b$ for all $n \geq 2017$. Setting $n = 2025$ and $n = 2020$ respectively, we get

$$S(2025a + b) - S(2020a + b) = (aS(2025) + b) - (aS(2020) + b) = 9a - 4a = 5a.$$

On the other hand, (2.2) implies

$$S(2025a + b) = S((2020a + b) + 5a) \leq S(2020a + b) + S(5a).$$

These gives $5a \leq S(5a)$. As $a \geq 1$, this holds only when $a = 1$, in which case (2.1) reduces to $S(n + b) = S(n) + b$ for all $n \geq 2017$. Then we find that

$$S(n+1+b) - S(n+b) = (S(n+1) + b) - (S(n) + b) = S(n+1) - S(n). \tag{2.3}$$

If $b > 0$, we choose n such that $n + 1 + b = 10^k$ for some sufficiently large k. Note that all the digits of $n + b$ are 9's, so that the left-hand side of (2.3) equals $1 - 9k$. As n is a positive integer less than $10^k - 1$, we have $S(n) < 9k$. Therefore, the right-hand side of (2.3) is at least $1 - (9k - 1) = 2 - 9k$, which is a contradiction.

The case $b < 0$ can be handled similarly by considering $n + 1$ to be a large power of 10. Therefore, we conclude that $P(x) = x$, in which case (2.1) is trivially satisfied.

Case 3. $\deg P(x) \geq 2$.

Suppose the leading term of P is $a_d n^d$ where $a_d \neq 0$. Clearly, we have $a_d > 0$. Consider $n = 10^k - 1$ in (2.1). We get $S(P(n)) = P(9k)$. Note that $P(n)$ grows asymptotically as fast as n^d, so $S(P(n))$ grows asymptotically as no faster than a constant multiple of k. On the other hand, $P(9k)$ grows asymptotically as fast as k^d. This shows the two sides of the equation $S(P(n)) = P(9k)$ cannot be equal for sufficiently large k since $d \geq 2$.

Therefore, we conclude that $P(x) = c$, where $1 \leq c \leq 9$ is an integer; or $P(x) = x$.

3. Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

   - $m = 1$ and $l = 2k$; or
   - $l | k$ and $m = \frac{n^{k-l} - 1}{n^l - 1}$.

Solution. It is given that

$$n^k + mn^l + 1 | n^{k+l} - 1. \tag{3.1}$$

This implies

$$n^k + mn^l + 1 | (n^{k+l} - 1) + (n^k + mn^l + 1) = n^{k+l} + n^k + mn^l. \tag{3.2}$$

We consider 2 cases.

• **Case 1**. $l \geq k$.

Then (3.2) yields

$$n^k + mn^l + 1 | n^l + mn^{l-k} + 1.$$

Since $2(n^k + mn^l + 1) > 2mn^l + 1 > n^l + mn^{l-k} + 1$, it follows that $n^k + mn^l + 1 = n^l + mn^{l-k} + 1$,

That is,

$$m(n^l - n^{l-k}) = n^l - n^k.$$

If $m \geq 2$, then as $n > 1$ we have $m(n^l - n^{l-k}) \geq 2n^l - 2n^{l-k} \geq 2n^l - 2n^l > n^l - n^k$ giving a contradiction. Hence $m = 1$ and $l - k = k$, which means $m = 1$ and $l = 2k$.

• **Case 2**. $l < k$.

Then (3.2) yields

$$n^k + mn^l + 1 | n^k + mn^{k-l} + m.$$

Since $2(n^k + mn^l + 1) > 2n^k + m > n^k + n^{k-l} + m$, it follows that $n^k + mn^l + 1 = n^k + n^{k-l} + m$. This gives $m = \frac{n^{k-l}-1}{n^l-1}$. Note that $n^l - 1 | n^{k-l} - 1$ implies $l | (k - l)$ and hence $l | k$. The proof is thus completed.

4. A 101×101 grid has a real number written in each of its cells.

   - If the number in a cell is greater than the numbers in the cells to its immediate left and immediately below it, we say that the cell is a **good cell**. A cell in the bottom row and the leftmost column is not a good cell.
   - If the number in a cell is less than the numbers in the cells to its immediate right and immediately above it, we say that the cell is a **bad cell**. A cell in the top row or the rightmost column is not a bad cell.

   Note that a cell can be both a good cell and a bad cell. If G is the number of good cells and B is the number of bad cells, what is the maximum value of $G - B$?

   Solution. We claim the maximum number is 5050. Let (i, j) denote the cell in the i-th row (from the top) and j-th column (from the left). First let us construct the maximum:

   - If i is odd, and $i < j$, put j in the cell, otherwise 0.
   - If i is even, and $i \geq j$, put $j - 1$ in the cell, otherwise 0.

0	2	3	4	5	...	97	98	99	100	101
0	1	0	0	0	...	0	0	0	0	0
0	0	0	4	5	...	97	98	99	100	101
0	1	2	3	0	...	0	0	0	0	0
0	0	0	0	0	...	97	98	99	100	101
...	...	...	...	...	...	...	...	...	...	...
0	0	0	0	0	...	0	98	99	100	101
0	1	2	3	4	...	96	97	0	0	0
0	0	0	0	0	...	0	0	0	100	101
0	1	2	3	4	...	96	97	98	99	0
0	0	0	0	0	...	0	0	0	0	0

The first row and last column will not be bad cells by definition. Any other cell is also not a bad cell since there is a 0 either in the cell above or to the right of it.

The good cells are those with positive integers inside them. Hence there are a total of $101 \times 50 = 5050$ good cells, and thus $G - B = 5050$.

Now we show that $G - B \leq 5050$. Define a cell to be a **very good cell** if it is a good cell and it does not have a bad cell directly below it. The number of very good cells is equal to the number of good cells minus the number of bad cells which has a good cell directly above it. Hence the number of very good cells is at least $G - B$.

Now we prove that if $1 \leq k \leq 50$ and $2 \leq l \leq 100$, amongst $(2k-1, l)$ and $(2k, l+1)$, there is at most 1 very good cell. Suppose not for the sake of contradiction. Then since they are good cells, the number in $(2k, l)$ is smaller than the numbers in the two cells $(2k-1, l)$ and $(2k, l+1)$. Therefore $(2k, l)$ is a bad cell, which contradicts the assumption that $(2k-1, l)$ is a very good cell.

If $i = 101$ or $j = 1$, (i, j) is not a very good cell since it is not a good cell.

Among the cells $1 \leq i \leq 100, 2 \leq j \leq 101$, only the cells of the form $(2m-1, 101)$ or $(2m, 2)$ (of which there are 100) cannot be expressed in either the form $(2k-1, l)$ or $(2k, l+1)$.

Taking away these cells, the remaining cells can all be paired into one of the cells $(2k-1, l)$ or $(2k, l+1)$. Hence there are at most $99 \times 50 = 4950$ very good cells amongst these, and thus at most a total of at most $100 + 4950 = 5050$ very good cells. Since the number of good cells is at least $G - B$, $G - B$ is at most 5050.

Day 1.

5. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

   Answer: The minimum number of guesses is 2 if $n = 2k$ and 1 if $n \neq 2k$.

   Solution. Let X be the binary string chosen by the leader and let X' be the binary string of length n every digit of which is different from that of X. The string written by the deputy leader are the same as those in the case when the leader's string is X' and k is changed to $n - k$. In view of this, we may assume $k \geq \frac{n}{2}$. Also, for the particular case $k = \frac{n}{2}$, this argument shows that the string X and X' cannot be distinguished, and hence in that case the contestant has to guess at least twice.

   It remains to show that the number of guesses claimed suffices. Consider any string Y which differs from X in m digits where $0 < m < 2k$. Without loss of generality, assume the first m digits of X and Y are distinct. Let Z be the binary string obtained from X by changing its first k digits. Then Z is written by the deputy leader. Note that Z differs from Y by $|m - k|$ digits where $|m - k| < k$ since $0 < m < 2k$. From this observation, the contestant must know that Y is not the desired string.

As we have assumed $k \geq \frac{n}{2}$, when $n < 2k$, every string $Y \neq X$ differs from X in fewer than $2k$ digits. When $n = 2k$, every string except X and X' differs from X in fewer than $2k$ digits. Hence the answer is as claimed.

6. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Solution 1. We are required to prove the inequality

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1 \tag{6.1}$$

First, for any positive real numbers x, y with $xy \geq 1$, we have

$$(x^2+1)(y^2+1) \leq \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2 \tag{6.2}$$

To prove (6.2), note that $xy \geq 1$ implies that $\left(\frac{x+y}{2}\right)^2 - 1 \geq xy - 1 \geq 0$. Thus $(x^2+1)(y^2+1) = (xy-1)^2 + (x+y)^2 \leq \left(\left(\frac{x+y}{2}\right)^2 - 1\right)^2 + (x+y)^2 = \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2$.
Let $f(x) = \ln(1+x^2)$. Then (6.1) is equivalent to

$$\frac{f(a)+f(b)+f(c)}{3} \leq f\left(\frac{a+b+c}{3}\right), \tag{6.3}$$

while (6.2) becomes

$$\frac{f(x)+f(y)}{2} \leq f\left(\frac{x+y}{2}\right) \tag{6.4}$$

for $xy \geq 1$.
Without loss of generality, assume $a \geq b \geq c$. By (6.4), we have

$$\frac{f(a)+f(b)+f(c)}{3} \leq \frac{f(a)+2f(\frac{b+c}{2})}{3}.$$

Note that $a \geq 1$ and $\frac{b+c}{2} \geq \sqrt{bc} \geq 1$. Since

$$f''(x) = \frac{2(1-x^2)}{(1+x^2)^2},$$

we know that f is concave downward on $[1, \infty)$. Then we can apply Jensen's theorem to get

$$\frac{f(a)+2f(\frac{b+c}{2})}{3} \leq f\left(\frac{a+2\cdot\frac{b+c}{2}}{3}\right) = \left(\frac{a+b+c}{3}\right).$$

This completes the proof.

Solution 2. Without loss of generality, assume $a \geq b \geq c$. As $\min\{ab, bc, ca\} \geq 1$, this implies $a \geq 1$. Let $d = \frac{1}{3}(a+b+c)$. Note that

$$ad = \frac{a(a+b+c)}{3} \geq \frac{1+1+1}{3} = 1.$$

Then we can apply (6.2) to the pair (a,d) and the pair (b,c) to get

$$(a^2+1)(d^2+1)(b^2+1)(c^2+1) \le \left(\left(\frac{a+d}{2}\right)^2+1\right)^2 \left(\left(\frac{b+c}{2}\right)^2+1\right)^2 \qquad (6.5)$$

Next from

$$\frac{a+d}{2}\cdot\frac{b+c}{2} \ge \sqrt{ad}\cdot\sqrt{bc} \ge 1,$$

we can apply (6.2) again to the pair $(\frac{a+d}{2}, \frac{b+c}{2})$. Together with (6.5), we have

$$(a^2+1)(d^2+1)(b^2+1)(c^2+1) \le \left(\left(\frac{a+b+c+d}{4}\right)^2+1\right)^4 = (d^2+1)^4.$$

Therefore, $(a^2+1)(b^2+1)(c^2+1) \le (d^2+1)^3$, and (6.1) follows by taking cube root of both sides.

7. Let a be a positive integer which is not a square number. Denote by A the set of all positive integers k such that

$$k = \frac{x^2-a}{x^2-y^2} \qquad (7.1)$$

for some integers x and y with $x > \sqrt{a}$. Denote by B the set of all positive integers k such that (7.1) is satisfied for some integers x and y with $0 \le x < \sqrt{a}$. Prove that $A = B$.

Solution 1. First we prove the following claim.

Claim. For fixed k, let x, y be integers satisfying (7.1). Then the numbers x_1, y_1 defined by

$$x_1 = \frac{1}{2}\left(x-y+\frac{(x-y)^2-4a}{x+y}\right), \quad y_1 = \frac{1}{2}\left(x-y-\frac{(x-y)^2-4a}{x+y}\right)$$

are integers satisfying (7.1) (with x, y replaced by x_1, y_1 respectively.)

Proof. Since $x_1 + y_1 = x - y$ and

$$x_1 = \frac{x^2-xy-2a}{x+y} = -x + \frac{2(x^2-a)}{x+y} = -x + 2k(x-y),$$

both x_1 and y_1 are integers. Let $u = x+y$ and $v = x-y$. The relation (7.1) can be rewritten as

$$u^2 - (4k-2)uv + (v^2-4a) = 0.$$

By Vieta's theorem, the number $z = \frac{v^2-4a}{u}$ satisfies

$$v^2 - (4k-2)vz + (z^2-4a) = 0.$$

Since x_1 and y_1 are defined so that $v = x_1 + y_1$ and $z = x_1 - y_1$, we can reverse the process and verify (7.1) for x_1, y_1. This completes the proof of the claim.

We first show that $B \subset A$. Take any $k \in B$ so that (7.1) is satisfied for some integers x, y with $0 \le x < \sqrt{a}$. Clearly $y \ne 0$ as $k > 0$ and we may assume y is positive. Since a is not a square, we have $k > 1$. Hence, we get $0 \le x < y < \sqrt{a}$. Define

$$x_1 = \frac{1}{2}\left|x - y + \frac{(x-y)^2 - 4a}{x+y}\right|, \ y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right).$$

By the claim, x_1, y_1 are integers satisfying (7.1). Also we have

$$x_1 = -\frac{1}{2}\left(x - y + \frac{(x-y)^2 - 4a}{x+y}\right) = \frac{2a + x(y-x)}{x+y} \geq \frac{2a}{x+y} > \sqrt{a}.$$

this implies $k \in A$ and hence $B \subset A$.

Next, we show that $A \subset B$. Take any $k \in A$ so that (7.1) is satisfied for some integers x, y with $x > \sqrt{a}$. Again we may assume y is positive. Among all such representations of k, we choose the one with the smallest $x + y$. Define

$$x_1 = \frac{1}{2}\left|x - y + \frac{(x-y)^2 - 4a}{x+y}\right|, \ y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right).$$

By the claim, x_1, y_1 are integers satisfying (7.1). Since $k > 1$, we get $x > y > \sqrt{a}$. Therefore, we have $y_1 > \frac{4a}{x+y} > 0$ and $\frac{4a}{x+y} < x + y$. It follows that

$$x_1 + y_1 \leq \max\left\{x - y, \frac{4a - (x-y)^2}{x+y}\right\} < x + y.$$

If $x_1 > \sqrt{a}$, we get a contradiction due to the minimality of $x + y$. Therefore, we must have $0 \leq x_1 < \sqrt{a}$, which means $k \in B$ so that $A \subset B$.

The two subset relations combine to give $A = B$.

Solution 2. The relation (7.1) is equivalent to

$$ky^2 - (k-1)x^2 = a. \tag{7.2}$$

Claim. If (x_0, y_0) is a solution to (7.2), then $((2k-1)x_0 \pm 2ky_0, (2k-1)y_0 \pm 2(k-1)x_0)$ is also a solution to (7.2).

Proof. We check directly that

$$\begin{aligned} &k((2k-1)y_0 \pm 2(k-1)x_0)^2 - (k-1)((2k-1)x_0 \pm 2ky_0)^2 \\ &= (k(2k-1)^2 - (k-1)(2k)^2)y_0^2 + (k(2(k-1))^2 - (k-1)(2k-1)^2)x_0^2 \\ &= ky_0^2 - (k-1)x_0^2 = a. \end{aligned}$$

If (7.2) is satisfied for some $0 \leq x < \sqrt{a}$ and nonnegative integer y, then clearly (7.1) implies $y > x$ so that $y \geq x + 1$. Also, we have $k > 1$ since a is not square number. By the claim, consider another solution to (7.2) defined by

$$x_1 = (2k-1)x + 2ky, \ \ y_1 = (2k-1)y + 2(k-1)x.$$

It satisfies $x_1 \geq (2k-1)x + 2k(x+1) = (4k-1)x + 2k > x$. Then we can replace the old solution by a new one which has a larger value in x. After a finite number of replacements, we must get a solution with $x > \sqrt{a}$. This shows $B \subset A$.

If (7.2) is satisfied for some $x > \sqrt{a}$ and nonnegative integer y, by the claim we consider another solution to (7.2) defined by

$$x_1 = |(2k-1)x - 2ky|, \quad y_1 = (2k-1)y - 2(k-1)x.$$

From (7.2), we get $\sqrt{k}y > \sqrt{k-1}x$. This implies $ky > \sqrt{k(k-1)}x > (k-1)x$ and hence $(2k-1)x - 2ky < x$. On the other hand, the relation (7.1) implies that $x > y$. Then it is clear that $(2k-1)x - 2ky > -x$. These combine to give $x_1 < x$, which means we have found a solution to (7.2) with x a smaller absolute value. After a finite number of steps, we shall obtain a solution with $0 \le x < \sqrt{a}$. This shows $A \subset B$.

The desired result follows from $B \subset A$ and $A \subset B$.

Day 2.

8. Let ABC be a triangle with circumcircle Γ and incentre I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to side BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Solution. Let AM meet Γ again at Y and XY meet BC at D'. It suffices to show $D' = D$. We shall apply the following fact.

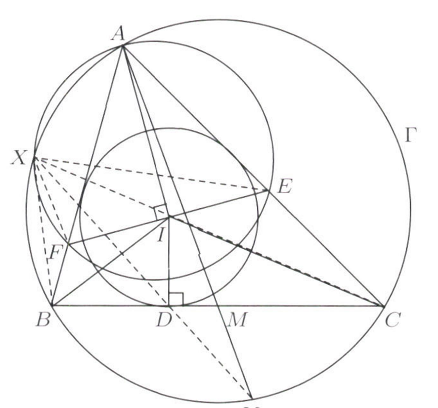
Claim. For any cyclic quadrilateral $PQRS$ whose diagonals meet at T, we have

$$\frac{QT}{TS} = \frac{PQ \cdot QR}{PS \cdot SR}.$$

Proof. We use $[W_1W_2W_2]$ to denote the area of $W_1W_2W_3$. Then

$$\frac{QT}{TS} = \frac{[PQR]}{[PSR]} = \frac{\frac{1}{2}PQ \cdot QR \sin\angle PQR}{\frac{1}{2}PS \cdot SR \sin\angle PSR} = \frac{PQ \cdot QR}{PS \cdot SR}.$$

This proves the claim.



Applying the claim to $ABYC$ and $XBYC$ respectively, we have $1 = \frac{BM}{MC} = \frac{AB \cdot BY}{AC \cdot CY}$ and $\frac{BD'}{D'C} = \frac{XB \cdot BY}{XC \cdot CY}$. These combine to give

$$\frac{BD'}{CD'} = \frac{XB}{XC} \cdot \frac{BY}{CY} = \frac{XB}{XC} \cdot \frac{AC}{AB} \tag{8.1}$$

Next, we use directed angles to find that $\angle XBF = \angle XBA = \angle XCA = \angle XCE$ and $\angle XFB = \angle XFA = \angle XEA = \angle XEC$. This shows that triangles XBF and XCE are directly similar. In particular, we have

$$\frac{XB}{XC} = \frac{BF}{CE}. \tag{8.2}$$

Let $\beta = \frac{1}{2}\angle ABC$ and $\gamma = \frac{1}{2}\angle ACB$. Observe that $\angle FIB = \angle AIB - 90^\circ = \gamma$. Hence, $\frac{BF}{FI} = \frac{\sin \angle FIB}{\sin \angle IBF} = \frac{\sin \gamma}{\sin \beta}$. Similarly, $\frac{CE}{EI} = \frac{\sin \beta}{\sin \gamma}$. As $EI = FI$, we get

$$\frac{BF}{CE} = \frac{BF}{FI} \cdot \frac{EI}{CE} = \left(\frac{\sin \gamma}{\sin \beta}\right)^2 \tag{8.3}$$

Together with (8.1) and (8.2), we find that

$$\frac{BD'}{CD'} = \frac{AC}{AB} \cdot \left(\frac{\sin \gamma}{\sin \beta}\right)^2 = \frac{\sin 2\beta}{\sin 2\gamma} \cdot \left(\frac{\sin \gamma}{\sin \beta}\right)^2 = \frac{\tan \gamma}{\tan \beta} = \frac{ID/CD}{ID/BD} = \frac{BD}{CD}.$$

This shows that $D' = D$ and the result follows.

9. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integers $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Answer. $f(n) = n^2$ for any $n \in \mathbb{N}$.

Solution. It is given that

$$f(m) + f(n) - mn \mid mf(m) + nf(n) \tag{9.1}$$

Taking $m = n = 1$ in (9.1), we have $2f(1) - 1 \mid 2f(1)$. Then $2f(1) - 1 \mid 2f(1) - (2f(1) - 1) = 1$ and hence $f(1) = 1$.

Let $p \geq 7$ be a prime. Taking $m = p$ and $n = 1$ in (9.1), we have $f(p) - p + 1 \mid pf(p) + 1$ and hence

$$f(p) - p + 1 \mid pf(p) + 1 - p(f(p) - p + 1) = p^2 - p + 1.$$

If $f(p) - p + 1 = p^2 - p + 1$, then $f(p) = p^2$. If $f(p) - p + 1 \neq p^2 - p + 1$, as $p^2 - p + 1$ ia an odd positive integer, we have $p^2 - p + 1 \geq 3(f(p) - p + 1)$, that is

$$f(p) \leq \frac{1}{3}(p^2 + 2p - 2) \tag{9.2}$$

Taking $m = n = p$ in (9.1), we have $2f(p) - p^2 \mid 2pf(p)$. This implies

$$2f(p) - p^2 \mid 2pf(p) - p(2f(p) - p^2) = p^3.$$

By (9.2) and $f(p) \geq 1$, we get

$$-p^2 \leq 2f(p) - p^2 \leq \frac{2}{3}(p^2 + 2p - 2) - p^2 < -p$$

since $p \geq 7$. This contradicts the fact that $2f(p) - p^2$ is a factor of p^3. Thus we have proved that $f(p) = p^2$ for all primes $p \geq 7$.

Let n be a fixed positive integer. Choose a sufficiently large prime p. Consider $m = p$ in (9.1). We obtain

$$f(p) + f(n) - pn \mid pf(p) + nf(n) - n(f(p) + f(n) - pn) = pf(p) - nf(p) + pn^2.$$

As $f(p) = p^2$, this implies $p^2 - pn + f(n) \mid p(p^2 - pn + n^2)$. As p sufficiently large and n is fixed, p cannot divide $f(n)$, and so $(p, p^2 - pn + f(n)) = 1$. It follows that $p^2 - pn + f(n) \mid p^2 - pn + n^2$ and hence

$$p^2 - pn + f(n) \mid p^2 - pn + n^2 - (p^2 - pn + f(n)) = n^2 - f(n).$$

Note that $n^2 - f(n)$ is fixed while $p^2 - pn + f(n)$ is chosen to be sufficiently large. Therefore, we must have $n^2 - f(n) = 0$ so that $f(n) = n^2$ for any positive integer n. Finally, we check that when $f(n) = n^2$ for any positive integer n, we have

$$f(m) + f(n) - mn = m^2 + n^2 - mn,$$

and

$$mf(m) + nf(n) = m^3 + n^3 = (m + n)(m^2 + n^2 - mn).$$

The latter expression is divisible by the former for any positive integers m, n. This shows that $f(n) = n^2$ is the only solution.

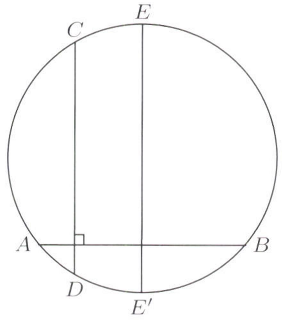
10. Let $n \geq 3$ be an integer. Find the maximum number of diagonals of a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

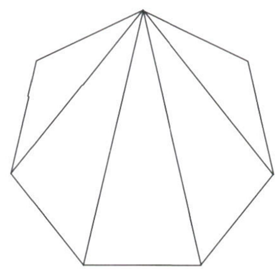
Answer. $n - 2$ if n is even and $n - 3$ if n is odd.

Solution 1. We consider two cases according to the parity of n.

Case 1. n is odd.

We first claim that no pair of diagonals is perpendicular. Suppose A, B, C, D are vertices where AB and CD are perpendicular, and let E be the vertex lying on the perpendicular bisector of AB. Let E' be the opposite point of E on the circumcircle of the regular polygon. Since $EC = E'D$ and C, D, E are vertices of the regular polygon, E' should also belong to the polygon. This contradicts the fact that a regular polygon with an odd number of vertices does not contain opposite points on the circumcircle.



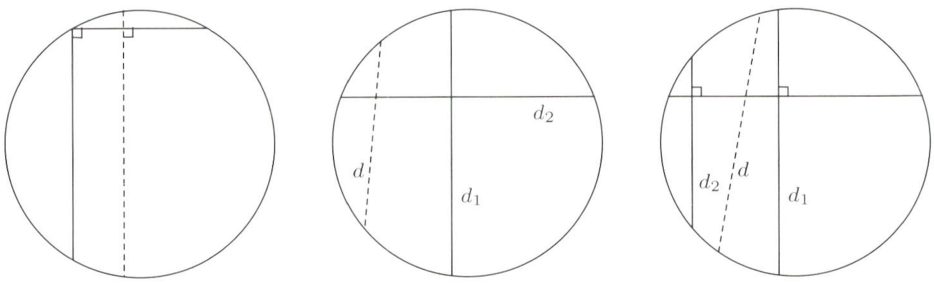


Therefore in the odd case we can only select diagonals which do not intersect. In the maximal case these diagonals should divide the regular n-gon into $n-2$ triangles, so we can select at most $n-3$ diagonals. This can be done, for example, by selecting all diagonals emanated from a particular vertex.

Case 2. n is even.

If there is no intersection, then the proof in the odd case works. Suppose there are 2 perpendicular diagonals selected. We consider the set S of all selected diagonals parallel to one of them which intersect with some selected diagonals. Suppose S contains k diagonals and the number of distinct endpoints of the k diagonals is ℓ.

Firstly, consider the longest diagonal in one of the two directions in S. No other diagonal in S can start from either endpoint of that diagonal, since otherwise it has to meet another longer diagonal in S. The same holds true for the other direction. Ignoring these 2 longest diagonals and their 4 endpoints, the remaining $k-2$ diagonals share $\ell-4$ endpoints where each endpoint can belong to at most 2 diagonals. This gives $2(\ell-4) \geq 2(k-2)$, so that $k \leq \ell - 2$.

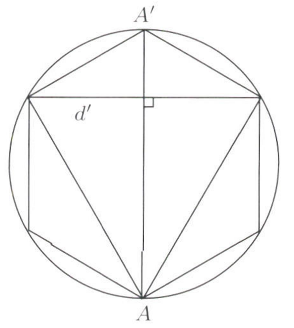


Consider a group of consecutive vertices of the regular n-gon so that each of the two outermost vertices is an endpoint of a diagonal in S, while the interior points are not. There are ℓ such groups. We label these groups $P_1, P_2, \ldots, P_\ell$ in this order. We claim that each selected diagonal outside S must connect vertices of the same group P_i. Consider any diagonal d joining vertices from distinct groups P_i and P_j. Let d_1 and d_2 be two diagonals in S each having one of the outermost points of P_i as endpoint. Then d must meet either d_1, d_2 or a diagonal in S which is perpendicular to both d_1 and d_2. In any case, d should belong to S by definition, which is a contradiction.

Within the same group P_i, there are no perpendicular diagonals since the vertices belong to the same side of a diameter of the cicumcircle. Hence there can be at most $|P_i|-2$ selected diagonals within P_i, including the one joining the two outermost points of P_i, when $|P_i| > 2$. Therefore, the maximum number of diagonals selected is

$$\sum_{i=1}^{\ell}(|P_i|-2)+k=\sum_{i=1}^{\ell}|P_i|-2\ell+k=(n+\ell)-2\ell+k=n-\ell+k\leq n-2.$$

This upper bound can be attained as follows. We take any vertex A and let A' be the vertex for which AA' is a diameter of the circumcircle. If we select all diagonals emanated from A together with the diagonals d' joining the two neighbouring vertices of A', then the only pair of diagonals that meet each other is AA' and d', which are perpendicular to each other. In total we can take $n-2$ diagonals.



Solution 2. Let's consider the even case and we shall prove that there are no more than $n-2$ diagonals that satisfy the requirement. Now suppose that there are intersecting diagonals. (When we say 2 diagonals meet, it means that they meet at the interior.) Without loss of generality, let a pair of intersecting diagonals be an horizontal-vertical pair. Then every pair of intersecting diagonals must also be a horizontal-vertical pair (This is easy to see).

Let S be the set of vertical and horizontal diagonals such that each must intersect another. Suppose L is the set of vertices such that each belongs to a member of S. Let $|L| = \ell$. Let A be an endpoint of the longest horizontal diagonal, Then A does not belong to any vertical diagonal in S. This is because if it does, then this vertical diagonal must intersect another horizontal diagonal in S and this horizontal diagonal would be longer than the one we started with.

This means that the number of vertical diagonal is at most $\lfloor(\ell-2)/2\rfloor$. Similarly, the number of horizontal diagonals in S is also $\lfloor(\ell-2)/2\rfloor$. Hence $|S| \leq \ell - 2$.

The vertices in L divide the vertices of the polygon into ℓ groups where a group consists of two consecutive vertices in L and the vertices between them. Let $n_1, \ldots, n_\ell$ be the number of vertices in the groups. It is clear that the diagonal joining the two vertices in L of each group is not in S, (except the one joining the 2 outermost vertices of the group) and that the diagonals in each group do not intersect. Moreover, there are no diagonals that join two vertices not in L and belong to two different groups. Therefore the total number of diagonals in the groups are $n_i - 2$. Hence the total number of diagonals is $\leq \sum_{i=1}^{\ell}(n_i - 2) + |S| = n + \ell - 2\ell + (\ell - 2) = n - 2$.