Trigonometric Identities

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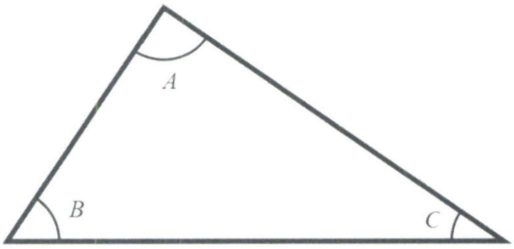
1 Introduction

It is a well-known result that [1]:

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

where

$$A + B + C = \pi$$



The inequality is true for any $\triangle ABC$.
We generalise the above result to all n-gons $A_1 A_2 \cdots A_n$ as follows:
For $n \geq 3$ and $\sum\limits_{i=1}^{n} A_i = (n-2)\pi$ where $0 < A_i < 2\pi$, let

$$S_n = \sum_{i=1}^{n} \sin A_i,$$

$\alpha = 4.49340945790906$, $v_1 = \lfloor n - \frac{\pi(n+2)}{\alpha} \rfloor$, $v_2 = \lceil n - \frac{\pi(n+2)}{\alpha} \rceil$. Then

$$S_n \begin{cases} \leq n \sin\left(\frac{2\pi}{n}\right) & \text{for all convex } n\text{-gons or } n \leq 6 \\ < \max\{(n-v_1)\sin\frac{\pi(2+v_1)}{n-v_1}, (n-v_2)\sin\frac{\pi(2+v_2)}{n-v_2}\} & \text{for all } n\text{-gons when } n > 6 \end{cases}$$

Note that the constant α is a numerical approximation as we were not able to find the exact form of α.

2 Concepts and Theorems

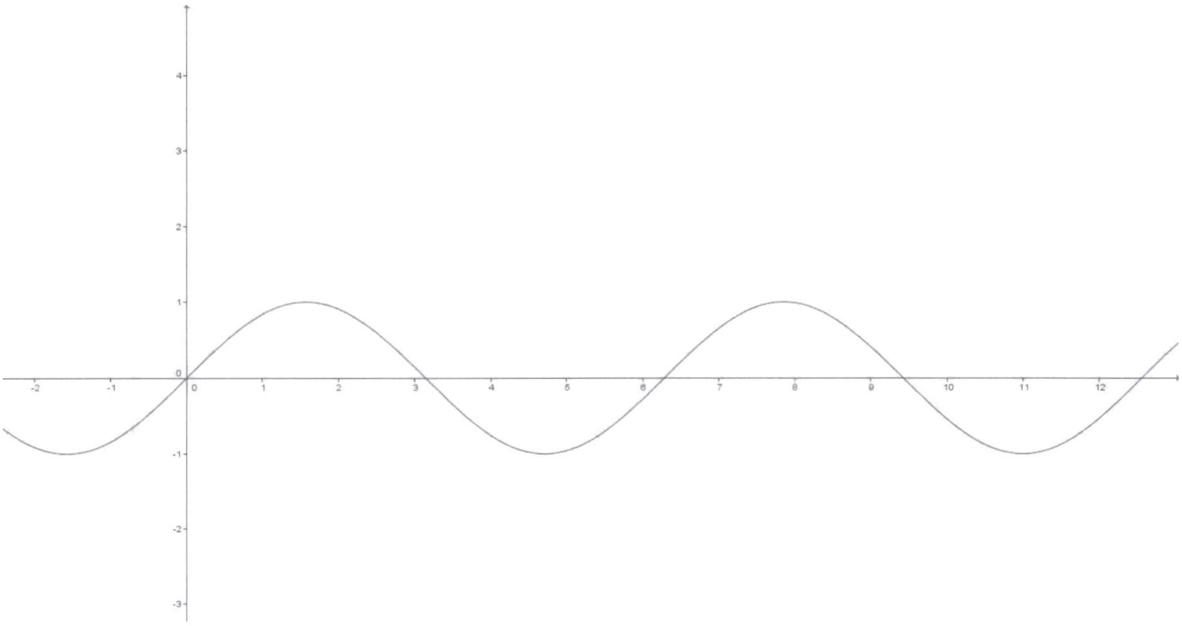
2.1 Concave functions

[2] Given a continuous function $f(x)$, it is concave downwards in the interval $[a, b]$ if for any $x_1, x_2 \in [a, b]$ such that

$$f[\lambda x_1 + (1 - \lambda)x_2] \geq \lambda f(x_1) + (1 - \lambda) f(x_2)$$

Another way to determine if a function is concave downwards is that its second derivative (if it exists in the interval $[a, b]$) is non-positive.
Another method of determining if a function is concave downwards in the interval $[a, b]$ is by drawing a line from $f(x_1)$ to $f(x_2)$ for any $x_1, x_2 \in [a, b]$ and that it always lies below the graph.

2.2 Sine function



In the interval of $[0, \pi]$, $\sin(x) \geq 0$

$$\sin'(x) = \cos(x)$$

$$\sin''(x) = \cos'(x) = -\sin(x) \leq 0$$

As such, $\sin(x)$ is a function that is concave downwards on the interval $[0, \pi]$.
Also, when x is reflex (i.e. $x \in (\pi, 2\pi)$), then $\sin x < 0$. $\sin x$ increases in the interval $[0, \frac{\pi}{2}]$, and the maximum of $\sin x$ is 1 when $x = \frac{\pi}{2}$.

2.3 Jensen's Inequality

[3] Given a continuous function $f(x)$ that is concave downwards and $x_1, \ldots, x_n \in \mathbb{R}$, we have

$$\sum_{i=1}^{n} \frac{f(x_i)}{n} \leq f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)$$

Equality holds if $x_1 = x_2 = \cdots = x_n$ or if $f(x)$ is linear.

3 Proof of Inequality for S_3

The function $f(\theta) = \sin(\theta)$ is a function that is concave downwards for $0 \leq \theta \leq \pi$. Hence, by Jensen's Inequality

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A+B+C}{3}\right) = \frac{\sqrt{3}}{2}$$

Multiplying by 3 on both sides will give the desired inequality. Equality holds if and only if $A = B = C = \frac{\pi}{3}$, that is, the triangle is equilateral.

4 Upper bound for S_4

The upper bound of the inequality for a quadrilateral is:

$$\sin A + \sin B + \sin C + \sin D \leq 4$$

Equality holds when $A = B = C = D = \frac{\pi}{2}$.

This is the maximum as the range of $\sin x$ is [-1,1], and changing any angle will definitely result in the decrease of the value of the whole expression.

5 Upper bound for convex polygons

In a convex polygon $A_1A_2\cdots A_n$, each angle is less than π, and since the function $f(\theta)=\sin\theta$ is a function that is concave downwards for $0\le\theta\le\pi$, we can apply Jensen's Inequality and obtain:

$$\frac{1}{n}\sum_{i=1}^{n}\sin A_i \le \sin\left(\frac{(n-2)\pi}{n}\right)=\sin\left(\pi-\frac{2\pi}{n}\right)=\sin\left(\frac{2\pi}{n}\right)$$

6 Consideration of reflex angles in a polygon

For a polygon with number of sides ≥ 4, there may exist at least 1 reflex angle.

Lemma 6.1. *For a polygon with n sides, there are at most $n-3$ reflex angles.*

Proof. Suppose otherwise (i.e. there can be $n-2$ or more reflex angles).
The sum of all the interior angles in the polygon is $(n-2)\pi$.
Now with $n-2$ or more reflex angles, the sum of angles is now more than $(n-2)\pi$, a contradiction. □

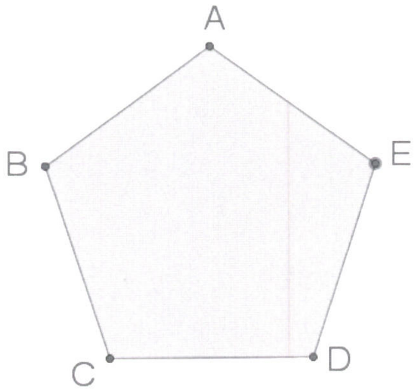
7 Upper bound for S_5

We claim that the upper bound of the inequality for a pentagon is:

$$\sin A+\sin B+\sin C+\sin D+\sin E\le 5\sin\left(\frac{3\pi}{5}\right)=\frac{5\sqrt{10+2\sqrt{5}}}{4}$$

We will first assume a convex pentagon, then let one or two of the angles be reflex. Note that it is impossible to have 3 or more reflex angles in a pentagon.

7.1 Case 1: Convex Pentagon



Since the function $f(\theta) = \sin(\theta)$ is a function that is concave downwards for $0 \leq \theta \leq \pi$, thus we can use Jensen's Inequality and obtain:

$$\frac{\sin A + \sin B + \sin C + \sin D + \sin E}{5} \leq \sin\left(\frac{A+B+C+D+E}{5}\right) = \sin\left(\frac{3\pi}{5}\right)$$

Multiplying both sides by 5 will give us the desired inequality. Equality holds when all the angles are equal to $\frac{3\pi}{5}$.

7.2 Case 2: Concave Pentagon with 1 reflex angle

Without loss of generality, let E be the reflex angle. Since $\pi < E < 2\pi$, then

$$\pi < A + B + C + D < 2\pi$$

As such, using Jensen's Inequality, we obtain

$$\frac{\sin A + \sin B + \sin C + \sin D}{4} \leq \sin\left(\frac{A+B+C+D}{4}\right)$$

Observe that $\frac{\pi}{4} < \frac{A+B+C+D}{4} < \frac{\pi}{2}$, as such, we can conclude that

$$\sin\left(\frac{A+B+C+D}{4}\right) < 1$$

Also, since E is reflex, then $\sin E < 0$, as such

$$\sin A + \sin B + \sin C + \sin D + \sin E < \sin A + \sin B + \sin C + \sin D < 4 < 5\sin\left(\frac{3\pi}{5}\right)$$

7.3 Case 3: Concave Pentagon with 2 reflex angles

Without loss of generality, assume D and E are the reflex angles. Since $\pi < D, E < 2\pi$, then

$$0 < A + B + C < \pi$$

As such, using Jensen's Inequality, we obtain

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A+B+C}{3}\right)$$

Observe that $0 < \frac{A+B+C}{3} < \frac{\pi}{3}$, as such, we can conclude that

$$\sin\left(\frac{A+B+C}{3}\right) < \frac{\sqrt{3}}{2}$$

Also, since D and E are reflex, then $\sin D, \sin E < 0$, as such

$$\sin A + \sin B + \sin C + \sin D + \sin E < \sin A + \sin B + \sin C < \frac{3\sqrt{3}}{2} < 5\sin\left(\frac{3\pi}{5}\right)$$

8 Upper bound for S_6

We claim that the upper bound of the inequality for a hexagon is:

$$\sin A + \sin B + \sin C + \sin D + \sin E + \sin F \leq 6 \sin\left(\frac{2\pi}{3}\right) = 3\sqrt{3}$$

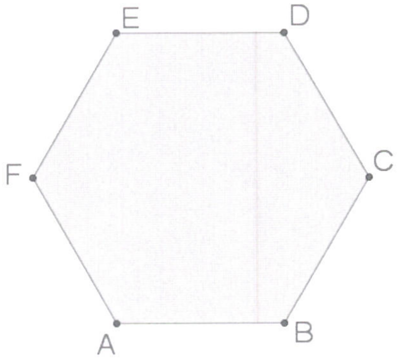
We will first assume a convex hexagon, then let one, two and three of the angles be reflex.

8.1 Case 1: Convex Hexagon

Since sin is a function that is concave downwards on the interval $[0, \pi]$, then we can use Jensen's Inequality and obtain:

$$\frac{\sin A + \sin B + \sin C + \sin D + \sin E + \sin F}{6} \leq \sin\left(\frac{A+B+C+D+E+F}{6}\right) = \sin\left(\frac{2\pi}{3}\right)$$

Multiplying both sides by 6 will give us the desired inequality.



8.2 Case 2: Concave Hexagon with 1 reflex angle

Without loss of generality, assume F is the reflex angle. Since $\pi < F < 2\pi$, then

$$2\pi < A + B + C + D + E < 3\pi$$

As such, using Jensen's Inequality, we obtain

$$\frac{\sin A + \sin B + \sin C + \sin D + \sin E}{5} \leq \sin\left(\frac{A + B + C + D + E}{5}\right)$$

Observe that $\frac{2\pi}{5} < \frac{A+B+C+D+E}{5} < \frac{3\pi}{5}$, as such, we can conclude that

$$\sin\left(\frac{A + B + C + D + E}{5}\right) \leq 1$$

Also, since F is reflex, then $\sin F < 0$, as such

$$\sin A+\sin B+\sin C+\sin D+\sin E+\sin F < \sin A+\sin B+\sin C+\sin D+\sin E \leq 5 < 3\sqrt{3}$$

8.3 Case 3: Concave Hexagon with 2 reflex angles

Without loss of generality, assume E and F are the reflex angles. Since $\pi < E, F < 2\pi$, then

$$0 < A + B + C + D < 2\pi$$

As such, using Jensen's Inequality, we obtain

$$\frac{\sin A + \sin B + \sin C + \sin D}{4} \leq \sin\left(\frac{A + B + C + D}{4}\right)$$

Observe that $0 < \frac{A+B+C+D}{4} < \frac{\pi}{2}$, as such, we can conclude that

$$\sin\left(\frac{A + B + C + D}{4}\right) < 1$$

Also, since E and F are reflex, then $\sin E, \sin F < 0$, as such

$$\sin A+\sin B+\sin C+\sin D+\sin E+\sin F < \sin A+\sin B+\sin C+\sin D < 4 < 3\sqrt{3}$$

This is a lower upper bound than before.

8.4 Case 4: Concave Hexagon with 3 reflex angles

Without loss of generality, assume D,E and F are the reflex angles. Since $\pi < D, E, F < 2\pi$, then

$$0 < A + B + C < \pi$$

As such, using Jensen's Inequality, we obtain

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A + B + C}{3}\right)$$

Observe that $0 < \frac{A+B+C}{3} < \frac{\pi}{3}$, as such, we can conclude that

$$\sin\left(\frac{A+B+C}{3}\right) < \frac{\sqrt{3}}{2}$$

Also, since D,E and F are reflex, then $\sin D, \sin E, \sin F < 0$, as such

$$\sin A+\sin B+\sin C+\sin D+\sin E+\sin F < \sin A+\sin B+\sin C < \frac{3\sqrt{3}}{2} < 3\sqrt{3}$$

This is a lower upper bound than before.

9 Upper bound for S_n

In this section, we will find the upper bound for the sum of sines for all integers n.

9.1 Global upper bound for an equiangular n-gon

Observation 9.1. *The sum of sines in an equiangular n-gon always increases as n increases.*

Proof. We want to show that the function $f(x) = x\sin(\pi - \frac{2\pi}{x})$ is always increasing.

$$\begin{aligned} f(x) &= x\sin\left(\pi - \frac{2\pi}{x}\right) \\ &= x\sin\left(\frac{2\pi}{x}\right) \end{aligned}$$

Now we compute $f'(x)$.

$$\begin{aligned} f'(x) &= x\frac{d}{dx}\left[\sin\left(\frac{2\pi}{x}\right)\right] + \sin\left(\frac{2\pi}{x}\right)\frac{d}{dx}[x] \\ &= \sin\left(\frac{2\pi}{x}\right) - \frac{2\pi\cos\left(\frac{2\pi}{x}\right)}{x} \end{aligned}$$

We want to show that $f'(x) \geq 0$.

Lemma 9.2. $\tan(x) \geq x$ *if* $0 \leq x \leq \pi$.

Proof. Let $g(x) = x - \tan(x)$, $0 < x < \pi/2$.

$$g'(x) = 1 - \sec^2(x) = -\tan^2(x) \leq 0$$

Hence $g(x)$ decreases on $0 < x < \pi/2$. Thus we have $g(0) > g(x)$ or $g(x) < 0$. ☐

Now for $x \geq 4$, $\frac{2\pi}{x} \leq \frac{\pi}{2}$. From this, we get $\cos(\frac{2\pi}{x}) \geq 0$. Hence

$$\begin{aligned} \tan\left(\frac{2\pi}{x}\right) &\geq \frac{2\pi}{x} \\ \sin\left(\frac{2\pi}{x}\right) &\geq \frac{2\pi}{x}\cos\left(\frac{2\pi}{x}\right) \\ \sin\left(\frac{2\pi}{x}\right) - \frac{2\pi\cos\left(\frac{2\pi}{x}\right)}{x} &\geq 0 \\ f'(x) &\geq 0 \end{aligned}$$

Hence $f'(x) \geq 0$, showing that $f(x)$ is increasing on the interval $[4, \infty)$. Thus the sum of sines of an equiangular n-gon always increases as n increases. ☐

Lemma 9.3.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Theorem 9.4. *The sum of sines in an equiangular n-gon is $\leq 2\pi$.*

Proof. We know that the sum of sines in a equiangular n-gon increases as n increases. The maximum value of the sum of sines in an n-gon is defined by the formula $n\sin\left(\pi - \frac{2\pi}{n}\right) = n\sin\frac{2\pi}{n}$.

$$\begin{aligned} \lim_{n\to\infty} n\sin\frac{2\pi}{n} &= 2\pi \lim_{n\to\infty} \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} \\ &= 2\pi \lim_{x\to 0} \frac{\sin x}{x} \\ &= 2\pi \cdot 1 = 2\pi \end{aligned}$$

Thus the sum of sines is bounded by 2π, showing that the global upper bound of the sum of sines of a convex n-gon is 2π. ☐

9.2 Upper bound of S_n for a concave n-gon

Although our earlier pattern shows that for any n-gon, the sum of sines of the individual angles is maximised when the n-gon is equiangular, this is not the case for a concave n-gon when $n \geq 7$.

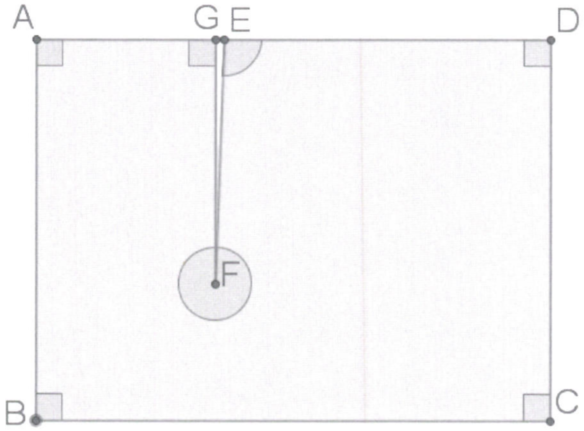
9.2.1 Upper bound for S_7

We see that a convex heptagon will have

$$\sum_{i=1}^{7} \sin A_i \leq 7\sin\left(\frac{5\pi}{7}\right) \approx 5.473$$

However, it is possible to obtain a even higher upper bound by considering a concave heptagon.

Consider the following diagram:



$\angle GFE$ can go arbitrarily close to 2π, and $\angle FED$ can go arbitrarily close to $\frac{\pi}{2}$. Hence, the sum of sines of the individual angles in the above heptagon will be

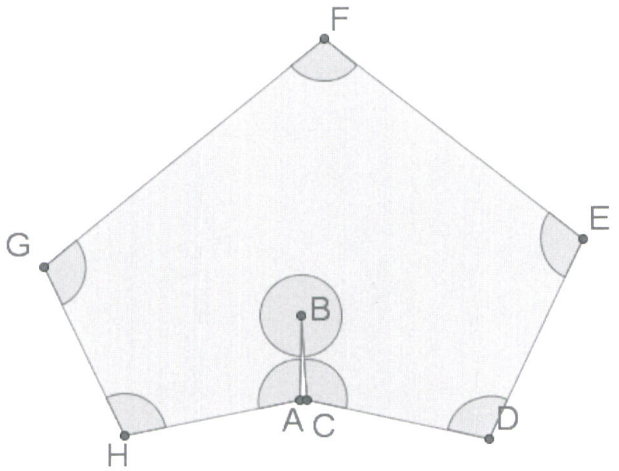
$$\begin{aligned}\sum_{i=1}^{7} \sin A_i &= 5\sin\frac{\pi}{2} + \sin\angle GFE + \sin\angle FED \\ &\approx 5 + 1 + 0 = 6\end{aligned}$$

which is a higher upper bound than 5.473.
Notice that we can get arbitrarily close to 6 by making the angle $\angle FED$ closer to $\frac{\pi}{2}$, but we can never be able to reach the exact value of 6.
This is indeed a higher upper bound than the one obtained using an equiangular heptagon.
Notice that there are many right angles in the figure, hence maximizing the sum of sines of individual angles. Also, the reflex angles are close to π or 2π. As such, they are very close to 0.
Is 6 the upper bound for a heptagon?
Since we have considered the cases where there are no reflex angles and 1 reflex angle, then we need to consider the cases with ≥ 2 reflex angles. However, this would mean that there are ≤ 5 non-reflex angles and as such, the sum of Sines must be less than 5. Therefore, the upper bound is 6.

9.2.2 Upper bound for S_8

As with the case $n = 7$, we can exceed the bound for a convex octagon of $8\sin\left(\frac{2\pi}{8}\right) = 4\sqrt{2}$.

Consider the following diagram:



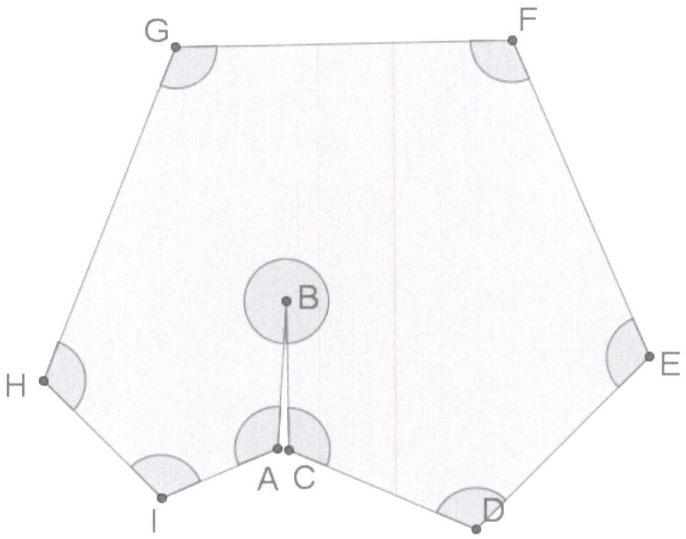
In the diagram above, $\angle BAH = \angle AHG = \angle HGF = \angle GFE = \angle FED = \angle EDC = \frac{4\pi}{7}$. $\angle ABC$ can go arbitrarily close to 2π and $\angle BCD$ can go arbitrarily close to $\frac{4\pi}{7}$. This is maximum for one reflex angle, as by using Jensen's inequality, we get that the sum of sines is maximised when all the angles are equal. There are 7 angles, each being $\frac{4\pi}{7}$, and the last angle being 2π. The sum of the sines of individual angles is

$$\begin{aligned}\sum_{i=1}^{8} \sin A_i &= 6\sin\frac{4\pi}{7} + \sin\angle ABC + \sin\angle BCD \\ &\approx 7\sin\frac{4\pi}{7} \approx 6.824\end{aligned}$$

which is higher than $4\sqrt{2} \approx 5.657$. Since we have considered the cases where there are no reflex angles and 1 reflex angle, then we need to consider the cases with ≥ 2 reflex angles. However, this would mean that there are ≤ 6 non-reflex angles and as such, the sum of Sines must be less than 6. Hence the upper bound for $n = 8$ is around 6.824.

9.2.3 Upper bound for S_9

As before, we can exceed the bound for a convex nonagon of $9\sin\left(\frac{2\pi}{9}\right)$ Consider the following diagram:



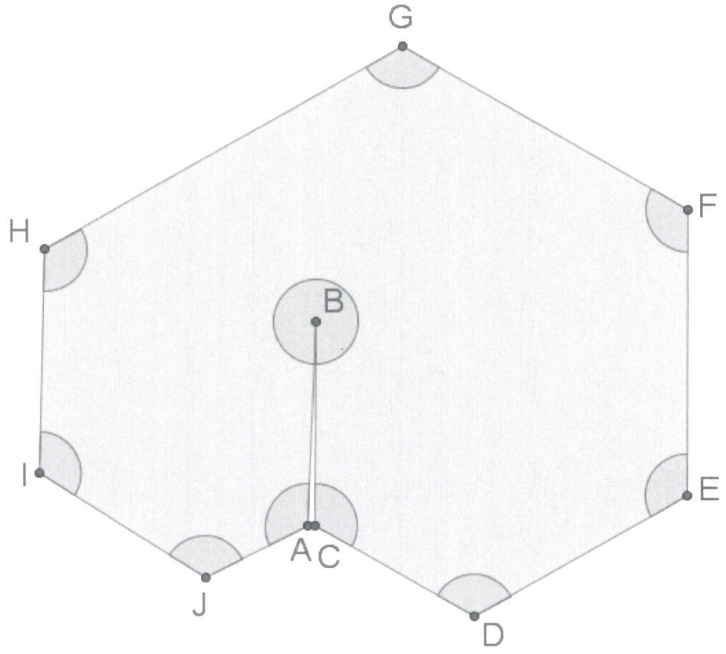
In the diagram, $\angle ABC$ can go arbitrarily close to 2π, and $\angle BCD$ can go arbitrarily close to $\frac{5\pi}{8}$. The rest of the angles are $\frac{5\pi}{8}$. The sum of sines is thus

$$\begin{aligned}\sum_{i=1}^{9} \sin A_i &= 7 \sin \frac{5\pi}{8} + \sin \angle ABC + \sin \angle BCD \\ &\approx 8 \sin \frac{5\pi}{8} \approx 7.391\end{aligned}$$

which exceeds the upper bound for a convex nonagon ≈ 5.785. For the cases with ≥ 2 reflex angles, the sum of sines < 7 which is lower than the upper bound for 1 reflex angle.

9.2.4 Upper bound for S_{10}

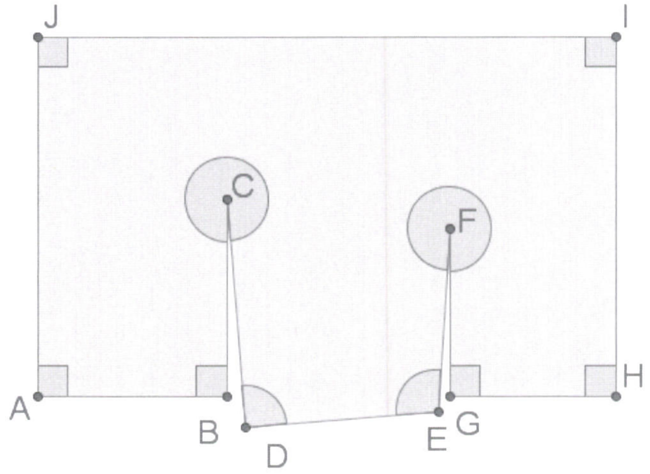
Consider the following diagram:



All the angles except $\angle ABC$ and $\angle BCD$ are equal to $\frac{2\pi}{3}$. $\angle ABC$ can go arbitrarily close to 2π, and $\angle BCD$ can go arbitrarily close to $\frac{2\pi}{3}$. The sum of sines is thus

$$\begin{aligned}\sum_{i=1}^{10} \sin A_i &= 8\sin\frac{2\pi}{3} + \sin\angle ABC + \sin\angle BCD \\ &\approx 9\sin\frac{2\pi}{3} \approx 7.794\end{aligned}$$

However, consider the following diagram with 2 reflex angles:



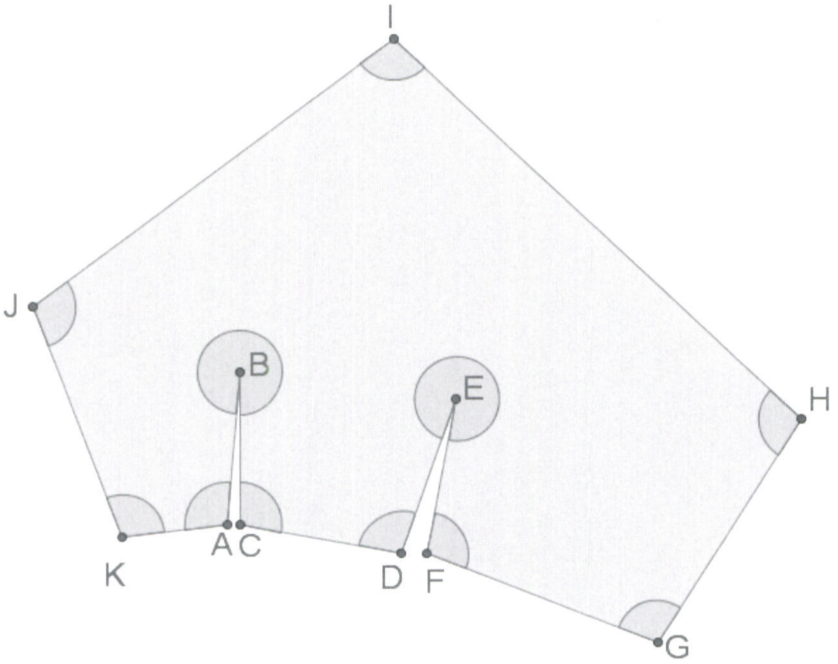
$\angle BCD$ and $\angle EFG$ can go arbitrarily close to 2π, and $\angle CDE$ and $\angle DEF$ can go arbitrarily close to $\frac{\pi}{2}$. The sum of sines is thus

$$\begin{aligned}\sum_{i=1}^{10} \sin A_i &= 6\sin\frac{\pi}{2} + \sin\angle BCD + \sin\angle CDE + \sin\angle DEF + \sin\angle EFG \\ &\approx 8\sin\frac{\pi}{2} \approx 8\end{aligned}$$

This is a higher bound than that with one reflex angle. For more than 2 reflex angles, there are ≤ 7 angles that are $\leq \pi$, thus the sum of sines of the angles is ≤ 7. Hence the upper bound for $n = 10$ is obtained when there are 2 reflex angles.

9.2.5 Upper bound for S_{11}

The diagram below for $n = 11$ has 2 angles approaching 2π and all the other angles are around $\frac{5\pi}{9}$.

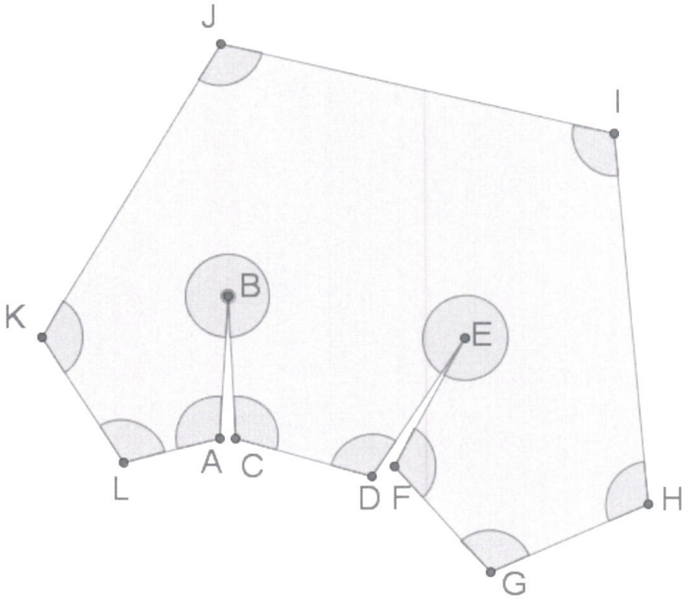


The sum of sines is

$$\begin{aligned}\sum_{i=1}^{11} \sin A_i &= 7\sin\frac{5\pi}{9} + \sin\angle BCD + \sin\angle CDE + \sin\angle DEF + \sin\angle ABC \\ &\approx 9\sin\frac{5\pi}{9} \approx 8.863\end{aligned}$$

9.2.6 Upper bound for S_{12}

Similarly, the diagram for $n = 12$ has 2 angles approaching 2π and all the other angles are around $\frac{3\pi}{5}$.



The sum of sines is

$$\sum_{i=1}^{12} \sin A_i = 8\sin\frac{3\pi}{5} + \sin\angle BCD + \sin\angle CDE + \sin\angle DEF + \sin\angle ABC$$
$$\approx 10\sin\frac{3\pi}{5} \approx 9.511$$

9.2.7 Upper bound for general S_n

In general, we have the upper bound for general n (when we assume that all reflex angles are very close to 2π) which is

$$\max_{0\le x\le n-2} (n-x)\sin\frac{\pi(2+x)}{n-x}$$

We obtained this result by the following: Suppose we have x reflex angles.

$$\begin{aligned}\sum_{i=1}^{n} \sin A_i &= \sum_{i=1}^{n-x} \sin A_i + \sum_{i=n-x+1}^{n} \sin A_i \\ &< \sum_{i=1}^{n-x} \sin A_i \\ &< (n-x)\sin\frac{(n-2)\pi - 2x\pi}{n-x} \\ &= (n-x)\sin\frac{\pi(2+x)}{n-x}\end{aligned}$$

Below is a table of the maximum sum of sines for small n, as well as the number of reflex angles to achieve this value.

n	**Max sum of sines**	**Approx. value**	**No. of reflex angles**
7	$6\sin\frac{\pi}{2}$	6	1
8	$7\sin\frac{3\pi}{7}$	6.824	1
9	$8\sin\frac{3\pi}{8}$	7.391	1
10	$8\sin\frac{\pi}{2}$	8	2
11	$9\sin\frac{4\pi}{9}$	8.863	2
12	$10\sin\frac{2\pi}{5}$	9.511	2
13	$11\sin\frac{4\pi}{11}$	10.006	2
14	$11\sin\frac{5\pi}{11}$	10.888	3
15	$12\sin\frac{5\pi}{12}$	11.591	3
16	$13\sin\frac{5\pi}{13}$	12.155	3
17	$13\sin\frac{6\pi}{13}$	12.905	4
18	$14\sin\frac{3\pi}{7}$	13.649	4
19	$15\sin\frac{2\pi}{5}$	14.266	4
20	$15\sin\frac{7\pi}{15}$	14.918	5
21	$16\sin\frac{7\pi}{16}$	15.693	5
22	$17\sin\frac{7\pi}{17}$	16.351	5
23	$17\sin\frac{8\pi}{17}$	16.928	6
24	$18\sin\frac{4\pi}{9}$	17.727	6
25	$19\sin\frac{8\pi}{19}$	18.419	6
26	$20\sin\frac{2\pi}{5}$	19.021	6

Now, we will investigate the graph of $(n-x)\sin\frac{\pi(2+x)}{n-x}$ and find a closed form for x.

9.3 The closed form for x

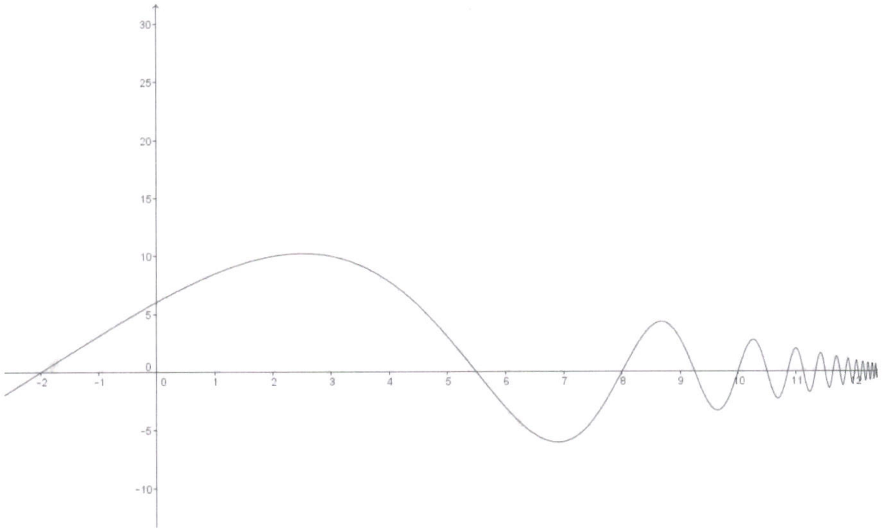


Figure 1: Graph of $(n-x)\sin\frac{\pi(2+x)}{n-x}$ for $n = 13$

Now, looking at the graph of $(n-x)\sin\frac{\pi(2+x)}{n-x}$ for $0 \le x \le n-3$ (as there can only be at most $n-3$ reflex angles), we observe that the graph increases and reaches a maximum point before decreasing and reaching 0 when $x = \frac{n}{2} - 1$. Therefore, it suffices to find the first value of x such that the gradient at x will be 0, and as such, indicating the highest point on the graph. We can do so by calculating the value of x such that the first derivative is equal to 0. That is,

$$\begin{aligned}
-\sin\frac{\pi(2+x)}{n-x} - \frac{\pi(n+2)\cos\frac{\pi(2+x)}{n-x}}{x-n} &= 0 \\
-(x-n)\sin\frac{\pi(2+x)}{n-x} - \pi(n+2)\cos\frac{\pi(2+x)}{n-x} &= 0 \\
(n-x)\sin\frac{\pi(2+x)}{n-x} &= \pi(n+2)\cos\frac{\pi(2+x)}{n-x} \\
\tan\frac{\pi(2+x)}{n-x} &= \frac{\pi(n+2)}{n-x} \\
\tan\left(\frac{\pi(n+2)}{n-x} - \pi\right) &= \frac{\pi(n+2)}{n-x} \\
\tan\frac{\pi(n+2)}{n-x} &= \frac{\pi(n+2)}{n-x}
\end{aligned}$$

Now, we have to solve the equation $\tan\alpha = \alpha$ where $\alpha = \frac{\pi(n+2)}{n-x}$

9.3.1 Finding the value of α

We use the Newton-Raphson method to find α.

$$\begin{aligned}\tan\alpha &= \alpha\\ \sin\alpha &= \alpha\cos\alpha\end{aligned}$$

Let $f(x) = \sin x - x\cos x$ and let the initial estimate x_0 be $\frac{3\pi}{2}$.

$$\begin{aligned}f'(x) &= x\sin x\\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)}\\ &\approx 4.49341195430395\\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)}\\ &\approx 4.4934094579317\\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)}\\ &\approx 4.4934094579091\end{aligned}$$

As such, we found that $\alpha = 4.49340945790906$ (approximate). Therefore, we can solve for x

$$\begin{aligned}\frac{\pi(n+2)}{n-x} &= \alpha\\ \pi(n+2) &= \alpha(n-x)\\ \alpha x &= \alpha n - \pi(n+2)\\ x &= n - \frac{\pi(n+2)}{\alpha}\end{aligned}$$

Let $g(x) = x - \frac{\pi(x+2)}{\alpha}$.
We can get the number of reflex angles required for an n-gon by obtaining the maximum among v_1 and v_2 where $v_1 = \lfloor g(x)\rfloor$ and $v_2 = \lceil g(x)\rceil$.

9.4 Proving the assumption

To get the upper bounds that we have obtained above, we have made an assumption.
We assumed that for the sum of sines of non-reflex angles to be maximised, the reflex angles have to be as close as 2π as possible. To prove this, we consider 2 cases. We define x to be the number of reflex angles.

9.4.1 Case 1: $x \le \frac{n-4}{3}$

When $x \le \frac{n-4}{3}$, the expression $\sum_{i=1}^{n} \sin A_i$ reaches the maximum when all the reflex angles are as close to 2π.
This is because the sum of remaining non-reflex angles is less than $(n-2)\pi - x\pi$ (when the reflex angles are as close to π as possible) but more than $(n-2)\pi - 2x\pi$ (when the reflex angles are as close to 2π as possible).

$$n - x \ge \frac{2n+4}{3}$$

$$\frac{(n-2-2x)\pi}{n-x} = 2\pi - \frac{(n+2)\pi}{n-x}$$

$$2\pi - \frac{(n+2)\pi}{n-x} \ge 2\pi - \frac{(3n+6)\pi}{2n+4} = \frac{\pi}{2}$$

Hence, it means that when the reflex angles are as close to 2π as possible, then the sum of sines of the remaining angles will increase as the angles will get closer to 2π

9.4.2 Case 2: $x > \frac{n-4}{3}$

We do not need to consider this case as the sum of sines for this case would definitely be less than $n - \frac{n-4}{3} = \frac{2n+4}{3}$, which can be achieved (arbitrarily close) from the previous case.
Also, we note that the number of reflex angles required is less than $\frac{n-4}{3}$.

$$2\alpha < 3\pi$$

$$2(n+2)\alpha < 3(n+2)\pi$$

$$\frac{2(n+2)\alpha}{3} < (n+2)\pi$$

$$n - \frac{n-4}{3} < \frac{(n+2)\pi}{\alpha}$$

$$n - \frac{(n+2)\pi}{\alpha} < \frac{n-4}{3}$$

10 Sum of cosines in a triangle

Now that we have found an upper bound for the sum of sines, we shall now find an upper bound for the sum of cosines in a triangle.

Observation 10.1.

$$\cos A - \cos B = 2\sin\frac{A+B}{2}\cos\frac{B-A}{2}$$

Proof.

$$\begin{aligned}\cos A - \cos B &= \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right)\\ &= \left(\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \sin\frac{A+B}{2}\sin\frac{A-B}{2}\right)\\ &- \left(\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \sin\frac{A+B}{2}\sin\frac{A-B}{2}\right)\\ &= -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}\\ &= 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}\end{aligned}$$

□

Observation 10.2.

$$\cos A = 1 - 2\sin^2\frac{A}{2}$$

Proof.

$$\begin{aligned}\cos A = \cos\frac{A}{2} + \frac{A}{2} = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} &= \left(1 - \sin^2\frac{A}{2}\right) - \sin^2\frac{A}{2}\\ &= 1 - 2\sin^2\frac{A}{2}\end{aligned}$$

□

Lemma 10.3. *In a triangle with angles A, B, C, we have*

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Proof.

$$\begin{aligned}(\cos A + \cos B) + \cos C &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C\\ &= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A-B}{2} + \cos C\\ &= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + \left(1 - 2\sin^2\frac{C}{2}\right)\\ &= 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\frac{C}{2}\right)\\ &= 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right)\\ &= 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\\ &= 1 + 2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2}\\ &= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\end{aligned}$$

□

Lemma 10.4. *In $\triangle ABC$,*

$$\sin\frac{A}{2} \leq \frac{a}{b+c}$$

where A is the angle at vertex A and a, b, c are the side lengths opposite the angles.

Proof. Let the area of $\triangle ABC$ be S.

$$S = \frac{1}{2}b(c\sin A) = \frac{1}{2}c(a\sin B) = \frac{1}{2}a(b\sin C).$$

Multiplying by $\frac{2}{bc}$ gives

$$\frac{2S}{bc} = \sin A$$

Similarly we can get

$$\frac{2S}{ac} = \sin B$$
$$\frac{2S}{ab} = \sin C$$

Now

$$\begin{aligned}\frac{a}{b+c} &= \frac{a\frac{2S}{abc}}{b\frac{2S}{abc}+c\frac{2S}{abc}}\\ &= \frac{\frac{2S}{bc}}{\frac{2S}{ac}+\frac{2S}{ab}}\\ &= \frac{\sin A}{\sin B+\sin C}\\ &= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}\\ &= \frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}} \geq \sin\frac{A}{2}\end{aligned}$$

by noting that $0 < \cos\frac{B-C}{2} \leq 1$, because $0 \leq |B-C| < \pi$. Similarly we get

$$\begin{aligned}\frac{b}{a+c} &\geq \sin\frac{B}{2}\\ \frac{c}{a+b} &\geq \sin\frac{C}{2}\end{aligned}$$

☐

Lemma 10.5. *For nonnegative real numbers a, b, we have*

$$a+b \geq 2\sqrt{ab}$$

Proof.

$$\begin{aligned}(a-b)^2 &\geq 0\\ a^2+b^2-2ab &\geq 0\\ a^2+b^2+2ab &\geq 4ab\\ (a+b)^2 &\geq 4ab\\ a+b &\geq 2\sqrt{ab}\end{aligned}$$

☐

Theorem 10.6. *In $\triangle ABC$,*

$$\cos A+\cos B+\cos C \leq \frac{3}{2}$$

Proof. By the above lemma,

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{abc}{(a+b)(b+c)(c+a)}$$

Lemma 10.5 yields

$$(a+b)(b+c)(c+a) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) = 8abc$$

Combing the last two inequalities, we get

$$\begin{aligned}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &\leq \frac{abc}{(a+b)(b+c)(c+a)} \\ &\leq \frac{abc}{8abc} = \frac{1}{8}\end{aligned}$$

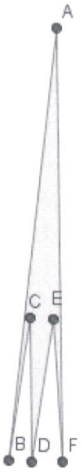
Thus

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq 1 + 4\cdot\frac{1}{8} = \frac{3}{2}$$

□

11 Sum of cosines in an n-gon

For even n, the sum of cosines in an n-gon can go arbitrarily close to n. The construction is as follows:
We have $\frac{n+2}{2}$ angles arbitrarily close to 0, and $\frac{n-2}{2}$ angles arbitrarily close to 2π. We construct the polygon using angles with value alternating between 0 and 2π. For example, the diagram for $n = 6$ is shown below:



This value is the maximum possible as it cannot go beyond n (as $\cos(x) \leq 1$). It also cannot reach n as to do so would result in the angles being 0 or 2π, both of which would not be possible.
We conjecture that the maximum sum of cosines for odd n is given by the formula

$$\frac{n-3}{2} + \frac{n+3}{2}\cos\left(\frac{2\pi}{n+3}\right)$$

12 Limitations

In this project, there were some limitations that we faced to get the formula for the upper bound of S_n.

1. Since α in the equation $\alpha = \tan\alpha$ has no exact form and an approximate value can only be found through the use of calculus, this may affect the results for large enough n.
2. Although the value of $(n-x)\sin\frac{\pi(2+x)}{n-x}$ seems to vary little in the domain $[v_1, v_2]$, we cannot round off $n - \frac{\pi(n+2)}{\alpha}$ to the nearest integer and use the obtained value as it may not be the correct answer.
   An example of this is $n = 13$. $n - \frac{\pi(n+2)}{\alpha}$ rounded to the nearest integer is 3 but the number of reflex angles that are needed to maximise the answer is actually 2.
3. As n grows larger, it becomes more difficult to construct the polygon that maximises the sum of sines due to the large number of angles that are arbitrarily close to 2π. We have not verified if it is possible to construct polygons for large n.

13 Conclusion

In conclusion, we have found the general upper bound for sum of sines in all n-gons. We define $v_1 = \lfloor n - \frac{\pi(n+2)}{\alpha} \rfloor$, $v_2 = \lceil n - \frac{\pi(n+2)}{\alpha} \rceil$.

$$S_n \begin{cases} \leq n\sin\left(\frac{2\pi}{n}\right) & \text{for all convex } n\text{-gons or } n \leq 6 \\ < \max\{(n-v_1)\sin\frac{\pi(2+v_1)}{n-v_1}, (n-v_2)\sin\frac{\pi(2+v_2)}{n-v_2}\} & \text{for all } n\text{-gons when } n > 6 \end{cases}$$

We also propose a conjecture for the general upper bound for sum of cosines in all n-gons.

$$\sum_{i=1}^{n} \cos A_i \begin{cases} < n & \text{for all even } n \\ < \frac{n-3}{2} + \frac{n+3}{2}\cos\left(\frac{2\pi}{n+3}\right) & \text{for all odd } n \end{cases}$$

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