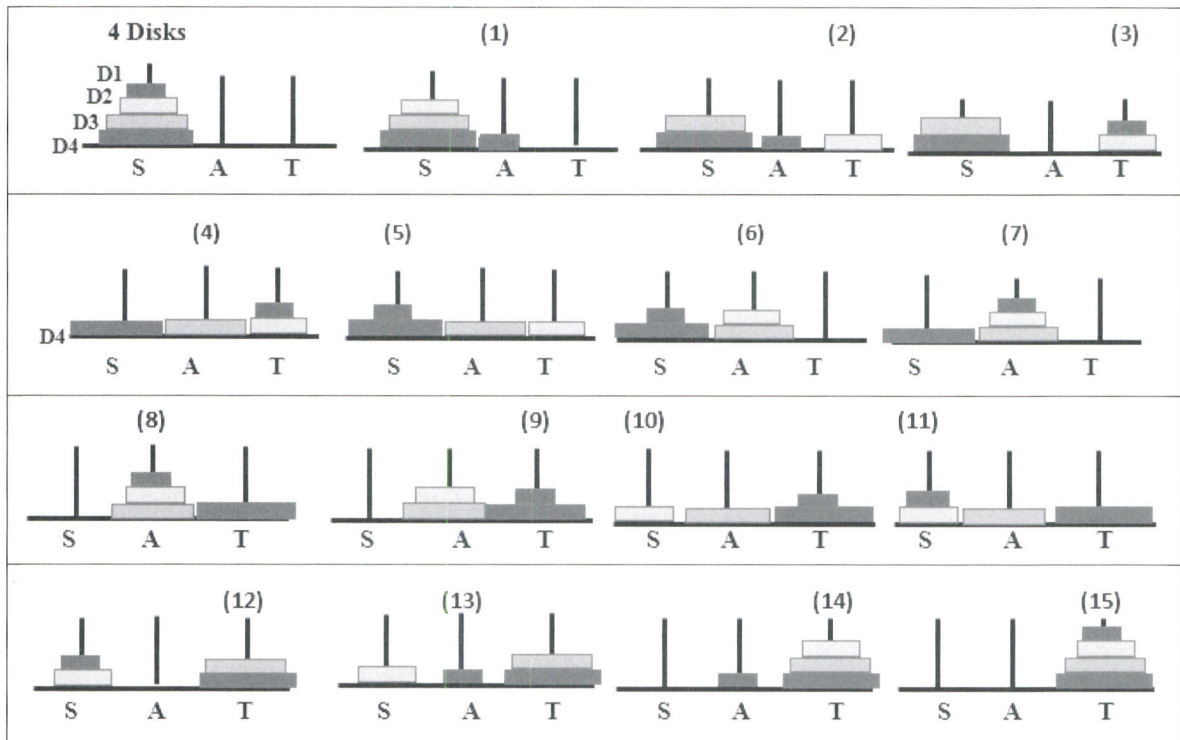


Tower of Hanoi and its Variations

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1 INTRODUCTION

According to the legend of the Tower of Hanoi (originally the "Tower of Brahma" in a temple in the Indian city of Benares), the temple priests are to transfer a tower consisting of 64 fragile disks of gold from one part of the temple to another one disk at a time. The disks are arranged in order, no two of them the same size, with the largest on the bottom and the smallest on top. Because of their fragility, a larger disk may never be placed on a smaller one, and there is only one intermediate location where disks can be temporarily placed. It is said that before the priests complete their task the temple will crumble into dust and the world will vanish in a clap of thunder.

Does this make mathematical sense?

The game "Towers of Hanoi" uses three rods. A number of disks is stacked with the largest disk at the bottom and the smallest one on top to form a conical tower.

The aim of the game is to move the tower of disks from one rod on the left to another rod on the right in the fewest possible moves.

The following rules have to be obeyed:

- In each step, only one disk can be moved.
- Only the most upper disk from one of the rods can be transferred in a move.
- The disk can be placed on another rod, if this rod is empty or if the most upper disk of this rod is larger than the one which is transferred.

2 OBJECTIVES

- To find an efficient and systematic move sequence that can achieve the optimal solution, that is the minimum number of moves required to transfer n disks from one rod to another rod for The Tower of Hanoi.
- To prove the general formulae for The Tower of Hanoi by Mathematical Induction.
- To look for patterns and relationships for Variations to The Tower of Hanoi for four and five rods and beyond.
- To prove the general formulae for Variations to The Tower of Hanoi by Mathematical Induction.

3 Tower of Hanoi

3.1 A Step-By-Step Approach:

Let us look for a pattern in the number of steps it takes to move just one, two, three or four disks. The disks are numbered starting with disk 1 on the top and the left rod as S (SOURCE), the right rod as T (TARGET) and the middle rod as A (AUX) which is needed as an auxiliary rod to temporarily deposit the disks.

The disk moves of each step for one, two, three or four disks are shown by the numbering above each disk in Figures 1 to 4 respectively.

1 Disk: 1 move

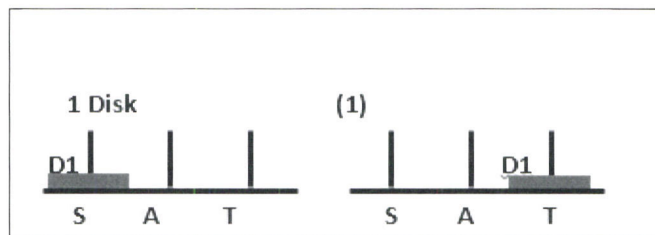


Figure 1

2 Disks: 3 moves

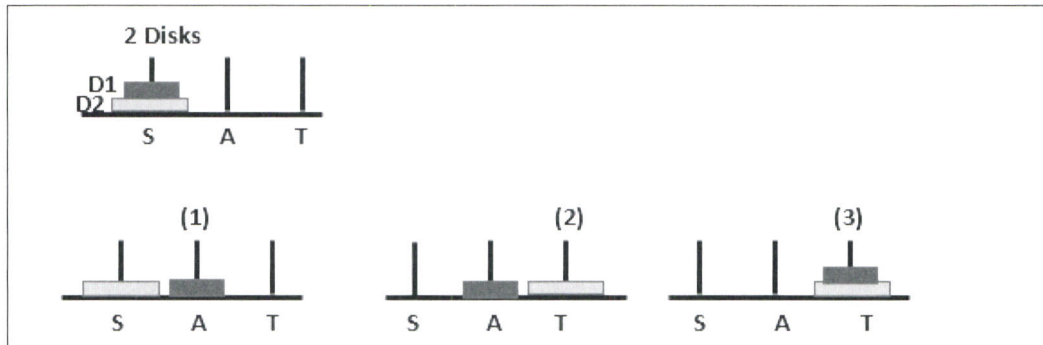


Figure 2

2 disks	Movement
Move 1	D1 to A
Move 2	D2 to T
Move 3	D1 to T
Total number of moves	3

Table 1

When $n = 2$, the choice of the first move is very crucial to successfully complete the game with the minimal number of moves.

3 Disks: 7 moves

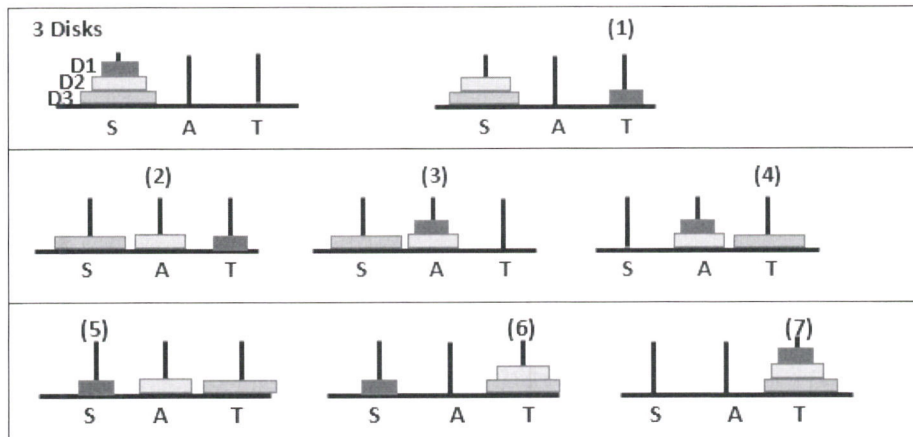


Figure 3

3 disks	Movement
Move 1	D_1 to T
Move 2	D_2 to A
Move 3	D_1 to A
Move 4	D_3 to T
Move 5	D_1 to S
Move 6	D_2 to T
Move 7	D_1 to T
Total number of moves	7

Table 2

Algorithm Development:

Let the smallest disk be labelled D_1 and the largest disk be labelled D_n . The moves are illustrated by the numbering above the respective rod in Figure 4.

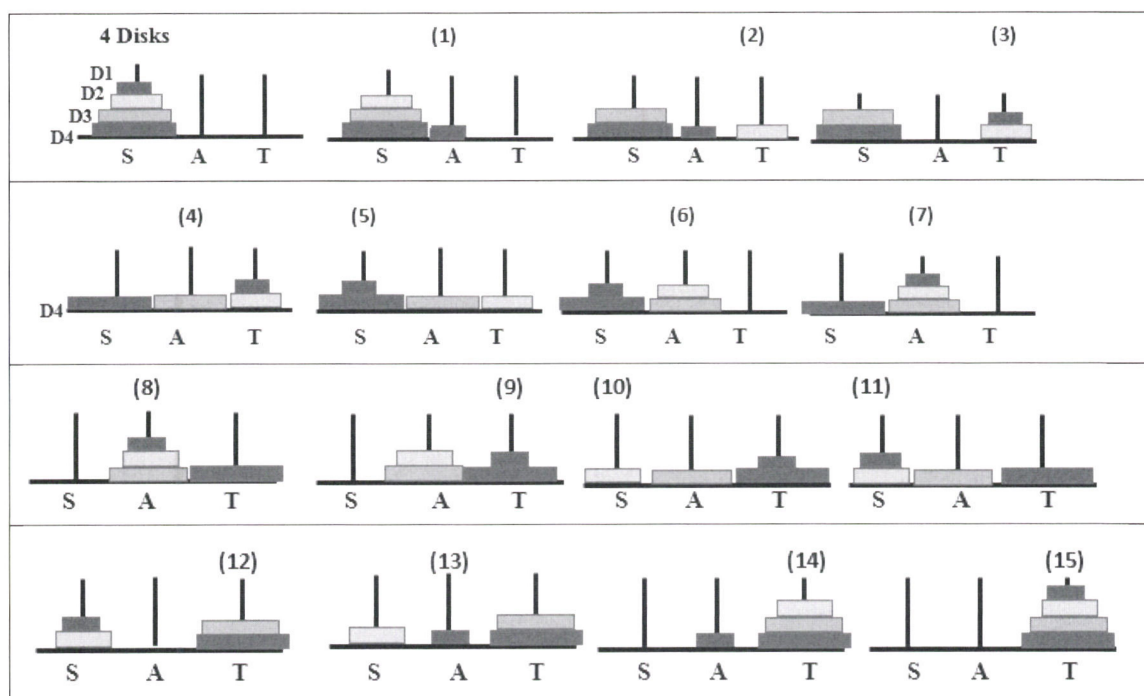


Figure 4

General pattern observed for placing a disk from the step-by-step approach:

- 1a) If the number of disks n is odd, the first top disk is moved to the TARGET rod (T).
 - b) If the number of disks n is even, the first top disk is moved to the AUX rod (A).
 - 2a) An odd disk cannot be placed directly on another odd disk.
 - b) An even disk cannot be placed directly on another even disk
 - 3) Only for even number of disks, when there are two possible rods: one with disks and the other empty, place the disk in the non-empty rod (as shown in Step 9 of Figure 4).
- The rest of the game continues similarly.

3.2 A Binary Perspective:

The moves of the Tower of Hanoi can be determined by reading the digits from right to left as the smallest (topmost) disk is first moved. Each digit is used to determine the movement of the respective disk.

For each move, there are two options: each disk is either being "moved" or "not moved" (i.e. "1" or "0"). Like the Step-by-Step Approach, the parity of the disk is a very important factor in the move sequence.

Let k be defined as the move number in denary form. For each step, k is written in its binary form. Starting from the right, we associate each place value with a disk number, starting with D_1 representing the smallest (topmost) disk.

The algorithm for the moves is as follows:

- 3.2.1. Locate the rightmost 1 in the binary expansion. Let the corresponding disk be the initial disk.
- 3.2.2. Locate the second 0 or 1 from the right, and let the corresponding disk be the location disk.
- (a) Between the initial and location disks, if there are NO different digits or an even number of different digits, then the initial disk is moved on top of the location disk.
- (b) If there are an ODD number of different digits between the initial and the location, then the initial disk is moved to the space that DOES NOT contain the location disk.

Disks	Movement	Reasons
D_3, D_2, D_1 0 1 1 ₂	move D_1 on top of D_2	No zeros between the consecutive ones
D_3, D_2, D_1 1 0 1 ₂	move D_1 to the space that DOES NOT contain D_3	Odd number of zeros between the ones
D_3, D_2, D_1 0 1 0 ₂		Odd number of ones between the zeros
D_5, D_4, D_3, D_2, D_1 0 1 0 0 1 ₂	move D_1 on top of D_4	Even number of zeros between the ones

Table 3

3.2.3. There is one binary digit for each disk. The leftmost digit represents the largest disk. Disk positions can be determined by reading the digits from left to right starting with the largest disk. Each binary digit can be used to determine the position of the respective disk.

A value of 0 indicates that the largest disk is on the Source rod, while a 1 indicates that it is on the Target rod.

The algorithm for determining the disk positions is as follows:

- A digit with the same value as the previous one means that the respective disk is stacked on top the previous disk on the same rod.
- A digit with a different value to the previous one means that the respective disk is on another rod, either Source (S), Aux (A) or Target (T) rods.
- From the patterns observed in Table 4 below, if there is an odd number of '0's or '1's in the initial string of '0's or '1's before the next different digit, the subsequent disk is on 'A'. From section 3.1, if the number of disks n is odd, the top disk is moved to the TARGET rod (T) and if the number of disks n is even, the top disk is moved to the AUX rod (A). Since the binary numbers shows the positions of the disks after they are moved, it can be concluded that if there is an odd number of disks at the Source or the Target rods, the subsequent disk will be at the AUX rod.

$n = 2$	2^1	2^0	Refer to Figure 2
k	$D2$	$D1$	Optimal Movement
0	0	0_2	
1	0	1_2	$D1$ to A [$\because n$ is even]
2	1	0_2	$D2$ to T
3	1	1_2	$D1$ onto $D2$

Table 4

$n = 3$	2^2	2^1	2^0	Refer to Figure 3	
k	$D3$	$D2$	$D1$	Optimal Movement	Final Disk Positions
0	0	0	0_2		S,S,S
1	0	0	1_2	$D1$ to T [$\because n$ is odd]	S,S,T
2	0	1	0_2	$D2$ to A	S,A,T
3	0	1	1_2	$D1$ onto $D2$ (A)	S,A,A

4	1	0	0 ₂	D3 to T	T,A,A
5	1	0	1₂	D1 to S [NOT onto D3]	T,A,S
6	1	1	0 ₂	D2 onto D3 (T)	T,T,S
7	1	1	1 ₂	D1 onto D2 (T)	T,T,T

Table 5

An example of how the above rules are applied is illustrated in Tables 6 and 7:

For $n = 6$,

k	D_6	D_5	D_4	D_3	D_2	D_1	Final Disk Positions
43	1	0	1	0	1	1 ₂	T,A,S,T,A,A

Table 6

	Position of Binary Digit when read from left to right	Position of Disks
1	leftmost digit is one	D_6 is at the Target rod
2	only one digit '1' before the digit '0'	D_5 is at the Aux rod
3	odd number of '0' between the previous '1' and the current '1'	D_4 is not stacked onto D_5
4	D_4 and D_5 have different digits	D_4 and D_5 are not on the same rod D_4 is on the Source rod
5	odd number of '1' between D_3 and D_5	D_3 and D_5 are not on the same rod
6	D_3 and D_4 have different digits	D_3 is not stacked onto D_4 D_3 is on the Target rod
7	odd number of '0' between D_2 and D_4	D_2 and D_4 are not on the same rod
8	D_2 and D_3 have different digits	D_2 is not stacked onto D_3 D_2 is on the Aux rod.
9	D_1 and D_2 have the same digits	D_1 is stacked onto D_2 D_1 is on the Aux rod

Table 7

3.3 Recursive Pattern

Let the disks be labelled as D_1 (smallest), D_2 and D_3 (largest) and the rods as S (SOURCE), A (AUX), T (TARGET). For the tower of two disks (disks D_1 and D_2), three moves were needed to move to A. Now D_3 was moved to T, where it was finally positioned. The last three moves moved the tower consisting of D_2D_1 from rod A to rod T to place them on top of D_3 .

There is a general rule for moving a tower of size n ($n > 1$) from the rod S to the rod T:

- i. Recursively move the top $n-1$ disks $D_{n-1}.....D_1$ from rod S to rod A. This takes a minimum of $M(n-1)$ moves. Disk D_n is left alone on rod S.

- ii. Move the n^{th} disk D_n to rod T. This requires only 1 move.
- iii. Recursively move the $n-1$ disks D_{n-1}, \dots, D_1 on rod A to rod T, i.e. on top of Disk D_n . Again, this takes a minimum of $M(n-1)$ moves. The number of moves will be the same as those needed to transfer $n-1$ disks from rod S to rod A.

Thus the minimum number of moves required to transfer a stack of n disks from rod S to rod T is $M(n-1)+1+M(n-1)$.

Hence the recursive formula for the relationship between $M(n)$ and $M(n-1)$ is

$$M(1)=1, \text{ for } n=1, \text{ and } M(n)=2M(n-1)+1, \text{ for } n>1.$$

Table 8 is an illustration of the use of the recursive formula for 5 disks.

No of disks (n)	No of moves from rod S to rod T
1	1
2	$2M(n-1)+1=2(1)+1=3$
3	$2M(n-1)+1=2(3)+1=7$
4	$2M(n-1)+1=2(7)+1=15$
5	$2M(n-1)+1=2(15)+1=31$

Table 8

A general formula where the value of n can be substituted to determine the minimum number of steps needed to solve the n -disk game would be more useful.

Let n be the number of disks, and let $M(n)$ be the minimum number of moves it takes to complete an n -disk game. We construct a recursive formula for $M(n)$.

$$\begin{aligned}
 M(n) &= 2M(n-1)+1 \\
 &= 2[2M(n-2)+1]+1 \\
 &= 2^2 M(n-2)+2+1 \\
 &= 2^2 [2M(n-3)+1]+2+1 \\
 &= 2^3 M(n-3)+2^2+2+1 \\
 &= 2^3 [2M(n-4)+1]+2^2+2+1 \\
 &= 2^4 M(n-4)+2^3+2^2+2+1 \\
 &\vdots \\
 &= 2^{n-1} M(1)+2^{n-2}+\dots+2^2+2+1 \\
 &= 2^{n-1}+2^{n-2}+\dots+2^2+2^1+2^0
 \end{aligned}$$

This is a geometric sum: $M(n) = 2^{n-1} + 2^{n-2} + \dots + 2^2 + 2^1 + 2^0 = \sum_{k=0}^{n-1} 2^k$

The general formula is $M(n) = \sum_{k=0}^{n-1} 2^k = \frac{2^n - 1}{2 - 1} = 2^n - 1$.

3.4 Explicit Patterns

No of disks (n)	Minimum number of moves $M(n)$ from rod S to rod T	Additional moves $A(n)$ as the number of disks is increased from $n-1$ to n
1	$2^1 - 1 = 1$	$1 = 2^0$
2	$2^2 - 1 = 4 - 1 = 3$	$2 = 2^1$
3	$2^3 - 1 = 8 - 1 = 7$	$4 = 2^2$
4	$2^4 - 1 = 16 - 1 = 15$	$8 = 2^3$
5	$2^5 - 1 = 32 - 1 = 31$	$16 = 2^4$
n	$2^n - 1$	2^{n-1}

Table 9

Claim 3.4.1:

The optimal solution (minimum number of steps) to transfer all the disks is $M_n = 2^n - 1$.

Proof: By Mathematical Induction.

Let p_n be the statement that $M_n = 2^n - 1, n \in \mathbb{Z}^+$.

When $n = 1$,

$$\begin{aligned} \text{LHS} &= M_1 = 1 \\ \text{RHS} &= 2^1 - 1 = 1 \end{aligned}$$

Since $\text{LHS} = \text{RHS}$, p_1 is true.

Assume that p_k is true for some $k \in \mathbb{Z}^+$, i.e. $M_k = 2^k - 1$ is true for some $k \in \mathbb{Z}^+$.

To prove: p_{k+1} is true based on the assumption that p_k is true.

$$\begin{aligned} \text{LHS of } p_{k+1} &= M_{k+1} \\ &= 2M_k + 1 \\ &= 2[2^k - 1] + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \\ &= \text{RHS of } p_{k+1} \end{aligned}$$

Hence p_k is true implies that p_{k+1} is true.

Since p_1 is true and p_k is true $\Rightarrow p_{k+1}$ is true, by mathematical induction, p_n is true for all $n \in \mathbb{Z}^+$.

From this formula if it only takes the priests one second to make each move, it will be $2^{64} - 1 \approx 1.844674407 \times 10^{19}$ seconds before the world will end.

Since there are $365 \times 24 \times 60 \times 60 = 31\,536\,000$ seconds in a year, this is $\frac{1.844674407 \times 10^{19}}{31,536,000} \approx 5.849424174 \times 10^{11}$ years ≈ 585 billion years!

Claim 3.4.2:

From the data of values in Table 9, as the number of disks increases from $n - 1$ to n , the number of additional moves $A(n)$ is $A(n) = 2^{n-1}$.

Proof:

$$\begin{aligned} A(n) &= M(n) - M(n-1) \\ &= (2^n - 1) - (2^{n-1} - 1) \\ &= 2^n - 2^{n-1} \\ &= 2^{n-1} \end{aligned}$$

We will now investigate the precise move for any disk for three rods. For this, the disks are enumerated from top to bottom by 1, 2, 3, 4, ..., n .

If $n = 2$, the sequence of moves is $S_2 = (1, 2, 1)$. This means that disk 1 is moved first, then disk 2 and finally disk 1 again.

For $n = 3$, the sequence of moves is obtained by first transferring disks 1 and 2 according to S_2 , then transferring disk 3 and finally applying S_2 to the top 2 disks. Therefore the sequence of moves is $S_3 = (S_2, 3, S_2)$. In general, we have $S_n = (S_{n-1}, n, S_{n-1})$.

The sequence of moves with its frequency and at which point a disk at an optimal sequence of moves is moved up till $n = 6$ is shown in Table 10.

$S_1 = (1)$
$S_2 = (1, 2, 1)$
$S_3 = (1, 2, 1, 3, 1, 2, 1)$
$S_4 = (1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1)$
$S_5 = (1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1)$
$S_6 = (1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 6, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1)$

Table 10

In particular, disk 1 is moved in every other move, more precisely in the odd moves 1, 3,

Disk Number (k)	Starting Point of Move for Disk k	Subsequent Move of Disk k	Frequency of Moves for Disk k
1	1 st move	Every $2^1 = 2^{\text{nd}}$ move	$32 = 2^5 = 2^{6-1}$
2	$2^{2-1} = 2^{\text{nd}}$ move	Every $2^2 = 4^{\text{th}}$ move	$16 = 2^4 = 2^{6-2}$
3	$2^{3-1} = 4^{\text{th}}$ move	Every $2^3 = 8^{\text{th}}$ move	$8 = 2^3 = 2^{6-3}$
4	$2^{4-1} = 8^{\text{th}}$ move	Every $2^4 = 16^{\text{th}}$ move	$4 = 2^2 = 2^{6-4}$
5	$2^{5-1} = 16^{\text{th}}$ move	Every $2^5 = 32^{\text{rd}}$ move	$2 = 2^1 = 2^{6-5}$
6	$2^{6-1} = 31^{\text{st}}$ move	-	$1 = 2^0 = 2^{6-6}$
S_k	2^{k-1} th move	Every 2^k move	2^{n-k}

Table 11

Claim 3.4.3:

In this way it is possible to determine the starting and subsequent moves a particular disk.

4. Variation 1: Four-Rod 'Tower of Hanoi'

Triangular Number (t_x)	No of disks (n)	Minimum number of moves from rod S to rod T	Difference, $2^a, a \in \mathbb{Z}^+$	Number of 2^a
0	0	0	$2^0=1$	1
1	1	1	$2^1 = 2$	2
	2	3	2	
2	3	5	$2^2 = 4$	3
	4	9	4	
	5	13	4	
3	6	17	$2^3 = 8$	4
	7	25	8	
	8	33	8	
	9	41	8	
4	10	49	$2^4 = 16$	5
	11	65	16	
	12	81	16	
	13	97	16	
	14	113	16	
5	15	129	$2^5 = 32$	6
	16	161	32	
	17	193	32	
	18	225	32	
	19	257	32	
	20	289	32	
6	21	321	64	
	22	385		

Table 12

Claim 4.1:

As seen from Table 12, if n and $n-1$ are two of the x numbers in the range $t_{x-1} < n < t_x$, then the additional number of moves from $n-1$ to n is 2^{x-1} .

Claim 4.2:

There is a general rule for moving a tower of n ($n > 1$) disks from the source rod (S) to the target rod (T) via the auxiliary rods (A):

Based on an integer j satisfying $1 \leq j \leq n$,

1. Recursively transfer the top $n - j$ smallest disks D_{n-j}, \dots, D_1 from S to A, using all four rods in the process. The $(n - j)^{\text{th}}$ disk is underlined in Table 12.
2. Transfer the remaining j largest disk from rod S to rod T, using the standard three rod algorithm, with the top $n - j$ smaller disks at one of the Aux rod.
If the remaining number of j disks is odd, then the topmost disk at rod S is moved to rod T, otherwise it is moved to rod A.
3. Recursively transfer the smallest $n - j$ disks D_{n-j}, \dots, D_1 on rod A to rod T, again using all four rods in the process.
4. Figures 5 and 6 show the two possible optimal moves for 5 disks.

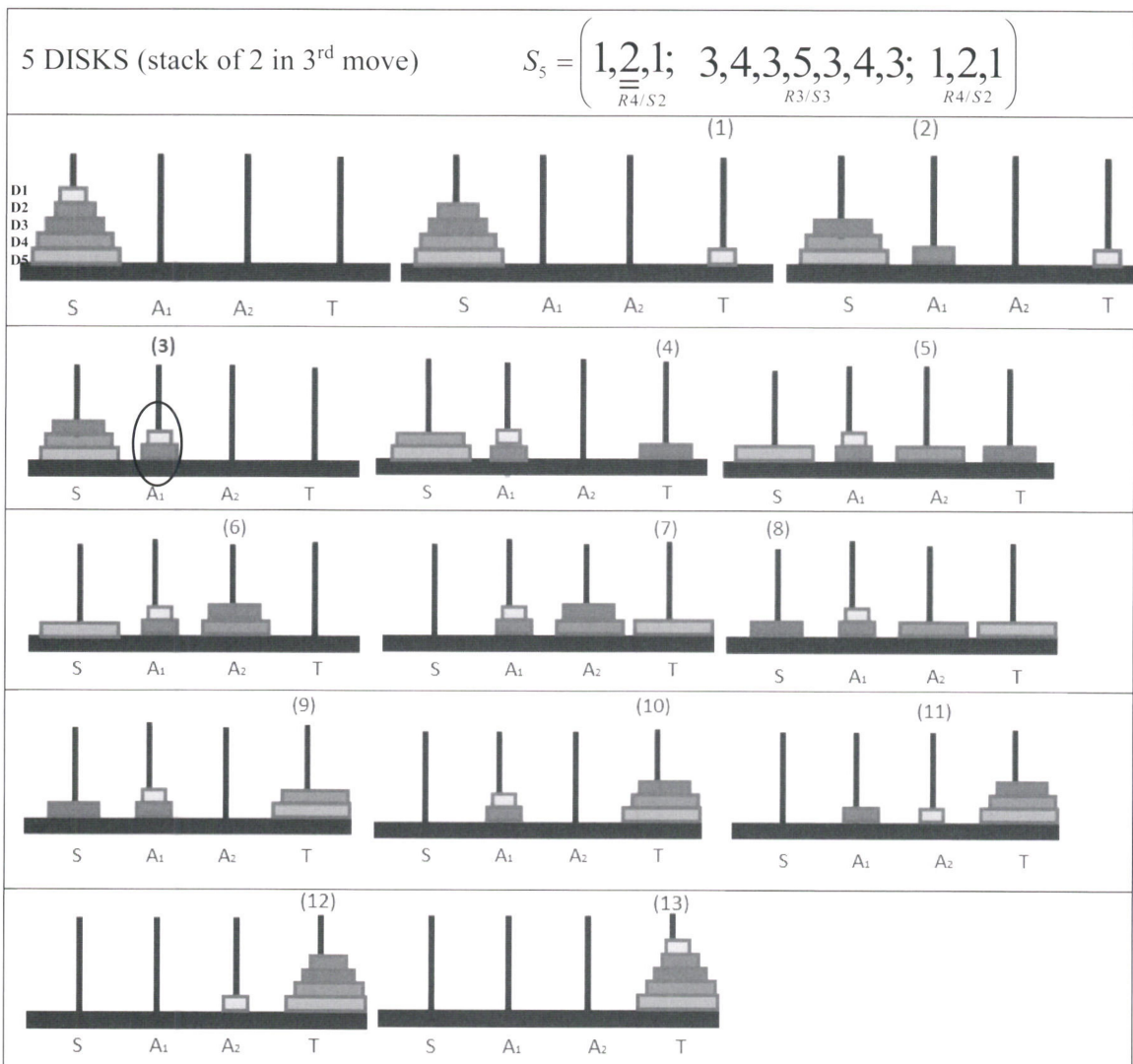


Figure 5

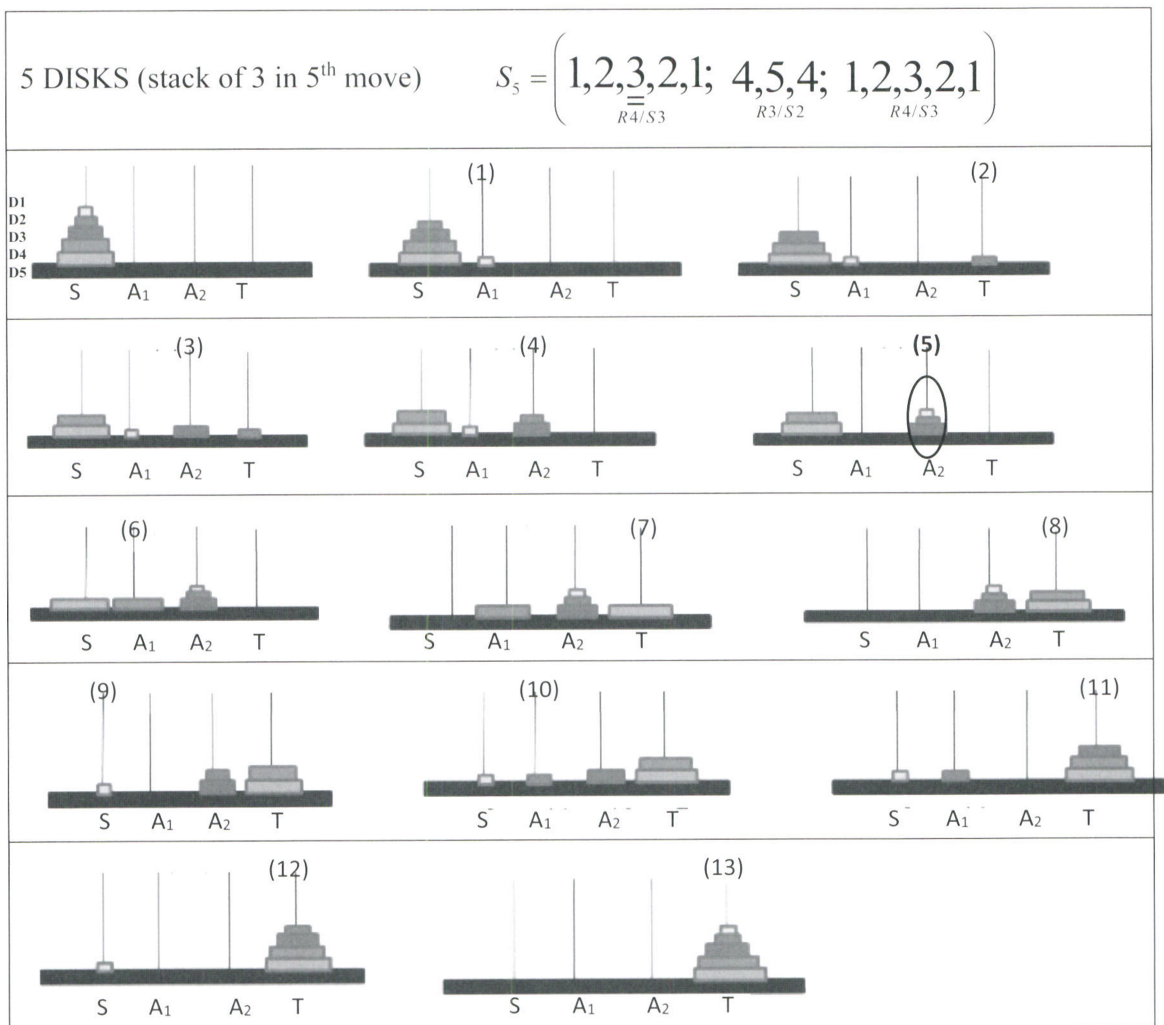


Figure 6

In Figure 5, the topmost two disks are moved to an AUX rod (as shown in step 3). Moving D3 is an important deciding factor to ensure the optimal solution.

D3 is moved to rod T because there are an odd number of remaining rods at rod S. Steps 4 – 10 of the moves follow the three rod algorithm.

Moves 11 – 13 is exactly a repetition of the first three steps to complete the game in the minimum number of moves.

In Figure 6, the topmost three disks are moved to an AUX rod (as shown in step 5). Moving D4 is an important deciding factor to ensure the optimal solution.

D4 is moved to AUX rod because there are an even number of remaining rods at rod S. Steps 6 – 8 of the moves follow the three rod algorithm.

Moves 9 – 13 is exactly a repetition of the first five steps to complete the game in the minimum number of moves.

Other combinations of moves will not yield the optimal solution.

Let \triangle_x be the x^{th} triangular number, n and r be the number of disks and rods respectively.
 Let Rr/Sn be the number of disks n using the r rod algorithm and $\underline{y}, y \in \mathbb{N}$ be the optimal number of disks that must be moved to another rod. The sequence of moves for $1 \leq n \leq 10$ are shown in Table 13.

\triangle_x	S_n
\triangle_1	$S_1 = (1)$
\triangle_2	$S_2 = (1, 2, 1)$
\triangle_3	$S_3 = \left(\underline{1}; \underline{2, 3, 2}; \underline{1} \right)$ <small>$R4/S1$ $R3/S2$ $R4/S1$</small>
	$S_4 = \left(\underline{1, 2, 1}; \underline{3, 4, 3}; \underline{1, 2, 1} \right)$ or $S_4 = \left(\underline{1, 2, 3, 2, 4, 2, 3, 2, 1} \right)$ <small>$R4/S2$ $R3/S2$ $R4/S2$ $R4/S4$</small>
	$S_5 = \left(\underline{1, 2, 1}; \underline{3, 4, 3, 5, 3, 4, 3}; \underline{1, 2, 1} \right)$ or $S_5 = \left(\underline{1, 2, 3, 2, 1}; \underline{4, 5, 4}; \underline{1, 2, 3, 2, 1} \right)$ <small>$R4/S2$ $R3/S3$ $R4/S2$ $R4/S3$ $R3/S2$ $R4/S3$</small>
\triangle_5	$S_6 = \left(\underline{1, 2, 3, 2, 1}; \underline{4, 5, 4, 6, 4, 5, 4}; \underline{1, 2, 3, 2, 1} \right)$ <small>$R4/S3$ $R3/S3$ $R4/S3$</small>
	$S_7 = \left(\underline{1, 2, 3, 2, 1}; \underline{4, 5, 4, 6, 4, 5, 4, 7, 4, 5, 4, 6, 4, 5, 4}; \underline{1, 2, 3, 2, 1} \right)$ or <small>$R4/S3$ $R3/S4$ $R4/S3$</small>
	$S_7 = \left(\underline{1, 2, 3, 2, 4, 2, 3, 2, 1}; \underline{5, 6, 5, 7, 5, 6, 5}; \underline{1, 2, 3, 2, 4, 2, 3, 2, 1} \right)$ <small>$R4/S4$ $R3/S3$ $R4/S4$</small>
	$S_8 = \left(\underline{1, 2, 3, 2, 1, 4, 5, 4, 1, 2, 3, 2, 1}; \underline{6, 7, 6, 8, 6, 7, 6}; \underline{1, 2, 3, 2, 1, 4, 5, 4, 1, 2, 3, 2, 1} \right)$ or <small>$R4/S5$ $R3/S3$ $R4/S5$</small>
	$S_8 = \left(\underline{1, 2, 3, 1, 4, 3, 1, 2, 1}; \underline{5, 6, 5, 7, 5, 6, 5, 8, 5, 6, 5, 7, 5, 6, 5}; \underline{1, 2, 3, 1, 4, 3, 1, 2, 1} \right)$ <small>$R4/S4$ $R3/S4$ $R4/S4$</small>
	$S_9 = \left(\underline{1, 2, 3, 2, 1, 4, 5, 4, 1, 2, 3, 2, 1}; \underline{6, 7, 6, 8, 6, 7, 6, 9, 6, 7, 6, 8, 6, 7, 6}; \underline{1, 2, 3, 2, 1, 4, 5, 4, 1, 2, 3, 2, 1} \right)$ <small>$R4/S5$ $R3/S4$ $R4/S5$</small>
	$S_9 = \left(\underline{1, 2, 3, 2, 1, 4, 5, 4, 6, 4, 5, 4, 1, 2, 3, 2, 1}; \underline{7, 8, 7, 9, 7, 8, 7}; \underline{1, 2, 3, 2, 1, 4, 5, 4, 6, 4, 5, 4, 1, 2, 3, 2, 1} \right)$ <small>$R4/S6$ $R3/S3$ $R4/S6$</small>
\triangle_4	$S_{10} = \left(\underline{1, 2, 3, 2, 1, 4, 5, 4, 6, 4, 5, 4, 1, 2, 3, 2, 1}; \underline{7, 8, 7, 9, 7, 8, 7, 10, 7, 8, 7, 9, 7, 8, 7}; \underline{1, 2, 3, 2, 1, 4, 5, 4, 6, 4, 5, 4, 1, 2, 3, 2, 1} \right)$ <small>$R4/S6$ $R3/S4$ $R4/S6$</small>

Table 13

Claim 4.3:

4.3.1 The choice of move for the topmost j^{th} disk is of great importance as it follows the three rod algorithm.

Let $t_x = \frac{x(x+1)}{2}$ denote the x^{th} triangular number and $M(n)$ be the number of moves needed to move a tower of $n = t_x$ disks.

t_x	$S_n = M(t_x)$
t_1	$S_1 = (1)$ $M(1) = M(t_1)$ $= 1$ move
t_2	$S_3 = \left(\begin{array}{c} \underline{1}; \quad 2,3,2; \quad 1 \\ R4/S1 \quad R3/S2 \quad R4/S1 \end{array} \right)$ $M(3) = M(t_2)$ $= (1 + 3 + 1)$ moves $= 5$ moves
t_3	$S_6 = \left(\begin{array}{c} \underline{1,2,3,2,1}; \quad 4,5,4,6,4,5,4; \quad 1,2,3,2,1 \\ R4/S3 \quad R3/S3 \quad R4/S3 \end{array} \right)$ $M(6) = M(t_3)$ $= (5 + 7 + 5)$ moves $= 17$ moves
t_4	$S_{10} = \left(\begin{array}{c} \underline{1,2,3,2,1,4,5,4,6,4,5,4,1,2,3,2,1}; \quad 7,8,7,9,7,8,7,10,7,8,7,9,7,8,7; \\ R4/S6 \quad R3/S4 \\ 1,2,3,2,1,4,5,4,6,4,5,4,1,2,3,2,1 \\ R4/S6 \end{array} \right)$ $M(10) = M(t_4)$ $= (17 + 15 + 17)$ moves $= 49$ moves

Table 14

4.3.2

$$M(1) = M(t_1) = 1$$

$$M(3) = M(t_2) = (2 \times 1) + (2^2 - 1) = 5$$

$$M(6) = M(t_3) = (2 \times 5) + (2^3 - 1) = 17$$

$$M(10) = M(t_4) = (2 \times 17) + (2^4 - 1) = 49$$

If $n = t_x$, then $M(t_x) = 2M(t_{x-1}) + 2^x - 1$, and the optimizing choice for j is $j = x$.

4.3.3

Using recursion, we can also obtain the general formula for $M_4(t_x)$:

$$\begin{aligned}
 M_4(t_x) &= 2M(t_{x-1}) + 2^x - 1 \\
 &= 2[2M(t_{x-2}) + 2^{x-1} - 1] + 2^x - 1 \\
 &= 2^2 M(t_{x-2}) + 2(2^x) - (2^1 + 2^0) \\
 &= 2^2 [2M(t_{x-3}) + 2^{x-2} - 1] + 2(2^x) - (2^1 + 2^0) \\
 &= 2^3 M(t_{x-3}) + 3(2^x) - (2^2 + 2^1 + 2^0) \\
 &= 2^3 [2M(t_{x-4}) + 2^{x-3} - 1] + 3(2^x) - (2^2 + 2^1 + 2^0) \\
 &= 2^4 M(t_{x-4}) + 4(2^x) - (2^3 + 2^2 + 2^1 + 2^0) \\
 &\vdots \\
 &= 2^{x-1} M(1) + (x-1)(2^x) - \sum_{n=0}^{x-2} 2^n \\
 &= 2^{x-1} + 2^x(x-1) - 2^{x-1} + 1 \\
 &= 2^x(x-1) + 1
 \end{aligned}$$

The general formula is $M_4(t_x) = 2^x(x-1) + 1$

4.3.4

The number of differences of $2^0, 2^1, 2^2, 2^3$ and so on are consecutive natural numbers. Let $\{d_4(i)\}_{i \in \mathbb{N}}$ be the sequence that consists of

1 of 2^0 , 2 of 2^1 , 3 of 2^2 , 4 of 2^3 , 5 of 2^4 , 6 of 2^5 and so on.

$$\begin{aligned}
 \{d_4\} &= \{1, 2, 2, 4, 4, 4, 8, 8, 8, 8, 16, 16, 16, 16, 16, 32, 32, 32, 32, 32, \dots\} \\
 &= \{2^0, 2^1, 2^1, 2^2, 2^2, 2^2, 2^3, 2^3, 2^3, 2^3, 2^4, 2^4, 2^4, 2^4, 2^4, 2^5, 2^5, 2^5, 2^5, 2^5, \dots\} \\
 &= \{1 \text{ of } 2^0, 2 \text{ of } 2^1, 3 \text{ of } 2^2, 4 \text{ of } 2^3, 5 \text{ of } 2^4, 6 \text{ of } 2^5, \dots\}
 \end{aligned}$$

When $n = t_x$, then $M(t_x) = \sum_{i=1}^x i2^{i-1} = (x-1)2^x + 1$.

Proof: By Mathematical Induction.

Let p_x be the statement that $\sum_{i=1}^x i2^{i-1} = (x-1)2^x + 1, x \in \mathbb{N}$.

When $x = 1$,

$$\text{LHS} = 1 \times 2^0 = 1$$

$$\text{RHS} = (1-1)2^1 + 1 = 1$$

Since LHS = RHS, p_1 is true.

Assume that p_k is true for some $k \in \mathbb{N}$, i.e. $\sum_{i=1}^k i2^{i-1} = (k-1)2^k + 1$.

To prove: p_{k+1} is true based on the assumption that p_k is true.

$$\begin{aligned} \text{LHS of } p_{k+1} &= \sum_{i=1}^{k+1} i2^{i-1} \\ &= [2^k(k-1)+1] + (k+1)2^{k+1-1} \\ &= 2^k(k-1)+1 + (k+1)2^k \\ &= 2^k(2k)+1 \\ &= k \cdot 2^{k+1} + 1 \\ &= ((k+1)-1)2^{k+1} + 1 \\ &= \text{RHS of } p_{k+1} \end{aligned}$$

Hence p_k is true implies that p_{k+1} is true.

Since p_1 is true and p_k is true $\Rightarrow p_{k+1}$ is true, by mathematical induction, p_x is true for all $x \in \mathbb{N}$.

4.3.5 If the number of disks, n , is NOT a triangular number, t_x , then,

$M(n) = M(t_x) - 2^{x-1}(t_x - n)$ iff $t_{x-1} < n < t_x$, and both $x-1$ and x are optimizing choice for j .

From **4.3.2**, $M_4(t_x) = 2^x(x-1)+1$

$$\begin{aligned} M(n) &= M(t_x) - 2^{x-1}(t_x - n) \\ &= 2^x(x-1)+1 - 2^{x-1}(t_x - n) \\ &= 2^{x-1} \left[2(x-1) - \frac{x(x+1)}{2} + n \right] + 1 \\ &= 2^{x-1} \left[\frac{3x - x^2 - 4}{2} + n \right] + 1 \\ &= 2^x \left[\frac{2n + 3x - x^2 - 4}{4} \right] + 1 \end{aligned}$$

The general formula is $M_4(n) = 2^x \left[\frac{2n + 3x - x^2 - 4}{4} \right] + 1$

4.3.6 The recursive formula for solutions to the Tower of Hanoi problem with n disks and 4 rods is as follows:

$$M_4(n) = \begin{cases} 2M_4(n-j) + M_{k-1}(j) & n > 0, j = x \text{ if } n = t_x \\ 2M_4(n-j) + M_{k-1}(j) & j = x \text{ or } j = x-1 \text{ if } t_{x-1} < n < t_x \end{cases}$$

5 Variation 2: Five-Rod ‘Tower of Hanoi’

As will be shown in column 4 (highlighted in red) of Table 16 on page 24, every new difference begins with a consecutive tetrahedral number.

Let T_x be the x^{th} tetrahedral number such that $T_x = \frac{x(x+1)(x+2)}{6}$ and let S_n be the n^{th} sequence of the Tower of Hanoi such that S_x has x number of disks.

Table 15 shows the number of moves $M(n)$ needed to move a tower of $n = T_x$ disks.

T_1	$S_1 = (1)$ $M_1 = 1$ move
T_2	$S_4 = \left(\begin{array}{c} \underline{1}; \quad 2,3,4,3,2; \quad 1 \\ R5/S1 \quad R4/S3 \quad R5/S1 \end{array} \right)$ $M_4 = M(T_2)$ $= (1 + 5 + 1) = 7$ moves
T_3	$S_{10} = \left(\begin{array}{c} \underline{1,2,3,4,3,2,1}; \quad 5,6,7,6,5,8,9,8,10,8,9,8,5,6,7,6,5; \quad 1,2,3,4,3,2,1; \\ R5/S4 \quad R4/S6 \quad R5/S4 \end{array} \right)$ $M_{10} = M(T_3)$ $= (7 + 17 + 7) = 31$ moves
T_4	$S_{20} = \left(\begin{array}{c} 1,2,3,4,3,2,1,5,6,7,6,5,8,9,8,10,8,9,8,5,6,7,6,5,1,2,3,4,3,2,1; \\ R5/S10 \\ 11,12,13,12,11,14,15,14,16,14,15,14,11,12,13,12,11,17,18,17,19,17,18,17, \\ R4/S10 \\ 20,17,18,17,19,17,18,17,11,12,13,12,11,14,15,14,16,14,15,11,12,13,12,11; \\ R4/S10 \\ 1,2,3,4,3,2,1,5,6,7,6,5,8,9,8,10,8,9,8,5,6,7,6,5,1,2,3,4,3,2,1; \\ R5/S10 \end{array} \right)$ $M_{20} = M(T_4)$ $= (31 + 49 + 31) = 111$ moves

Table 15

Claim 5.1:

From Table 14, we can obtain the recursive formula for number of moves $M(n)$ needed to move a tower of $n = T_x$ disks where T_x is a tetrahedral number for the 5-rod ‘Tower of Hanoi’.

$$\begin{aligned}
 M_5(T_x) &= 2M_5(T_{x-1}) + M_4(T_x - T_{x-1}) \\
 &= 2M_5(T_{x-1}) + M_4(t_x) \\
 &= 2M_5(T_{x-1}) + 2^x(x-1) + 1
 \end{aligned}$$

The recursive formula is $M_5(T_x) = M_5(T_{x-1}) + 2^x(x-1) + 1$

Claim 5.2: Using recursion, we can also obtain the general formula for $M_5(T_x)$:

$$\begin{aligned}
 M_5(T_x) &= 2M_5(T_{x-1}) + 2^x(x-1) + 1 \\
 &= 2[2M_5(T_{x-2}) + 2^{x-1}(x-2) + 1] + 2^x(x-1) + 1 \\
 &= 2^2 M_5(T_{x-2}) + 2^x[(x-1) + (x-2)] + 2^1 + 2^0 \\
 &= 2^2 [2M_5(T_{x-3}) + 2^{x-2}(x-3) + 1] + 2^x[(x-1) + (x-2)] + 2^1 + 2^0 \\
 &= 2^3 M_5(T_{x-3}) + 2^x[(x-1) + (x-2) + (x-3)] + 2^2 + 2^1 + 2^0 \\
 &= 2^3 [2M_5(T_{x-4}) + 2^{x-3}(x-4) + 1] + 2^x[(x-1) + (x-2) + (x-3)] + 2^2 + 2^1 + 2^0 \\
 &= 2^4 M_5(T_{x-4}) + 2^x[(x-1) + (x-2) + (x-3) + (x-4)] + 2^3 + 2^2 + 2^1 + 2^0 \\
 &\vdots \\
 &= 2^{x-1} M_5(1) + 2^x[(x-1) + (x-2) + \dots + 2 + 1] + \sum_{n=0}^{x-1} 2^n \\
 &= 2^{x-1} + 2^x \left(\frac{x(x-1)}{2} \right) + 2^{x-1} - 1 \\
 &= 2^x + 2^x \left(\frac{x(x-1)}{2} \right) - 1 \\
 &= 2^x \left[\frac{x(x-1) + 2}{2} \right] - 1 \\
 &= 2^{x-1} [x(x-1) + 2] - 1
 \end{aligned}$$

The general formula is $M_5(T_x) = 2^{x-1} [x(x-1) + 2] - 1$.

Claim 5.3

The number of differences of $2^0, 2^1, 2^2, 2^3$ and so on are consecutive triangular numbers with

the general formula $T_n = \frac{n(n+1)}{2} = \binom{n+1}{2}$ where $\binom{a}{b}$ is the binomial coefficient.

$$\begin{aligned}
 \{d_5\} &= \{1, 2, 2, 2, 4, 4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, 8\} \\
 &= \{2^0, 2^1, 2^1, 2^1, 2^2, 2^2, 2^2, 2^2, 2^2, 2^2, 2^3, 2^3, 2^3, 2^3, 2^3, 2^3, 2^3, 2^3, 2^3\} \\
 &= \{1 \text{ of } 2^0, 3 \text{ of } 2^1, 6 \text{ of } 2^2, 10 \text{ of } 2^3\}
 \end{aligned}$$

When $n = t_x$, $M_5(t_x) = \sum_{i=1}^x \frac{i(i+1)}{2} 2^{i-1} = 2^{x-1} [x(x-1) + 2] - 1$.

Proof: By Mathematical Induction

Let p_x be a statement that $\sum_{i=1}^x \frac{i(i+1)}{2} 2^{i-1} = 2^{x-1} [x(x-1)+2] - 1; x \in \mathbb{N}$

When $x = 1$,

$$\text{LHS} = \frac{1(1+1)}{2} (2^1) = 1$$

$$\text{RHS} = 2^{1-1} (1(1-1)+2) - 1 = 1$$

Since LHS = RHS, p_1 is true.

Assume that p_k is true for some $k \in \mathbb{N}$, ie $\sum_{i=1}^k \frac{i(i+1)}{2} 2^{i-1} = 2^{k-1} [k(k-1)+2] - 1; k \in \mathbb{N}$.

To prove: p_{k+1} based on the assumption that p_k is true

$$\begin{aligned} \text{LHS of } p_{k+1} &= \sum_{i=1}^{k+1} \frac{i(i+1)}{2} 2^{i-1} \\ &= 2^{k-1} [k(k-1)+2] - 1 + 2^k \left(\frac{(k+1)(k+2)}{2} \right) \\ &= 2^{k-1} [k(k-1) + (k+1)(k+2) + 2] - 1 \\ &= 2^{k-1} [2k^2 + 2k + 4] - 1 \\ &= 2^k [(k+1)k + 2] - 1 \\ &= \text{RHS of } p_{k+1} \end{aligned}$$

Hence p_k is true implies that p_{k+1} is true.

Since p_1 is true and p_k is true $\Rightarrow p_{k+1}$ is true, by mathematical induction, p_x is true for all $x \in \mathbb{N}$.

Claim 5.4:

We can also obtain the recursive formula for any non-tetrahedral number n such that for $T_{x-1} < n < T_x, n \in \mathbb{N}$,

$$M_5(n) = M_5(T_x) - 2^{x-1} (T_x - n)$$

Claim 5.5: Using the above formula in Claim 5.4, we can obtain the general formula for $M_5(n)$

$$\begin{aligned}
 M_5(n) &= M_5(T_x) - 2^{x-1}(T_x - n) \\
 &= 2^{x-1} [x(x-1) + 2] - 2^{x-1}(T_x - n) - 1 \\
 &= 2^{x-1} [x(x-1) + 2] - 2^{x-1} \left(\frac{x(x+1)(x+2)}{6} - n \right) - 1 \\
 &= 2^{x-1} [x(x-1) + 2] - 2^{x-1} \left(\frac{x(x+1)(x+2) - 6n}{6} \right) - 1 \\
 &= 2^{x-1} \left[x(x-1) + 2 - \frac{x(x+1)(x+2) - 6n}{6} \right] - 1 \\
 &= 2^{x-1} \left(\frac{6x(x-1) - x(x+1)(x+2) + 6n + 12}{6} \right) - 1 \\
 &= 2^{x-1} \left(\frac{6x^2 - 6x - x^3 - 3x^2 - 2x + 6n + 12}{6} \right) - 1 \\
 &= 2^{x-1} \left(\frac{3x^2 - 8x - x^3 + 6n + 12}{6} \right) - 1
 \end{aligned}$$

The general formula is $M_5(n) = 2^{x-1} \left(\frac{3x^2 - 8x - x^3 + 6n + 12}{6} \right) - 1$.

6 Variation 3: Six-Rod ‘Tower of Hanoi’

As shown in column 5 (highlighted in blue) of Table 17, consecutive pentatope numbers start a new difference. Let P_x be a x^{th} pentatope number.

The move sequence for pentatope numbers are as shown in Table 16.

P_1	$S_1 = \{1\}$ $M_6(P_1) = 1$ moves
P_2	$S_5 = \{1; 2, 3, 4, 5, 4, 3, 2; 1\}$ $M_6(P_2) = 1 + 7 + 1$ $= 9$ moves
P_3	$S_{15} = \left\{ 1, 2, 3, 4, 5, 4, 3, 2, 1; 6, 7, 8, 9, 8, 7, 6, 10, 11, 12, 11, 10, 13, \right\}$ $\left\{ 14, 13, 15, 13, 14, 13, 10, 11, 12, 11, 10; 1, 2, 3, 4, 5, 4, 3, 2, 1 \right\}$ $M_6(P_3) = 9 + 31 + 9$ $= 49$ moves

Table 16

The general formula for pentatope and non-pentatope numbers are respectively shown below.

General formula for pentatope numbers:

$$\begin{aligned}
 M_6(P_x) &= 2M_6(P_{x-1}) + 2^{x-1} [(x-1)x + 2] - 1 \\
 &= 2[2M_6(P_{x-2}) + 2^{x-2} [(x-2)(x-1) + 2] - 1] + 2^{x-1} [(x-1)x + 2] - 1 \\
 &= 2^2 M_6(P_{x-2}) + 2^{x-1} [(x-2)(x-1) + (x-1)x + 2(2)] - (2^1 + 2^0) \\
 &= 2^2 [2M_6(P_{x-3}) + 2^{x-3} [(x-3)(x-2) + 2] - 1] + 2^{x-1} [(x-2)(x-1) + (x-1)x + 2(2)] - (2^1 + 2^0) \\
 &= 2^3 M_6(P_{x-3}) + 2^{x-1} [(x-3)(x-2) + (x-2)(x-1) + (x-1)x + 3(2)] - (2^2 + 2^1 + 2^0) \\
 &\vdots \\
 &= 2^{x-1} M_6(1) + 2^{x-1} [1(2) + 2(3) + \dots + (x-2)(x-1) + (x-1)x + 2x - 2] - \sum_{n=0}^{x-2} 2^n \\
 &= 2^{x-1} + 2^{x-1} \left[2x - 2 + \sum_{n=1}^{x-1} n(n+1) \right] - 2^{x-1} + 1 \\
 &= 2^{x-1} \left[2x - 2 + \frac{x(x-1)(x+1)}{3} \right] + 1 \\
 &= 2^{x-1} \left[\frac{x(x-1)(x+1) + 6(x-1)}{3} \right] + 1 \\
 &= 2^{x-1} \left[\frac{(x-1)(x(x+1) + 6)}{3} \right] + 1
 \end{aligned}$$

The general formula is $M_6(P_x) = 2^{x-1} \left[\frac{(x-1)(x(x+1) + 6)}{3} \right] + 1$

General formula for any non-pentatope number such that $P_{x-1} < n < P_x$, $n \in \mathbb{N}$:

$$\begin{aligned}
 M_6(n) &= M_6(P_x) - 2^{x-1}(P_x - n) \\
 &= 2^{x-1} \left[\frac{(x-1)(x(x+1)+6)}{3} \right] - 2^{x-1}(P_x - n) + 1 \\
 &= 2^{x-1} \left[\frac{(x-1)(x(x+1)+6)}{3} \right] - 2^{x-1} \left[\frac{x(x+1)(x+2)(x+3) - 24n}{24} \right] + 1 \\
 &= 2^{x-1} \left[\frac{8(x-1)(x(x+1)+6) - x(x+1)(x+2)(x+3) + 24n}{24} \right] \\
 &= 2^x \left[\frac{-x^4 + 2x^3 - 11x^2 + 34x + 24n - 48}{48} \right] + 1
 \end{aligned}$$

The general formula is $M_6(n) = 2^x \left[\frac{-x^4 + 2x^3 - 11x^2 + 34x + 24n - 48}{48} \right] + 1$

7 Minimum Number of Moves for the Tower of Hanoi for $3 \leq r \leq 8$ and $0 \leq n \leq 28$

No of rods, r		3	4	5	6	7	8
No of disks, n							
	0	0	0	0	0	0	0
t_1	1	P_1, T_1	1	1	1	1	1
	2	3	3	3	3	3	3
t_2	3	7	5	5	5	5	5
	4	T_2	15	9	7	7	7
	5	P_2	31	13	11	9	9
t_3	6	63	17	15	13	11	11
	7	127	25	19	17	15	13
	8	255	33	23	21	19	17
	9	511	41	27	25	23	21
t_4	10	T_3	1023	49	31	27	25
	11	2047	65	39	33	31	29
	12	4095	81	47	37	35	33
	13	8191	97	55	41	39	37
	14	16383	113	63	45	43	41
t_5	15	P_3	32767	129	71	47	45
	16	65535	161	79	57	51	49
	17	131071	193	87	65	55	53
	18	262143	225	95	73	59	57
	19	524287	257	103	81	63	61
	20	T_4	1048575	289	111	67	65
t_6	21	2097151	321	127	97	71	69
	22	4194303	385	143	105	79	73
	:						:
	28						97

Table 17

Legend:

i indicates the number of data values with differences of 2, $2^2=4$, $2^3=8$,... $\{i : i \geq 1, i \in \mathbb{N}\}$

$\triangle t_x$, $\square T_x$ and $\text{pentagon } P_x$ represents the x^{th} triangular, tetrahedral and pentatope numbers respectively, $\{x : x \geq 1, x \in \mathbb{N}\}$.

The triangular, tetrahedral and pentatope numbers are found in the Pascal's Triangle as shown in Figure 5.

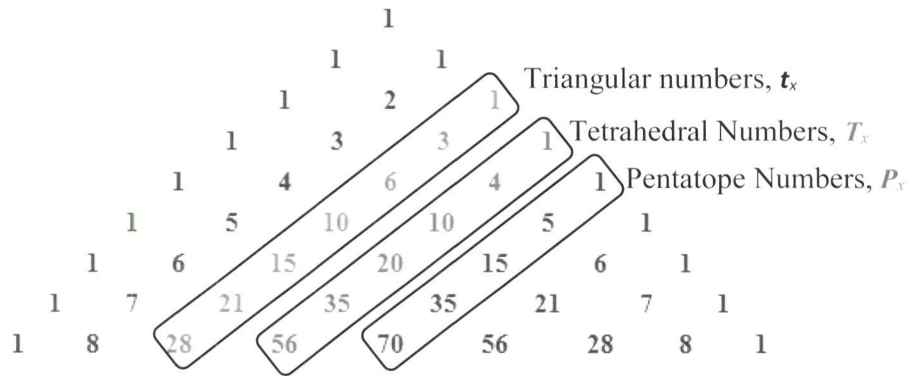


Figure 5

A power function graph for the Tower of Hanoi and piecewise linear function graphs for the variations show the connections between the numerical, symbolic and graphical representations. Since the number of disks and number of moves are discrete data, it is not appropriate to draw a line connecting the plotted points.

The graphical relationship between the number of moves versus the number of disks for $0 \leq n \leq 19, 3 \leq r \leq 6; n, r \in \mathbb{N}$ is illustrated in Figure 6.

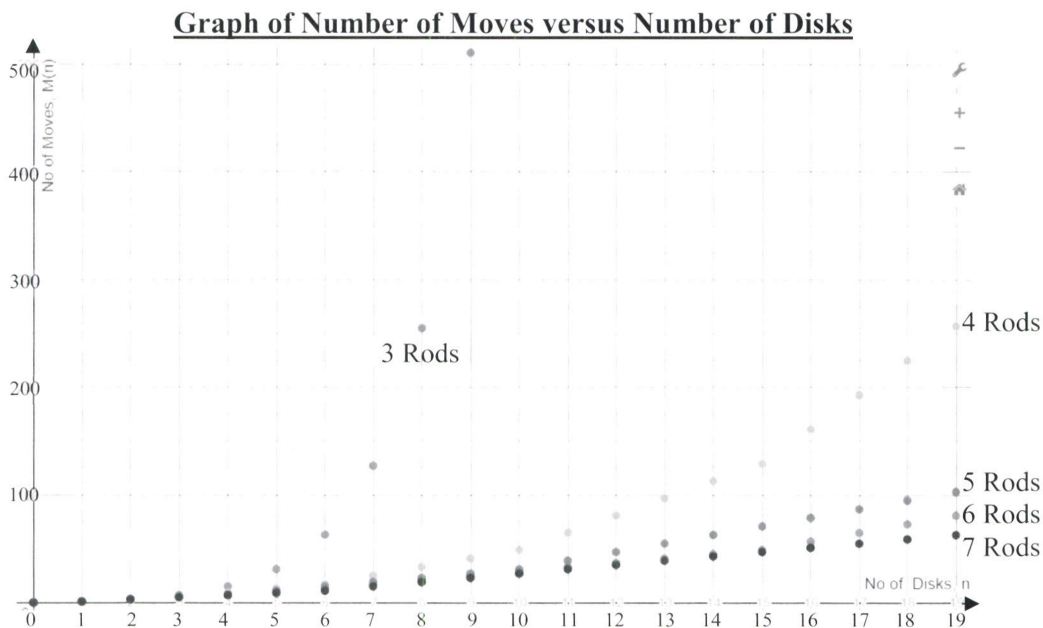


Figure 6

As seen from the graphs, as the number of rods increases, the number of moves decreases very sharply, particularly for three to four rods.

5 (T_x is a Tetrahedral number)	$M_5(n) = M_5(T_x)$ $= \sum_{i=1}^x \frac{i(i+1)}{2} 2^{i-1}$	$M_5(n) = M_5(T_x)$ $= 2^{x-1} [x(x-1)+2] - 1$
6 (P_x is a Pentatope number)	$M_6(n) = M_6(P_x)$ $= \sum_{i=1}^x \frac{i(i+1)(i+2)}{6} 2^{i-1}$	$M_6(n) = M_6(P_x)$ $= 2^{x-1} \left[\frac{(x-1)(x(x+1)+6)}{3} \right] + 1$
r	$\sum_{i=1}^x \binom{x+r-4}{r-3} 2^{i-1}$	$M_r \left(\binom{x+r-3}{r-2} \right) = 2M_r \left(\binom{x+r-4}{r-2} \right) + M_{r-1} \left(\binom{x+r-4}{r-3} \right)$

Table 19

Claim 8.3:

Number of Rods, $r, r \in \mathbb{N}$ Number of Disks, $n, n \in \mathbb{N}$	General Formula for Minimum Number of Moves, $M_r(n)$, such that $x, r, n \in \mathbb{N}$
3	$M_3(n) = 2^n - 1$
4 ($t_{x-1} < n < t_x$)	$M_4(n) = 2^x \left[\frac{-x^2 + 3x + 2n - 4}{4} \right] + 1$
5 ($T_{x-1} < n < T_x$)	$M_5(n) = 2^{x-1} \left(\frac{3x^2 - 8x - x^3 + 6n + 12}{6} \right) - 1$
6 ($P_{x-1} < n < P_x$)	$M_6(n) = 2^x \left[\frac{-x^4 + 2x^3 - 11x^2 + 34x + 24n - 48}{48} \right] + 1$
r $\binom{x+r-4}{r-2} < n < \binom{x+r-3}{r-2}$	$M_r(n) = M_r \left(\binom{x+r-3}{r-2} \right) - 2^{x-1} \left(\binom{x+r-3}{r-2} - n \right)$

Table 20

9 Applications of the Tower of Hanoi

- 9.1** The recursive algorithm makes it a popular game for beginning programming students and is taught in computer science course in first year university.
- 9.2** The game is used as a test on patients with dementia or stroke to see which areas of their brains have been impaired.
- 9.3** The game can be used as a tool for problem-solving among students in schools, providing an avenue for them to think critically and analytically as they look for number patterns and relationships in each move.
- 9.4** It can also be played by senior citizens as a past-time to slow down their chances of dementia as the neurons of their brains are activated.

10 CONCLUSION

The game allows us to make natural connections between the disk moves and the numeric expressions related to each method. Through this project we discover an efficient and systematic move sequence that can achieve the optimal solution, that is, the minimum number of moves required to transfer n disks from one rod to another rod for The Tower of Hanoi through different lens, namely the step-by-step strategy including the binary perspective and the recursion method that leads to the general formulae.

For three rods, the choice of the first move is extremely crucial depending on the parity of the disks at the Source rod. Disks of the same parity cannot be placed on top of each other. When there are two possible rods, the disk must be placed in the non-empty rod. Based on these disk moves, disk positions can be directly determined from the binary representation of the move number following certain rules.

For the Variation on the Four-Rod Tower of Hanoi, the optimal number of moves is achieved by transferring the top $n - j$ smallest disks D_{n-j}, \dots, D_1 from the Source (S) to Aux (A), using all four rods in the process. The choice of the $n - j - 1$ move is of great importance as it follows the three rod algorithm.

If $n = t_x$, the x^{th} triangular number, the optimizing choice for j is $j = x$.

If $t_{x-1} < n < t_x$, both $x - 1$ and x are optimizing choice for j .

Subsequently, the remaining j largest disk are moved from Source (S) to Target (T), using the standard three rod algorithm.

Similarly, for the Variation on the Five-Rod Tower of Hanoi, the optimal number of moves depends on T_x , the x^{th} tetrahedral number.

For the Variation on the Six-Rod Tower of Hanoi, the optimal number of moves depends on P_x , the x^{th} pentatope number.

The use of numeric, tabular and graphical representations for The Tower of Hanoi and its Variations gives us the opportunity to see patterns and relationships that we might not normally see if we were studying only one or two representations independent of each other. By looking at the patterns and relationships, we are able to prove the general formulae for The Tower of Hanoi and its 4-rod and 5-rod variations by Mathematical Induction.

Last but not least, this project has open up a new horizon and will definitely be a stepping stone for us to explore further by learning basic programming that involves recursion. We will also research deeper and try to prove some of our stated claims in our extension of this project.

11 REFERENCES

- https://en.wikipedia.org/wiki/Tower_of_Hanoi
- www.cs.wm.edu/~pkstoc/boca.pdf
- <http://cs.nyu.edu/courses/summer07/G22.2340-001/Presentations/McCann.pdf>