Reflection Within Conics

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Introduction

With the re-introduction of the Further Mathematics into the GCE 'A' Level syllabus, the study of conics in greater depth will now be part of the Further Mathematics student's learning. This article seeks to explore reflection of light within internally-reflecting conics, making use of properties that students will learn in the subject to prove all the relevant results. It is in hope that students taking Further Mathematics will see how content learnt can be applied in further ways.

1. Definitions and Properties of Conics

In this article, we will make use of various properties of conics in proving our results. As properties of reflection do not change with the position or orientation of the conic section on the Cartesian plane, we will make some assumptions without loss of generality in proving our results. These properties are summarised below:

Ellipses

- Cartesian equation: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where without loss of generality, $a \geq b > 0$ and $h = k = 0$.
- The parameters a, b, and e are related by the equation $b^2 = a^2(1 - e^2)$, where e is the eccentricity of the ellipse.

Hyperbolas

- Cartesian equation: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where without loss of generality, $a \geq b > 0$ and $h = k = 0$.
- The parameters a, b, and e are related by the equation $b^2 = a^2(e^2 - 1)$, where e is the eccentricity of the hyperbola.

Parabolas

- Cartesian equation: $y = ax^2 + bx + c$, where without loss of generality $a > 0$ and $b = c = 0$.

2. The Initial Conjecture

We did some rough drawings and simulations by following the law of reflection, which states that the angle of incidence is always equal to angle of reflection. It seemed clear that as a ray of light is allowed to continuously reflect within an ellipse, the path it forms seems to enclose (or strictly speaking, is tangent to) either an ellipse or a hyperbola, depending on where the initial ray was. In fact, it seemed likely that this enclosed ellipse or hyperbola shared the same foci as the original ellipse. As such, we make the initial conjecture as follows:

If the initial ray passes between a focus and the nearer x-intercept of ellipse ξ, then the light path is such that it is tangential to a confocal ellipse (see **Figure 2.1**). If the initial ray passes between the two foci of ξ, then the light path is such that it is tangential to a confocal hyperbola.

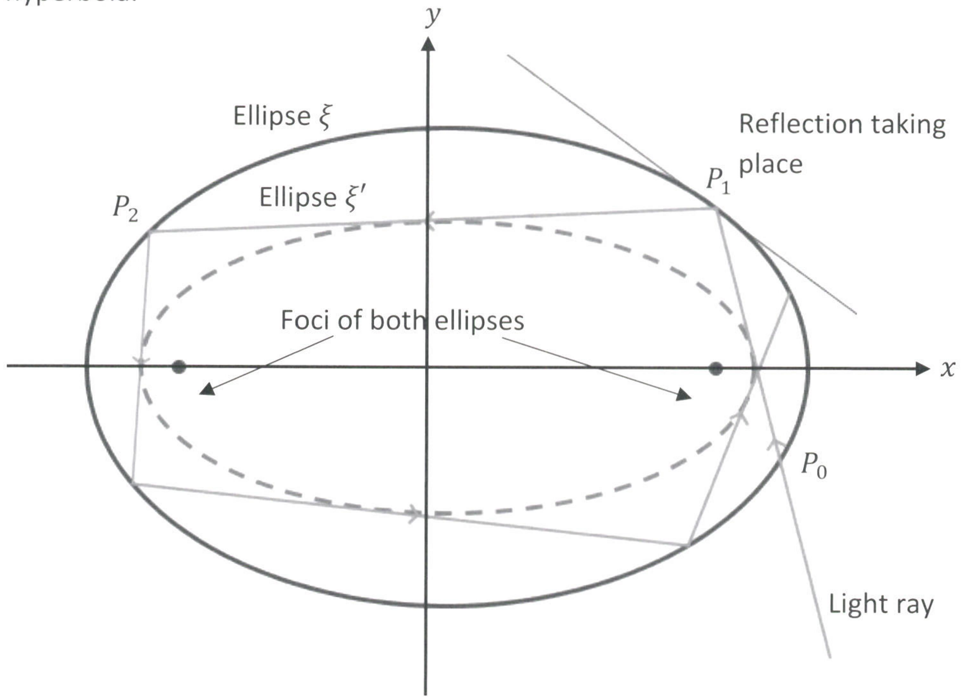


Figure 2.1

3. Proving the Initial Conjecture

Key Idea 1

The law of reflection states that given any reflection, the angle of incidence is always equal to the angle of reflection. Hence, it seems intuitive that proving the conjecture would require some work with angles. However, given "curvy" nature of conic sections, this approach may be difficult as it is hard to obtain information on the angles.

However, we recall from Additional Mathematics in Secondary School that the gradient of a line is actually equal to the tangent of the angle it makes with the x-axis. Hence, this gives us an avenue to **relate angles to Cartesian parameters**.

Key Idea 2

Let us look at the first part of the conjecture, where the initial ray passes between a foci and the nearer x-intercept. Let the equation of the internally reflecting ellipse ξ be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the equation of the confocal ellipse ξ' be $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$. Using the property that ξ and ξ' are confocal, which means that both ellipses have the same eccentricity e, we can obtain the relationship $a^2-b^2=p^2-q^2$. This now allows us to **relate the Cartesian parameters of the two confocal conics**.

Key Idea 3

We consider each ray to be a line segment on ξ, while letting the initial ray be one that intersects ξ at P_0 and P_1. The next reflected ray will then be P_1P_2, where P_2 is a point on ξ. Instead of assuming that P_0P_1 and P_1P_2 make the same angle to the normal at P_1 and trying to prove that these two rays are tangential to ξ', we do the reverse.

We find the two rays from P_1 that are tangential to ξ' and prove that these two rays make the same angle with the normal at P_1.

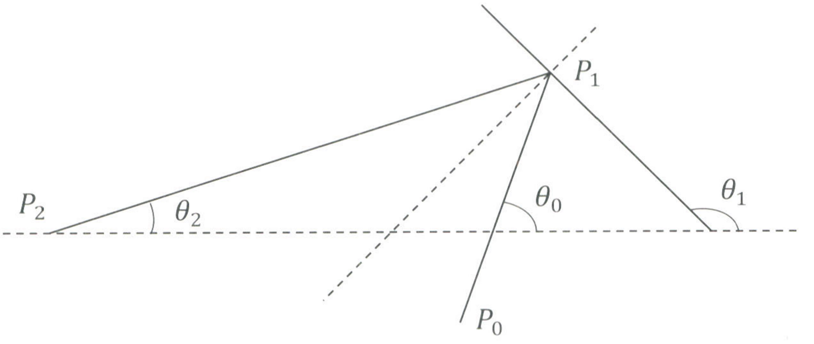


Figure 3.1

Sketch of Proof

1) Following Key Idea 1, we look at three key lines/line segments: P_0P_1, P_1P_2, and the tangent to ξ at P_1. We then look that the angles these three lines/line segments make with the x-intercept, naming them θ_0, θ_2 and θ_1 respectively as we try to relate them in Cartesian terms.

2) If we have that P_0P_1 and P_1P_2 make the same angle to the normal at P_1, we are able to obtain a relationship between the angles θ_0, θ_2 and θ_1. In fact, we will obtain $2\theta_1=\pi+\theta_0+\theta_2$ if and only if P_0P_1 and P_1P_2 make the same angle to the normal at P_1.

3) By taking tangents on both sides, and applying simple trigonometric properties and manipulations, we then obtain $\frac{2\tan\theta_1}{1-\tan^2\theta_1}=\frac{\tan\theta_0+\tan\theta_2}{1-\tan\theta_0\tan\theta_2}$. Thereafter, we notice that $\tan\theta_0$, $\tan\theta_2$ and $\tan\theta_1$ are the gradients of P_0P_1, P_1P_2, and the tangent to ξ at P_1 respectively. We now have a relationship between gradients instead of the angles, which we want to prove is true if P_0P_1 and P_1P_2 are tangential to ξ'.

4) Letting the coordinates of $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, we are able to express the LHS of the trigonometric equation in Step 3 terms of a, b, x_1 and y_1 by considering the gradient function at P_1.

5) Next, we consider all the lines that pass through P_1, which would have the general equation of $\frac{y-y_1}{x-x_1} = m$, where m is the gradient of the line. Consider the intersection of such lines with the confocal ellipse ξ' and we obtain a quadratic equation in x. When a line through P_1 is tangential to ξ', we have that the discriminant of the quadratic equation in x being equal to 0, which gives on simplification:

$$(p^2 - x_1^2)m^2 + 2x_1y_1m + q^2 - y_1^2 = 0.$$

6) We know the solutions of the quadratic equation in m found in Step 5 be $\tan\theta_0$ and $\tan\theta_2$. Using sum of roots and product of roots formula, we can express RHS of the trigonometric equation in Step 3 in terms of p, q, x_1 and y_1.

7) Referring to Key Idea 2, where we have a relationship between a, b, p and q, as well as Step 4 and Step 6, we can easily prove that LHS equals to RHS for the equation in Step 3 when P_0P_1 and P_1P_2 are tangential to ξ'. Hence, this means that P_0P_1 and P_1P_2 are tangential to ξ' if and only if they obey the law of reflection.

With this, we can inductively show that if the initial ray passes between a focus and the nearer x-intercept of ellipse ξ, then the light path is such that it is tangential to a confocal ellipse ξ'.

In an almost identical way, we can show as well that if the initial ray passes between the two foci of ξ, then the light path is such that it is tangential to a confocal hyperbola.

4. Extension of Initial Conjecture/Result

Having proven the initial conjecture (which is now a result), we look to extend it to cover reflection of light rays in the other conic sections – namely hyperbolas and parabolas. Due to the similar nature of conics in the way they are defined, we can prove corresponding results in a very similar way.

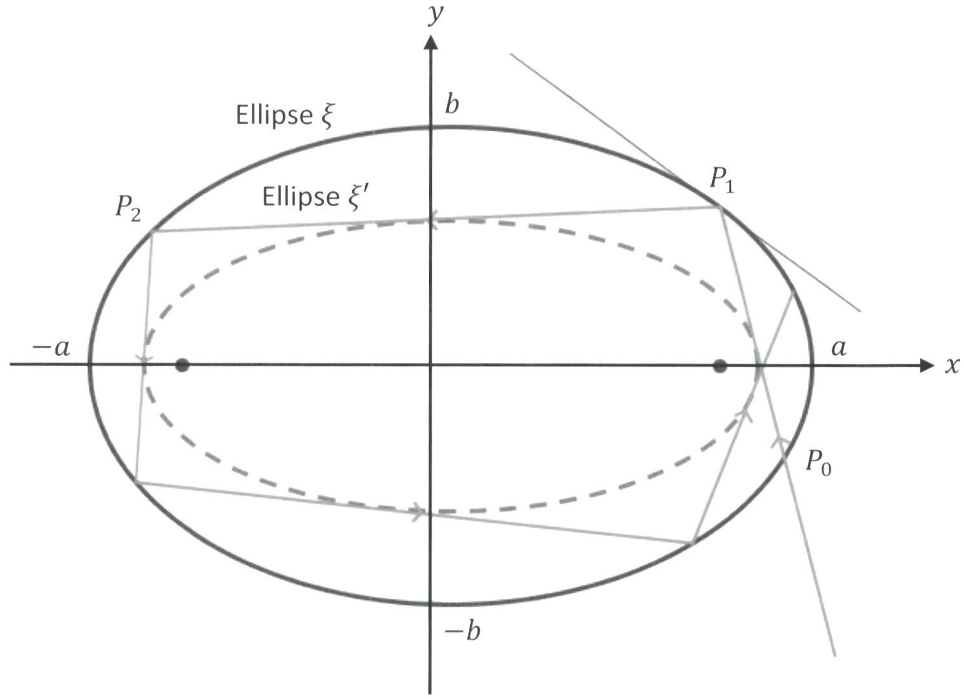
The results are summarised as follows:

- If the initial ray passes between a focus and the nearer x-intercept of ellipse ξ, then the light path is such that it is tangential to a confocal ellipse. If the initial ray passes between the two foci of ξ, then the light path is such that it is tangential to a confocal hyperbola.
- If the initial ray passes between a focus of the nearer x-intercept of a hyperbola ζ, then the light path is such that it is tangential to a confocal hyperbola. If the initial ray passes beyond the focus of ζ (ie. it is not between a focus and any x-intercept), then the light path is such that it is tangential to a confocal ellipse.
- The light path in an internally-reflecting parabola ρ is always tangential to another parabola if the initial ray does not pass through the focus of the parabola.

From here on, we will provide full Mathematical proofs for all the results mentioned.

5. Tracing the path of a light ray that initially passes between a focus of an ellipse and its nearer major vertex

This section investigates the path of a light ray that initially passes between a focus of an ellipse and its nearer major vertex as it is reflected within the ellipse. The proof here confirms the conjecture that indeed the light path, after each reflection, is always tangent to



a confocal ellipse within the internally reflecting ellipse.

Figure 5.1

Let the equation of the internally reflecting ellipse ξ be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the equation of confocal ellipse ξ' be $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$, with $p<a$ and $q<b$. Let e_ξ be the eccentricity of ξ and $e_{\xi'}$ be the eccentricity of ξ'.

Then we have the following:

(5.1) $b^2=a^2(1-e_\xi{}^2)$

(5.2) $q^2=p^2(1-e_{\xi'}{}^2)$ and

(5.3) $ae_\xi=pe_{\xi'}$ since the distance of the foci from the origin are given by ae_ξ and $pe_{\xi'}$.

Substituting $e_{\xi'}=\frac{ae_\xi}{p}$ in (5.2), we have

(5.4) $q^2=p^2\left(1-\frac{a^2e_\xi{}^2}{p^2}\right)=p^2-a^2e_\xi{}^2$.

Eliminating e_ξ from both (5.1) and (5.4), we have

(5.5) $a^2-b^2=p^2-q^2$.

Theorem 5.1

Let P_1P_0 and P_1P_2 be tangents to the ellipse ξ', where P_0, P_1, P_2 are points on the ellipse ξ. Then P_1P_0 and P_1P_2 make equal angles with the tangent to the ellipse ξ at P_1.

Proof of Theorem 5.1

Let the Cartesian coordinates of P_1 be (x_1, y_1).
Let the tangent to ξ at P_1 be ST and the normal to ξ at P_1 be MN.
Let the gradients of P_1P_0, P_1P_2 and ST be m_0, m_2 and m_1 respectively.
Let the angles θ_0, θ_2 and θ_1 be respectively the angles made between each of P_1P_0, P_1P_2 and ST, with the positive x-axis.

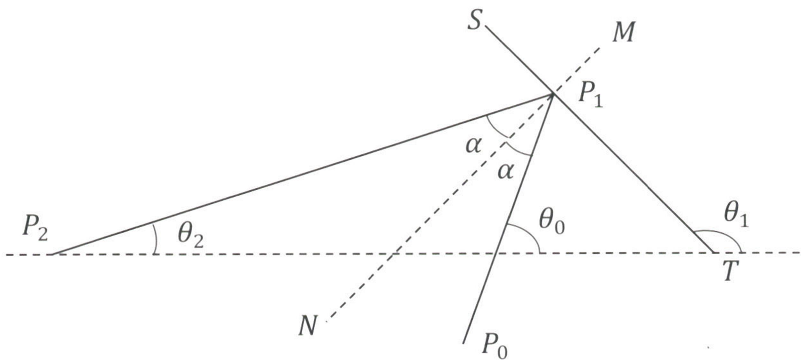


Figure 5.2

By the law of reflection, $\measuredangle P_2P_1N = \measuredangle P_0P_1N = \alpha$.
As such, we have $\measuredangle P_0P_1T = \frac{\pi}{2} - \alpha$ and $\theta_1 = \frac{\pi}{2} - \alpha + \theta_0$.
Also, $\theta_0 = 2\alpha + \theta_2$. Eliminating α, we have
(5.6) $\quad 2\theta_1 = \pi + \theta_0 + \theta_2$.

For the ellipse ξ, $\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{b^2x}{a^2y}$. Thus, at the point P_1, the gradient of the tangent ST is given by $m_1 = -\frac{b^2x_1}{a^2y_1} = \tan\theta_1$. This leads to

$$\tan 2\theta_1 = \frac{2\tan\theta_1}{1-\tan^2\theta_1} = \frac{-\frac{2b^2x_1}{a^2y_1}}{1-\frac{b^4x_1{}^2}{a^4y_1{}^2}} = -\frac{2a^2b^2x_1y_1}{a^4y_1{}^2-b^4x_1{}^2}.$$

However, $\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = 1$ or $a^2y_1{}^2 = a^2b^2 - b^2x_1{}^2$. Hence, we have

(5.7) $\quad \tan 2\theta_1 = -\frac{2a^2b^2x_1y_1}{a^4y_1{}^2-b^4x_1{}^2} = -\frac{2a^2b^2x_1y_1}{a^2(a^2b^2-b^2x_1{}^2)-b^4x_1{}^2} = -\frac{2a^2x_1y_1}{a^4-(a^2+b^2)x_1{}^2}.$

The equation of any straight line passing through the point P_1 is given by $\frac{y-y_1}{x-x_1} = m$ or $y = y_1 + m(x - x_1) = mx + y_1 - mx_1$, where m is the gradient of the line.

To find the points of intersection of any straight line through point P_1 and the ellipse ξ', it is equivalent to solving the pair of simultaneous equations
$y = mx + y_1 - mx_1$ and $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$.

Substituting $y = mx + y_1 - mx_1$ in $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ yields

$\frac{x^2}{p^2} + \frac{(mx+y_1-mx_1)^2}{q^2} = 1$ or $q^2x^2 + p^2(mx + y_1 - mx_1)^2 - p^2q^2 = 0$,

which simplifies to

$(p^2m^2 + q^2)x^2 + 2p^2m(y_1 - mx_1)x + p^2(y_1 - mx_1)^2 - p^2q^2 = 0$.

In the case of P_1P_2 being tangent to ξ', we have

$4p^4m^2(y_1 - mx_1)^2 - 4(p^2m^2 + q^2)[p^2(y_1 - mx_1)^2 - p^2q^2] = 0$ or

(5.8) $\quad (p^2 - {x_1}^2)m^2 + 2x_1y_1m + q^2 - {y_1}^2 = 0$.

The solutions to (5.8) must be m_0 and m_2, which are the gradients of P_1P_0 and P_1P_2 respectively.

Since m_0 and m_2 are the solutions to (5.8), we have

$m_0 + m_2 = -\frac{2x_1y_1}{p^2-{x_1}^2}$ and $m_0m_2 = \frac{q^2-{y_1}^2}{p^2-{x_1}^2}$.

Hence,

$$\tan(\pi + \theta_0 + \theta_2) = \tan(\theta_0 + \theta_2) = \frac{\tan\theta_0+\tan\theta_2}{1-\tan\theta_0\tan\theta_2} = \frac{m_0+m_2}{1-m_0m_2} = \frac{-\frac{2x_1y_1}{p^2-{x_1}^2}}{1-\frac{q^2-{y_1}^2}{p^2-{x_1}^2}} = -\frac{2x_1y_1}{p^2-{x_1}^2-q^2+{y_1}^2}.$$

However, from (5.5), $a^2 - b^2 = p^2 - q^2$. Thus,

$\tan(\theta_0 + \theta_2) = -\frac{2x_1y_1}{p^2-{x_1}^2-q^2+{y_1}^2} = -\frac{2x_1y_1}{a^2-b^2-{x_1}^2+{y_1}^2}$ or

(5.9) $\quad -2x_1y_1 = [\tan(\theta_0 + \theta_2)][a^2 - b^2 - {x_1}^2 + {y_1}^2]$.

From (5.7), we have $\tan 2\theta_1 = -\frac{2a^2x_1y_1}{a^4-(a^2+b^2){x_1}^2}$ and substituting for $-2x_1y_1$ yields

$\tan 2\theta_1 = \frac{a^2[\tan(\theta_0+\theta_2)][a^2-b^2-{x_1}^2+{y_1}^2]}{a^4-(a^2+b^2){x_1}^2} = \frac{[\tan(\theta_0+\theta_2)][a^4-a^2b^2-a^2{x_1}^2+a^2{y_1}^2]}{a^4-(a^2+b^2){x_1}^2}$.

However, $\frac{{x_1}^2}{a^2} + \frac{{y_1}^2}{b^2} = 1$ or $a^2{y_1}^2 = a^2b^2 - b^2{x_1}^2$. Hence,

$\tan 2\theta_1 = \frac{[\tan(\theta_0+\theta_2)][a^4-a^2b^2-a^2{x_1}^2+a^2b^2-b^2{x_1}^2]}{a^4-(a^2+b^2){x_1}^2} = \frac{[\tan(\theta_0+\theta_2)][a^4-(a^2+b^2){x_1}^2]}{a^4-(a^2+b^2){x_1}^2}$.

Thus, $\tan 2\theta_1 = \tan(\theta_0 + \theta_2) = \tan(\pi + \theta_0 + \theta_2)$ and the theorem is proven. □

From the results of **Theorem 5.1**, as P_0P_1 is an arbitrary chord of the ellipse ξ that intersects the x-axis between a focus and its nearer major vertex $(a, 0)$, it thus shows that for a light ray that travels the paths P_0P_1, P_1P_2, P_2P_3, ..., these paths form chords that are tangent to the confocal ellipse ξ'. By symmetry, any light ray that passes between a focus of an ellipse ξ and its nearer major vertex will travel along paths that are tangential to confocal ellipse ξ'.

6. Tracing the path of a light ray that initially passes between the foci of an ellipse

This section investigates the remaining case for ellipses, when the light ray initially passes between the foci of an ellipse as it is reflected within the ellipse. The proof here confirms that indeed the light path, after each reflection, is always tangent to a confocal hyperbola. The proof here turns out to be similar to that in **Section 5** and hence, many similar steps in the proof are omitted.

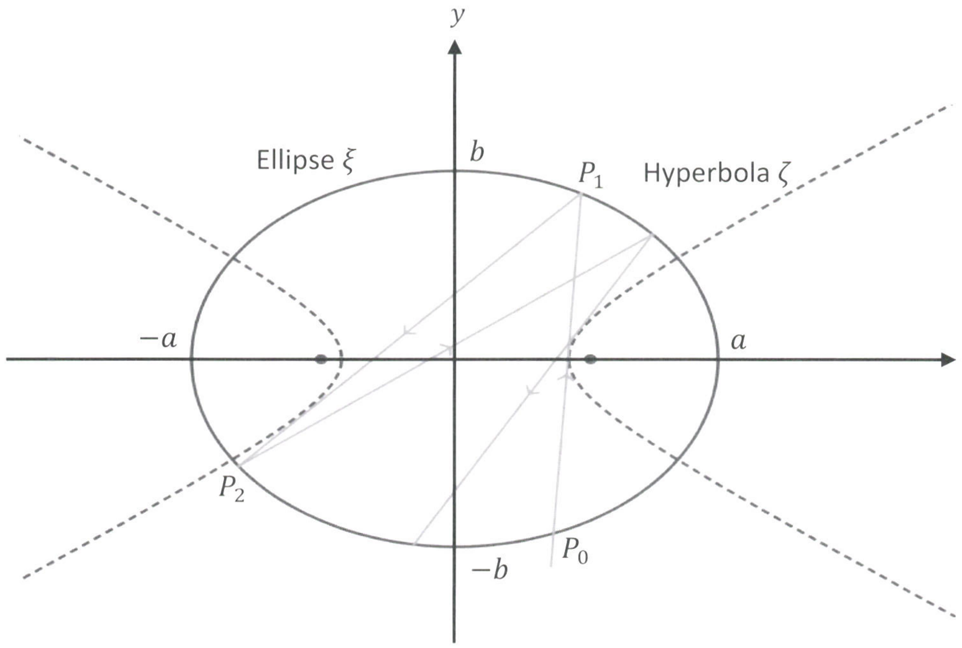


Figure 6.1

The equation of ellipse ξ is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the equation of hyperbola ζ is given by $\frac{x^2}{p^2}-\frac{y^2}{q^2}=1$ with $p<a$, where both ellipse and hyperbola share the same foci.
Let e_ξ be the eccentricity of ξ and e_ζ be the eccentricity of ζ.

Then, we have the following:
(6.1) $b^2=a^2(1-{e_\xi}^2)$
(6.2) $q^2=p^2({e_\zeta}^2-1)$ and
(6.3) $ae_\xi=pe_\zeta$, since the distance of the foci from the origin are given by ae_ξ and pe_ζ.

Substituting $e_\zeta=\frac{ae_\xi}{p}$ in (6.2), we have
(6.4) $q^2=p^2\left(\frac{a^2{e_\xi}^2}{p^2}-1\right)=a^2{e_\xi}^2-p^2$.

Eliminating e_ξ from both (6.1) and (6.4), we have
(6.5) $a^2-b^2=p^2+q^2$.

Note:
Without any loss of generality, Figure 5.2 will be used in the proofs of Theorems 6.1, 7.1, 8.1 and 9.1 for easy reference.

Theorem 6.1

Let P_1P_0 and P_1P_2 be tangents to the hyperbola ζ, where P_0, P_1 and P_2 are points on the ellipse ξ. Then P_1P_0 and P_1P_2 make equal angles with the tangent ST to the ellipse ξ at P_1.

Proof of Theorem 6.1

Let the Cartesian coordinates of P_1 be (x_1, y_1).
Let the tangent to ξ at P_1 be ST and the normal to ξ at P_1 be MN.
Let the gradients of P_1P_0, P_1P_2 and ST be m_0, m_2 and m_1 respectively.
Let the angles θ_0, θ_2 and θ_1 be respectively the angles made between each of P_1P_0, P_1P_2 and ST, with the positive x-axis.

By the law of reflection, $\measuredangle P_2P_1N = \measuredangle P_0P_1N = \alpha$.
As in the proof of **Theorem 5.1**, we have
(6.6) $2\theta_1 = \pi + \theta_0 + \theta_2$.
Also, similar to **Theorem 5.1**, we have
(6.7) $\tan 2\theta_1 = -\frac{2a^2x_1y_1}{a^4-(a^2+b^2){x_1}^2}$.

The equation of any straight line passing through the point P_1 is given by
$\frac{y-y_1}{x-x_1} = m$ or $y = y_1 + m(x - x_1) = mx + y_1 - mx_1$, where m is the gradient of the line.

To find the points of intersection of any straight line through point P_1 and the hyperbola ζ, it is equivalent to solving the pair of simultaneous equations
$y = mx + y_1 - mx_1$ and $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$.

Substituting $y = mx + y_1 - mx_1$ in $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$ yields
$\frac{x^2}{p^2} - \frac{(mx+y_1-mx_1)^2}{q^2} = 1$ or $q^2x^2 - p^2(mx + y_1 - mx_1)^2 - p^2q^2 = 0$,
which simplifies to
$(q^2 - p^2m^2)x^2 - 2p^2m(y_1 - mx_1)x - p^2(y_1 - mx_1)^2 - p^2q^2 = 0$.

In the case of P_1P_2 being tangent to ζ, we have
$4p^4m^2(y_1 - mx_1)^2 + 4(q^2 - p^2m^2)[p^2(y_1 - mx_1)^2 + p^2q^2] = 0$ or
(6.8) $(p^2 - {x_1}^2)m^2 + 2x_1y_1m - q^2 - {y_1}^2 = 0$.

The solutions to (6.8) must be m_0 and m_2, which are the gradients of P_1P_0 and P_1P_2 respectively.
Since m_0 and m_2 are the solutions to (6.8), we have
$m_0 + m_2 = -\frac{2x_1y_1}{p^2-{x_1}^2}$ and $m_0m_2 = -\frac{q^2+{y_1}^2}{p^2-{x_1}^2}$.

Hence,

$$\tan(\pi + \theta_0 + \theta_2) = \tan(\theta_0 + \theta_2) = \frac{\tan\theta_0+\tan\theta_2}{1-\tan\theta_0\tan\theta_2} = \frac{m_0+m_2}{1-m_0m_2} = \frac{-\frac{2x_1y_1}{p^2-{x_1}^2}}{1+\frac{q^2+{y_1}^2}{p^2-{x_1}^2}} = -\frac{2x_1y_1}{p^2-{x_1}^2+q^2+{y_1}^2}.$$

However, from (6.5), $a^2 - b^2 = p^2 + q^2$. Thus,
$\tan(\theta_0 + \theta_2) = -\frac{2x_1y_1}{p^2-{x_1}^2+q^2+{y_1}^2} = -\frac{2x_1y_1}{a^2-b^2-{x_1}^2+{y_1}^2}$ or
(6.9) $-2x_1y_1 = [\tan(\theta_0 + \theta_2)][a^2 - b^2 - {x_1}^2 + {y_1}^2]$.

From (6.7), we have $\tan 2\theta_1 = -\frac{2a^2x_1y_1}{a^4-(a^2+b^2)x_1^2}$ and substituting for $-2x_1y_1$ yields

$$\tan 2\theta_1 = \frac{a^2[\tan(\theta_0+\theta_2)][a^2-b^2-x_1^2+y_1^2]}{a^4-(a^2+b^2)x_1^2} = \frac{[\tan(\theta_0+\theta_2)][a^4-a^2b^2-a^2x_1^2+a^2y_1^2]}{a^4-(a^2+b^2)x_1^2}.$$

However, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ or $a^2y_1^2 = a^2b^2 - b^2x_1^2$. Hence,

$$\tan 2\theta_1 = \frac{[\tan(\theta_0+\theta_2)][a^4-a^2b^2-a^2x_1^2+a^2b^2-b^2x_1^2]}{a^4-(a^2+b^2)x_1^2} = \frac{[\tan(\theta_0+\theta_2)][a^4-(a^2+b^2)x_1^2]}{a^4-(a^2+b^2)x_1^2}.$$

Thus, $\tan 2\theta_1 = \tan(\theta_0 + \theta_2) = \tan(\pi + \theta_0 + \theta_2)$ and the theorem is proven. □

From the results of **Theorem 6.1**, as P_0P_1 is an arbitrary chord of the ellipse ξ that intersects the x-axis between the foci, it thus shows that for a light ray that travels the paths P_0P_1, P_1P_2, P_2P_3, ..., these paths form chords that are tangent to the confocal hyperbola ζ. In other words, any light ray that passes between the foci of an ellipse ξ will travel along paths that are tangential to confocal hyperbola ζ.

7. Tracing the path of a light ray that initially passes between a focus of a hyperbola and its nearer major vertex

This section investigates the path of a light ray that initially passes between a focus of a hyperbola and its nearer major vertex as it is reflected within the hyperbola. The proof here confirms the conjecture that indeed the light path, after each reflection, is always tangent to a confocal hyperbola contained within the original hyperbola.

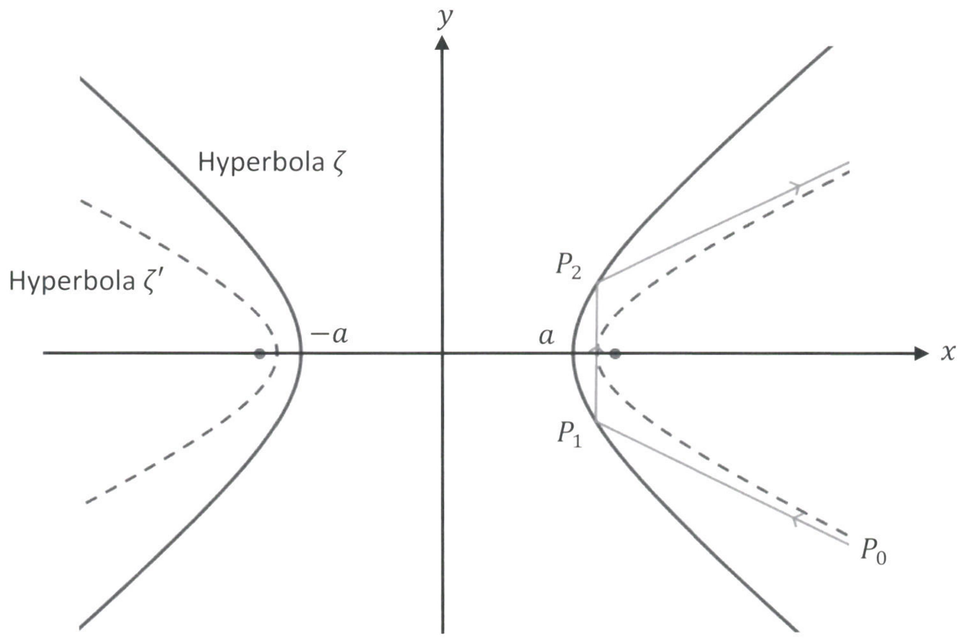


Figure 7.1

Let the equation of the internally reflecting hyperbola ζ be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the equation of confocal hyperbola ζ' be $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$, with $p > a$ and $q < b$. Let e_ζ be the eccentricity of ζ and $e_{\zeta'}$ be the eccentricity of ζ'.

Then we have the following:
(7.1) $b^2 = a^2(e_\zeta{}^2 - 1)$
(7.2) $q^2 = p^2(e_{\zeta'}{}^2 - 1)$ and
(7.3) $ae_\zeta = pe_{\zeta'}$ since the distance of the foci from the origin are given by ae_ζ and $pe_{\zeta'}$.

Substituting $e_{\zeta'} = \frac{ae_\zeta}{p}$ in (7.2), we have
(7.4) $q^2 = p^2\left(\frac{a^2 e_\zeta{}^2}{p^2} - 1\right) = a^2 e_\zeta{}^2 - p^2.$

Eliminating e_ζ from both (7.1) and (7.4), we have
(7.5) $a^2 + b^2 = p^2 + q^2.$

Theorem 7.1

Let P_1P_0 and P_1P_2 be tangents to the hyperbola ζ', where P_0, P_1 and P_2 are points on the hyperbola ζ. Then P_1P_0 and P_1P_2 make equal angles with the tangent ST to the hyperbola ζ at P_1.

Proof of Theorem 7.1

Let the Cartesian coordinates of P_1 be (x_1, y_1).
Let the tangent to ζ at P_1 be ST and the normal to ζ at P_1 be MN.
Let the gradients of P_1P_0, P_1P_2 and ST be m_0, m_2 and m_1 respectively.
Let the angles θ_0, θ_2 and θ_1 be respectively the angles made between each of P_1P_0, P_1P_2 and ST, with the positive x-axis.

By the law of reflection, $\measuredangle P_2P_1N = \measuredangle P_0P_1N = \alpha$.
As in the proof of **Theorem 5.1**, we have
(7.6) $2\theta_1 = \pi + \theta_0 + \theta_2$.
For the hyperbola ζ, $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{b^2x}{a^2y}$. Thus, at the point P_1, the gradient of the tangent ST is given by $m_1 = \frac{b^2x_1}{a^2y_1} = \tan\theta_1$. This leads to

$$\tan 2\theta_1 = \frac{2\tan\theta_1}{1-\tan^2\theta_1} = \frac{\frac{2b^2x_1}{a^2y_1}}{1-\frac{b^4x_1{}^2}{a^4y_1{}^2}} = \frac{2a^2b^2x_1y_1}{a^4y_1{}^2-b^4x_1{}^2}.$$

However, $\frac{x_1{}^2}{a^2} - \frac{y_1{}^2}{b^2} = 1$ or $a^2y_1{}^2 = b^2x_1{}^2 - a^2b^2$. Hence, we have
(7.7) $\tan 2\theta_1 = \frac{2a^2b^2x_1y_1}{a^4y_1{}^2-b^4x_1{}^2} = \frac{2a^2b^2x_1y_1}{a^2(b^2x_1{}^2-a^2b^2)-b^4x_1{}^2} = \frac{2a^2x_1y_1}{(a^2-b^2)x_1{}^2-a^4}.$

The equation of any straight line passing through the point P_1 is given by
$\frac{y-y_1}{x-x_1} = m$ or $y = y_1 + m(x - x_1) = mx + y_1 - mx_1$, where m is the gradient of the line.

To find the points of intersection of any straight line through point P_1 and the hyperbola ζ', it is equivalent to solving the pair of simultaneous equations
$y = mx + y_1 - mx_1$ and $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1.$

Substituting $y = mx + y_1 - mx_1$ in $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$ yields
$\frac{x^2}{p^2} - \frac{(mx+y_1-mx_1)^2}{q^2} = 1$ or $q^2x^2 - p^2(mx + y_1 - mx_1)^2 - p^2q^2 = 0$,
which simplifies to
$(q^2 - p^2m^2)x^2 - 2p^2m(y_1 - mx_1)x - p^2(y_1 - mx_1)^2 - p^2q^2 = 0.$

In the case of P_1P_2 being tangent to ζ', we have
$4p^4m^2(y_1 - mx_1)^2 + 4(q^2 - p^2m^2)[p^2(y_1 - mx_1)^2 + p^2q^2] = 0$ or
(7.8) $(p^2 - {x_1}^2)m^2 + 2x_1y_1m - q^2 - {y_1}^2 = 0.$

The solutions to (7.8) must be m_0 and m_2, which are the gradients of P_1P_0 and P_1P_2 respectively.
Since m_0 and m_2 are the solutions to (7.8), we have
$m_0 + m_2 = -\frac{2x_1y_1}{p^2-{x_1}^2}$ and $m_0m_2 = -\frac{q^2+{y_1}^2}{p^2-{x_1}^2}$.

Hence,

$$\tan(\pi + \theta_0 + \theta_2) = \tan(\theta_0 + \theta_2) = \frac{\tan\theta_0+\tan\theta_2}{1-\tan\theta_0\tan\theta_2} = \frac{m_0+m_2}{1-m_0m_2} = \frac{-\frac{2x_1y_1}{p^2-{x_1}^2}}{1+\frac{q^2+{y_1}^2}{p^2-{x_1}^2}} = -\frac{2x_1y_1}{p^2-{x_1}^2+q^2+{y_1}^2}.$$

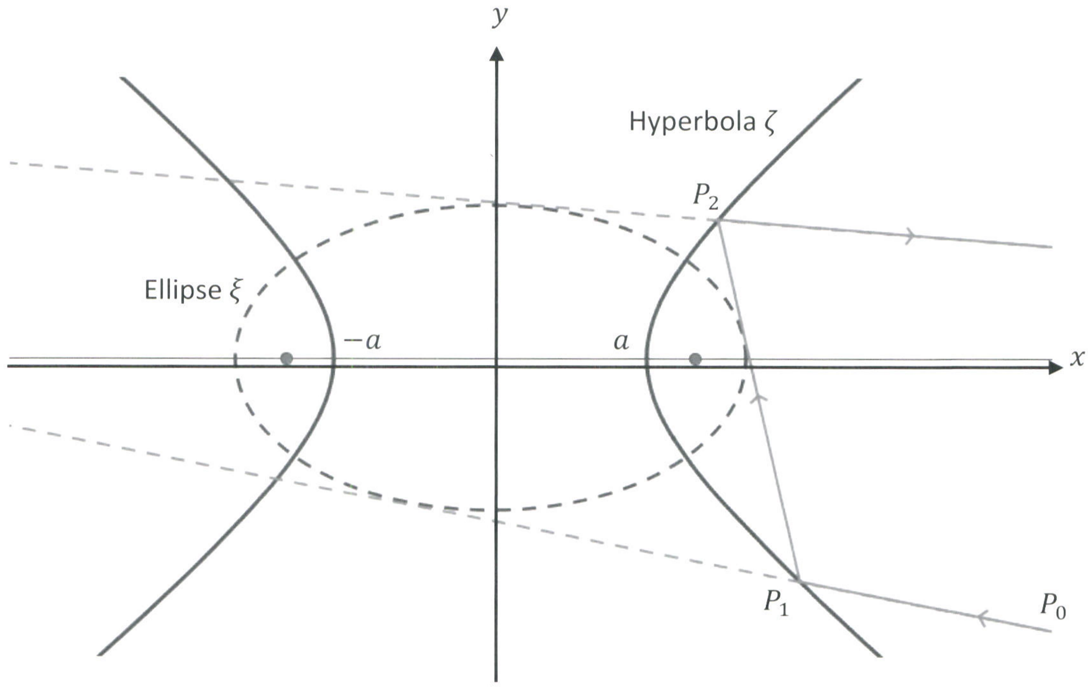
However, from (7.5), $a^2 + b^2 = p^2 + q^2$. Thus,
$\tan(\theta_0 + \theta_2) = -\frac{2x_1y_1}{p^2-{x_1}^2+q^2+{y_1}^2} = -\frac{2x_1y_1}{a^2+b^2-{x_1}^2+{y_1}^2}$ or
(7.9) $2x_1y_1 = -[\tan(\theta_0 + \theta_2)][a^2 + b^2 - {x_1}^2 + {y_1}^2].$

From (7.7), we have $\tan 2\theta_1 = \frac{2a^2x_1y_1}{(a^2-b^2){x_1}^2-a^4}$ and substituting for $2x_1y_1$ yields
$\tan 2\theta_1 = -\frac{a^2[\tan(\theta_0+\theta_2)][a^2+b^2-{x_1}^2+{y_1}^2]}{(a^2-b^2){x_1}^2-a^4} = -\frac{[\tan(\theta_0+\theta_2)][a^4+a^2b^2-a^2{x_1}^2+a^2{y_1}^2]}{(a^2-b^2){x_1}^2-a^4}$.
However, $\frac{{x_1}^2}{a^2} - \frac{{y_1}^2}{b^2} = 1$ or $a^2{y_1}^2 = b^2{x_1}^2 - a^2b^2$. Hence,
$\tan 2\theta_1 = -\frac{[\tan(\theta_0+\theta_2)][a^4+a^2b^2-a^2{x_1}^2+b^2{x_1}^2-a^2b^2]}{(a^2-b^2){x_1}^2-a^4} = -\frac{[\tan(\theta_0+\theta_2)][a^4-(a^2-b^2){x_1}^2]}{(a^2-b^2){x_1}^2-a^4}$.
Thus, $\tan 2\theta_1 = \tan(\theta_0 + \theta_2) = \tan(\pi + \theta_0 + \theta_2)$ and the theorem is proven. □

From the results of **Theorem 7.1**, as P_0P_1 is an arbitrary chord of the hyperbola ζ that intersects the x-axis between a focus and its nearer major vertex $(a, 0)$, it thus shows that for a light ray that travels the paths P_0P_1, P_1P_2, P_2P_3, ..., these paths form chords that are tangent to the confocal hyperbola ζ'. By symmetry, any light ray that passes between a focus of a hyperbola ζ and its nearer major vertex will travel along paths that are tangential to confocal hyperbola ζ'.

8. Tracing the path of a light ray that initially passes beyond a focus of a hyperbola

This section investigates the remaining case for hyperbolas, when the light ray initially passes beyond the focus of a hyperbola as it is reflected within the hyperbola. The proof here confirms that indeed the light path, after each reflection, is always tangent to a confocal ellipse. The proof here turns out to be similar to that in **Section 7** and hence, many



similar steps in the proof are omitted.

Figure 8.1

The equation of hyperbola ζ is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the equation of ellipse ξ is given by $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ with $p > a$, where both hyperbola and ellipse share the same foci.
Let e_ζ be the eccentricity of ζ and e_ξ be the eccentricity of ξ.

Then, we have the following:
(8.1) $b^2 = a^2(e_\zeta{}^2 - 1)$
(8.2) $q^2 = p^2(1 - e_\xi{}^2)$ and
(8.3) $ae_\zeta = pe_\xi$, since the distance of the foci from the origin are given by ae_ζ and pe_ξ.

Substituting $e_\xi = \frac{ae_\zeta}{p}$ in (8.2), we have
(8.4) $q^2 = p^2\left(1 - \frac{a^2 e_\zeta{}^2}{p^2}\right) = p^2 - a^2 e_\zeta{}^2$.

Eliminating e_ζ from both (8.1) and (8.4), we have
(8.5) $a^2 + b^2 = p^2 - q^2$.

Theorem 8.1

Let P_1P_0 and P_1P_2 be tangents to the ellipse ξ, where P_0, P_1 and P_2 are points on the hyperbola ζ. Then P_1P_0 and P_1P_2 make equal angles with the tangent ST to the hyperbola ζ at P_1.

Proof of Theorem 8.1

Let the Cartesian coordinates of P_1 be (x_1, y_1).
Let the tangent to ζ at P_1 be ST and the normal to ζ at P_1 be MN.
Let the gradients of P_1P_0, P_1P_2 and ST be m_0, m_2 and m_1 respectively.
Let the angles θ_0, θ_2 and θ_1 be respectively the angles made between each of P_1P_0, P_1P_2 and ST, with the positive x-axis.

By the law of reflection, $\measuredangle P_2P_1N = \measuredangle P_0P_1N = \alpha$.
As in the proof of **Theorem 7.1**, we have
(8.6) $\quad 2\theta_1 = \pi + \theta_0 + \theta_2$.
Also, similar to **Theorem 7.1**, we have
(8.7) $\quad \tan 2\theta_1 = \frac{2a^2x_1y_1}{(a^2-b^2){x_1}^2-a^4}$.

The equation of any straight line passing through the point P_1 is given by
$\frac{y-y_1}{x-x_1} = m$ or $y = y_1 + m(x - x_1) = mx + y_1 - mx_1$, where m is the gradient of the line.

To find the points of intersection of any straight line through point P_1 and the ellipse ξ, it is equivalent to solving the pair of simultaneous equations
$y = mx + y_1 - mx_1$ and $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$.

Substituting $y = mx + y_1 - mx_1$ in $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ yields
$\frac{x^2}{p^2} + \frac{(mx+y_1-mx_1)^2}{q^2} = 1$ or $q^2x^2 + p^2(mx + y_1 - mx_1)^2 - p^2q^2 = 0$,
which simplifies to
$(p^2m^2 + q^2)x^2 + 2p^2m(y_1 - mx_1)x + p^2(y_1 - mx_1)^2 - p^2q^2 = 0$.

In the case of P_1P_2 being tangent to ξ, we have
$4p^4m^2(y_1 - mx_1)^2 - 4(p^2m^2 + q^2)[p^2(y_1 - mx_1)^2 + p^2q^2] = 0$ or
(8.8) $\quad (p^2 - {x_1}^2)m^2 + 2x_1y_1m + q^2 - {y_1}^2 = 0$.

The solutions to (8.8) must be m_0 and m_2, which are the gradients of P_1P_0 and P_1P_2 respectively.
Since m_0 and m_2 are the solutions to (8.8), we have
$m_0 + m_2 = -\frac{2x_1y_1}{p^2-{x_1}^2}$ and $m_0m_2 = \frac{q^2-{y_1}^2}{p^2-{x_1}^2}$.

Hence,

$$\tan(\pi + \theta_0 + \theta_2) = \tan(\theta_0 + \theta_2) = \frac{\tan\theta_0+\tan\theta_2}{1-\tan\theta_0\tan\theta_2} = \frac{m_0+m_2}{1-m_0m_2} = \frac{-\frac{2x_1y_1}{p^2-{x_1}^2}}{1-\frac{q^2-{y_1}^2}{p^2-{x_1}^2}} = -\frac{2x_1y_1}{p^2-{x_1}^2-q^2+{y_1}^2}.$$

However, from (8.5), $a^2 + b^2 = p^2 - q^2$. Thus,
$\tan(\theta_0 + \theta_2) = -\frac{2x_1y_1}{p^2-{x_1}^2-q^2+{y_1}^2} = -\frac{2x_1y_1}{a^2+b^2-{x_1}^2+{y_1}^2}$ or
(8.9) $\quad 2x_1y_1 = -[\tan(\theta_0 + \theta_2)][a^2 + b^2 - {x_1}^2 + {y_1}^2]$.

From (8.7), we have $\tan 2\theta_1 = \frac{2a^2x_1y_1}{(a^2-b^2){x_1}^2-a^4}$ and substituting for $2x_1y_1$ yields

$\tan 2\theta_1 = -\frac{a^2[\tan(\theta_0+\theta_2)][a^2+b^2-{x_1}^2+{y_1}^2]}{(a^2-b^2){x_1}^2-a^4} = -\frac{[\tan(\theta_0+\theta_2)][a^4+a^2b^2-a^2{x_1}^2+a^2{y_1}^2]}{(a^2-b^2){x_1}^2-a^4}$.

However, $\frac{{x_1}^2}{a^2} - \frac{{y_1}^2}{b^2} = 1$ or $a^2{y_1}^2 = b^2{x_1}^2 - a^2b^2$. Hence,

$\tan 2\theta_1 = -\frac{[\tan(\theta_0+\theta_2)][a^4+a^2b^2-a^2{x_1}^2+b^2{x_1}^2-a^2b^2]}{(a^2-b^2){x_1}^2-a^4} = -\frac{[\tan(\theta_0+\theta_2)][a^4-(a^2-b^2){x_1}^2]}{(a^2-b^2){x_1}^2-a^4}$.

Thus, $\tan 2\theta_1 = \tan(\theta_0 + \theta_2) = \tan(\pi + \theta_0 + \theta_2)$ and the theorem is proven. □

From the results of **Theorem 8.1**, as P_0P_1 is an arbitrary chord of the hyperbola ζ that intersects the x-axis beyond a focus, it thus shows that for a light ray that travels the paths P_0P_1, P_1P_2, P_2P_3, ..., these paths form chords that are tangent to the confocal ellipse ξ. By symmetry, any light ray that initially passes beyond a focus of a hyperbola ζ will travel along paths that are tangential to confocal ellipse ξ.

9. Tracing the path of a light ray within a parabola

This section investigates the path of a light ray that reflects within a parabola. It is found that the light paths are tangent to another parabola after each reflection, if the light paths do not at any time pass through the focus of the original parabola.

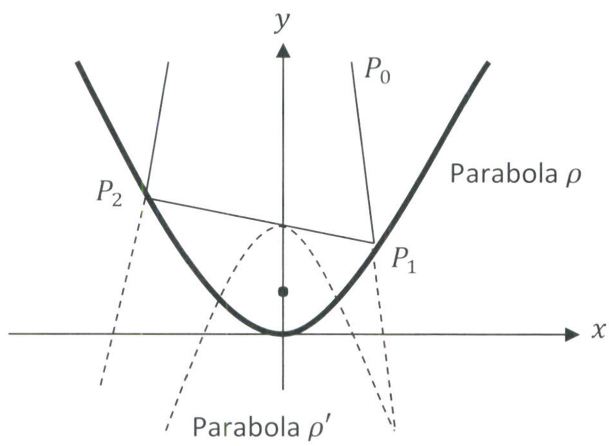


Figure 9.1

Let the equation of parabola ρ be $y = ax^2$ and the equation of parabola ρ' be $y = px^2 + k$.

Theorem 9.1

Let P_1P_0 and P_1P_2 be tangents to the parabola ρ', where P_0, P_1 and P_2 are points on the parabola ρ. Then P_1P_0 and P_1P_2 make equal angles with the tangent ST to the parabola ρ at P_1.

Proof of Theorem 9.1

Let the Cartesian coordinates of P_1 be (x_1, y_1).
Let the tangent to ρ at P_1 be ST and the normal to ρ at P_1 be MN.

Let the gradients of P_1P_0, P_1P_2 and ST be m_0, m_2 and m_1 respectively.
Let the angles θ_0, θ_2 and θ_1 be respectively the angles made between each of P_1P_0, P_1P_2 and ST, with the positive x-axis.

By the law of reflection, $\measuredangle P_2P_1N = \measuredangle P_0P_1N = \alpha$.
As in the proof of **Theorem 5.1**, we have
(9.1) $\quad 2\theta_1 = \pi + \theta_0 + \theta_2$.

For the parabola ρ, $\frac{\mathrm{dy}}{\mathrm{dx}} = 2ax$. Thus, at the point P_1, the gradient of the tangent ST is given by $m_1 = 2ax_1 = \tan\theta_1$. This leads to
(9.2) $\quad \tan 2\theta_1 = \frac{2\tan\theta_1}{1-\tan^2\theta_1} = \frac{4ax_1}{1-4a^2{x_1}^2} = \frac{4ax_1}{1-4ay_1}$.

The equation of any straight line passing through the point P_1 is given by
$\frac{y-y_1}{x-x_1} = m$ or $y = y_1 + m(x - x_1) = mx + y_1 - mx_1$, where m is the gradient of the line.

To find the points of intersection of any straight line through point P_1 and the parabola ρ', it is equivalent to solving the pair of simultaneous equations
$y = mx + y_1 - mx_1$ and $y = px^2 + k$.

Substituting $y = mx + y_1 - mx_1$ in $y = px^2 + k$ yields
$mx + y_1 - mx_1 = px^2 + k$ or
$px^2 - mx - y_1 + mx_1 + k = 0$.

In the case of P_1P_2 being tangent to ρ', we have
$m^2 - 4p(-y_1 + mx_1 + k) = 0$ or
(9.3) $\quad m^2 - 4px_1m + 4py_1 - 4pk = 0$.

The solutions to (9.3) must be m_0 and m_2, which are the gradients of P_1P_0 and P_1P_2 respectively.
Since m_0 and m_2 are the solutions to (9.3), we have
$m_0 + m_2 = 4px_1$ and $m_0m_2 = 4py_1 - 4pk$.

Hence,
$\tan(\pi + \theta_0 + \theta_2) = \tan(\theta_0 + \theta_2) = \frac{\tan\theta_0+\tan\theta_2}{1-\tan\theta_0\tan\theta_2} = \frac{m_0+m_2}{1-m_0m_2} = \frac{4px_1}{1-(4py_1-4pk)} = \frac{4px_1}{1-4py_1+4pk}$.

For solutions to $\tan 2\theta_1 = \tan(\pi + \theta_0 + \theta_2)$, we have
$\frac{4ax_1}{1-4ay_1} = \frac{4px_1}{1-4py_1+4pk}$, or
$4ax_1(1 - 4py_1 + 4pk) = 4px_1(1 - 4ay_1)$, or
$4ax_1 - 16apx_1y_1 + 16apx_1k = 4px_1 - 16apx_1y_1$, or
$p = \frac{a}{1-4ak}$, which exists for $k \neq \frac{1}{4a}$, and the theorem is proven. □

Note:
It is interesting to note that given a parabola ρ with equation $y = ax^2$, the focus of ρ is located at the point $(0, \frac{1}{4a})$. Furthermore, it is also a known result that for a light ray that passes through the focus of ρ, its next reflection will be such that it becomes vertical (and hence will not undergo any further reflection thereafter).

From the results of **Theorem 9.1**, we know that the light paths within a parabola ρ are tangential to parabola ρ' with equation $y = \frac{a}{1-4ak}x^2 + k$, if the light paths do not at any time pass through the focus.

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