

Language: **English**Day: **1**

Monday, July 11, 2016

Problem 1. Triangle BCF has a right angle at B . Let A be the point on line CF such that $FA = FB$ and F lies between A and C . Point D is chosen such that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF . Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that lines BD , FX , and ME are concurrent.

Problem 2. Find all positive integers n for which each cell of an $n \times n$ table can be filled with one of the letters I , M and O in such a way that:

- in each row and each column, one third of the entries are I , one third are M and one third are O ; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I , one third are M and one third are O .

Note: The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integers (i, j) with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of the first type consists of all cells (i, j) for which $i + j$ is a constant, and a diagonal of the second type consists of all cells (i, j) for which $i - j$ is a constant.

Problem 3. Let $P = A_1A_2 \dots A_k$ be a convex polygon in the plane. The vertices A_1, A_2, \dots, A_k have integral coordinates and lie on a circle. Let S be the area of P . An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n . Prove that $2S$ is an integer divisible by n .

Language: **English**Day: **2**

Tuesday, July 12, 2016

Problem 4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \dots, P(a+b)\}$$

is fragrant?

Problem 5. The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

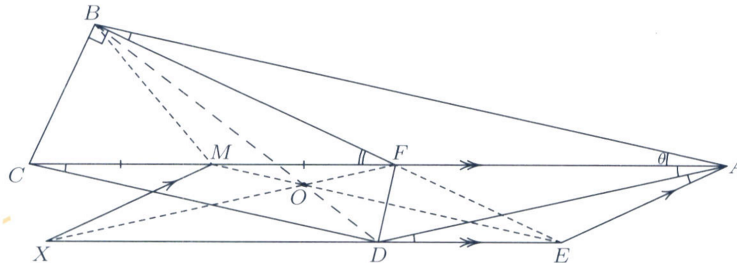
Problem 6. There are $n \geq 2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

- (a) Prove that Geoff can always fulfil his wish if n is odd.
 (b) Prove that Geoff can never fulfil his wish if n is even.

Language: *English*

Time: 4 hours and 30 minutes
Each problem is worth 7 points

1. *Solution by Tan Junyao Joel.* Let $\angle BAF = \theta$. All the marked angles in the figure are easily seen to be equal to θ . Note that $\angle XEA = 180^\circ - \angle MAE = 180^\circ - 2\theta = \angle DEA$ so that X, D, E are collinear.



Lemma. $AE = MF$.

Proof. First note that $\angle BFC = 2\theta$. Let $MF = a$. Then $CF = 2a$, $AF = BF = 2a \cos 2\theta$, and $CA = CF + AF = 2a(1 + \cos 2\theta) = 4a \cos^2 \theta$. Thus in the isosceles triangle ADC , we have $AD = \frac{CA}{2 \cos \theta} = 2a \cos \theta$. Also in the isosceles triangle AED , we have $AE = \frac{AD}{2 \cos \theta} = \frac{2a \cos \theta}{2 \cos \theta} = a = MF$.

As triangles BFA and CDA are similar, we have $BA/CA = AF/AD$. Also $\angle BAC = \angle FAD$. Thus triangles BAC and FAD are similar. Hence $\angle DFA = \angle CBA = 90^\circ + \theta$, and $\angle CFD = 90^\circ - \theta$. Hence $\angle BCD = \angle BCF + \angle DCF = (90^\circ - 2\theta) + \theta = 180^\circ - [2\theta + (90^\circ - \theta)] = 180^\circ - (\angle BFC + \angle CFD) = 180^\circ - \angle BFD$. Therefore, B, C, D, F are concyclic. Hence $\angle DBF = \angle DCF = \theta$ and $\angle DBA = 2\theta$.

As triangles BFA and DEA are similar, we have $BA/FA = DA/EA$. Also $\angle BAD = \angle FAE = 2\theta$. Thus triangles BAD and FAE are similar. Since the triangle BAD is isosceles with $AD = BD$, the triangle FAE is also isosceles with $AE = FE$. Consequently, $MX = AE = FE$. Note that $\angle MXE = 2\theta = \angle XEF$. Hence $MXEF$ is an isosceles trapezium. Also by the lemma, $MF = AE = FE$. Thus $\angle EMF = \frac{1}{2} \angle EFA = \theta$.

Let ME and XF intersect at O . Join BO and OD . Thus the isosceles triangles MOF and EOX are similar, and $\angle EOF = 2\theta = \angle MBF$. This shows that B, M, O, F are concyclic. Consequently, $\angle OBF = \angle OMF = \theta = \angle DBF$ so that B, O, D are collinear.

2. *Solution by Wang Peng Jun Bryan.* The answer is that n is a multiple of 9. The first condition implies $3|n$. When $n = 9$, we have the following construction.

I	M	O	O	I	M	M	O	I
I	M	O	O	I	M	M	O	I
I	M	O	O	I	M	M	O	I
O	I	M	M	O	I	I	M	O
O	I	M	M	O	I	I	M	O
O	I	M	M	O	I	I	M	O
M	O	I	I	M	O	O	I	M
M	O	I	I	M	O	O	I	M
M	O	I	I	M	O	O	I	M

For $n = 9m$, we can adjoin m^2 copies of the above 9×9 construction to give a configuration for the $n \times n$ board satisfying the conditions of the problem.

Write $n = 3k$. Suppose it is possible to fill in the letters I, M, O for the $n \times n$ board satisfying the conditions of the problem. We can subdivide the board into k^2 subboards of size 3×3 . Call the center entry of each 3×3 board a critical entry. So there are k^2 critical entries in the $n \times n$ board. Note that the number of entries of a diagonal is a multiple of three if and only if that diagonal contains at least a critical entry. If a diagonal, a row or a column contains a critical entry, we shall call it a critical diagonal, a critical row and a critical column respectively.

Let ω be a primitive cube root of unity. For instance we may take $\omega = e^{\frac{2\pi i}{3}}$. Assign to the letters I, M and O in the $n \times n$ board the corresponding values $1, \omega$ and ω^2 respectively. By the two given conditions and the relation $1 + \omega + \omega^2 = 0$, the sum of values of all entries along any row, any column and any critical diagonal is zero.

Sum of values of all entries along all critical diagonals + sum of values of all entries along all critical rows + sum of values of all entries along all critical columns = $3 \times$ sum of values of all critical entries + sum of values of all entries.

That is $0 + 0 + 0 = 3 \times$ sum of values of all critical entries + 0. Thus sum of values of all critical entries = 0.

Suppose there are a letters of I , b letters of M and c letters of O among the critical entries. Then sum of values of all critical entries = $a + b\omega + c\omega^2 = 0$. This implies $a = b = c$. [To see this: $0 = a + b\omega + c\omega^2 = a + b\omega + c\omega^2 - a(1 + \omega + \omega^2) = \omega((b-a) + (c-a)\omega)$ so that $(b-a) + (c-a)\omega = 0$. Since 1 and ω are linearly independent complex numbers over real numbers, we have $b-a = 0$ and $c-a = 0$. Thus $a = b = c$.] Therefore there are equal numbers of letters I, M and O among all the critical entries. Consequently $3|k^2$ and so $3|k$ and $9|n$.

3. *Solution by Ma Shao Yu.* Let $P = A_1A_2 \dots A_k$, and let $A_{k+i} = A_i$ for $i \geq 1$. Let the coordinates of A_i be (x_i, y_i) for $i = 1, \dots, k$. By the Shoelace formula, $2S = |\sum_{i=1}^k x_i y_{i+1} - x_{i+1} y_i|$ is an integer. It suffices to prove that for an odd prime power p^α , we have $p^\alpha | 2S$. We use induction on k . When $k = 3$, the polygon is a triangle. Let the side lengths of P be $\sqrt{p^\alpha a}, \sqrt{p^\alpha b}, \sqrt{p^\alpha c}$. By Heron's formula, $16S^2 = p^{2\alpha}(2ab + 2bc + 2ca - a^2 - b^2 - c^2)$. As p^α is odd, we have $p^\alpha | 2S$.

Lemma. Let a_1, \dots, a_r be positive integers. If $\sqrt{a_1} + \dots + \sqrt{a_r}$ is a rational number, then it is an integer.

Proof. Each $\sqrt{a_i}$ is a root of a monic polynomial with integer coefficients, namely $x^2 - a_i$. It is a well-known result (see [1]) that their sum $\sqrt{a_1} + \dots + \sqrt{a_r}$ also satisfies a monic polynomial with integer coefficients. It follows that if $\sqrt{a_1} + \dots + \sqrt{a_r}$ is a rational number, then it must be an integer.

Let the circumcenter and the circumradius of P be O and R respectively. Since O is the intersection of the perpendicular bisectors of the sides of P and the vertices of P have integer coordinates, the coordinates of O are rational numbers. Thus R^2 is also a rational number. For a rational number $z = \frac{a}{b}$, let $v_p(z)$ be the exponent of p in the prime factorization of a minus the exponent of p in the prime factorization of b . Let the length of the side A_iA_{i+1} of P be $\sqrt{e_i p^\alpha}$, $i = 1, \dots, k$, where e_1, \dots, e_k are positive integers. Let X_i be the midpoint of the side A_iA_{i+1} , $i = 1, \dots, k$.

Case 1: $v_p(R^2) \geq \alpha$. Write $R^2 = \frac{ep^\alpha}{g}$, where e, g are positive integers and $v_p(g) = 0$. For each $i = 1, \dots, k$, we have

$$OX_i^2 = R^2 - \left(\frac{A_iA_{i+1}}{2}\right)^2 = \frac{ep^\alpha}{g} - \frac{e_i p^\alpha}{4} = \frac{p^\alpha}{4g^2}(4eg - e_i g^2).$$

Thus, area of the triangle $OA_iA_{i+1} = \frac{1}{2}A_iA_{i+1} \times OX_i = \frac{1}{2}\sqrt{e_i p^\alpha} \sqrt{\frac{p^\alpha}{4g^2}(4eg - e_i g^2)} = \frac{1}{4g}p^\alpha \sqrt{a_i}$, where $a_i = ge_i(4e - e_i)$ is a positive integer. Therefore $S = \frac{1}{4g}p^\alpha(\sqrt{a_1} + \dots + \sqrt{a_k})$. By the lemma, $p^\alpha | 4gS$. Since $v_p(2g) = 0$, we have $p^\alpha | 2S$.

Case 2: $v_p(R^2) < \alpha$. Write $R^2 = \frac{up^\beta}{v}$, where u, v are positive integers and $v_p(u) = v_p(v) = 0$. Thus $v_p(R^2) = \beta < \alpha$. As $OX_1^2 = R^2 - \left(\frac{A_1A_2}{2}\right)^2$, we have $v_p(OX_1^2) = v_p(R^2) = \beta$. Similarly, $v_p(OX_2^2) = \beta$. By Ptolemy's theorem applied to the cyclic quadrilateral $OX_1A_2X_2$, we have $OA_2 \times X_1X_2 = OX_1 \times A_2X_2 + OX_2 \times A_2X_1$. Squaring both sides and noting that $OA_2^2 = R^2$, we have

$$R^2 \times X_1X_2^2 = OX_1^2 \times A_2X_2^2 + OX_2^2 \times A_2X_1^2 + 2OX_1 \times OX_2 \times A_2X_1 \times A_2X_2.$$

Since $v_p(A_2X_2^2) \geq \alpha$, $v_p(A_2X_1^2) \geq \alpha$, the right-hand side of the above equation can be written in the form $\frac{p^{\alpha+\beta}}{c}(\sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3})$, where b_1, b_2, b_3, c are positive integers with $(p, c) = 1$. Thus

$$A_1A_3^2 = 4X_1X_2^2 = \frac{4p^{\alpha+\beta}}{uc}(\sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3}).$$

By the lemma, $p^\alpha | ucA_1A_3^2$. Since $(u, p) = 1$ and $(c, p) = 1$, we have $p^\alpha | A_1A_3^2$. We then cut P along A_1A_3 and apply induction hypothesis to complete the proof.

[1] H. Pollard and H.G. Diamond, *The Theory of Algebraic Numbers*, The Carus Mathematical Monographs, No 9, 2nd edition, MAA, 1975.

4. *Solution by Toh Shan Hong Dylan.* The smallest size of a fragrant set is 6. Using Chinese remainder theorem, there exists a positive integer a such that $a \equiv 0 \pmod{3}$, $a \equiv 6 \pmod{7}$, $a \equiv 5 \pmod{19}$. For example, take $a = 195$. Then

$$\begin{aligned} P(a+1) &\equiv P(a+4) \equiv P(1) \equiv 0 \pmod{3}, \\ P(a+2) &\equiv P(7) \equiv 7^2 + 7 + 1 \equiv 0 \pmod{19}, \\ P(a+6) &\equiv P(11) \equiv 11^2 + 11 + 1 \equiv 0 \pmod{19}, \\ P(a+3) &\equiv P(9) \equiv 9^2 + 9 + 1 \equiv 0 \pmod{7}, \\ P(a+5) &\equiv P(11) \equiv 11^2 + 11 + 1 \equiv 0 \pmod{7}. \end{aligned}$$

Thus $3|P(a+1), P(a+4)$, $7|P(a+3), P(a+5)$ and $19|P(a+2), P(a+6)$. Therefore, the set $\{P(a+1), P(a+2), P(a+3), P(a+4), P(a+5), P(a+6)\}$ is fragrant.

Next we shall show that there does not exist a fragrant set of size 2, 3, 4, 5. Suppose there exists a fragrant set with at most 5 elements. We may assume it contains exactly 5 elements $P(a), P(a+1), P(a+2), P(a+3), P(a+4)$ since the following argument works with fewer elements too. We have the following calculations.

Firstly, $(P(n), P(n+1)) = (n^2 + n + 1, n^2 + 3n + 3) = (n^2 + n + 1, 2n + 2) = (n^2 + n + 1, n + 1) = (n(n+1) + 1, n + 1) = (1, n + 1) = 1$. The third equality holds because $2|n(n+1)$ but $2 \nmid n^2 + n + 1$. That is

$$(1) \quad (P(n), P(n+1)) = 1.$$

Secondly, $(P(n), P(n+2)) = (n^2 + n + 1, n^2 + 5n + 7) = (n^2 + n + 1, 4n + 6) = (n^2 + n + 1, 2n + 3) = (2(n^2 + n + 1), 2n + 3) = ((2n + 3(n-1)) + n + 5, 2n + 3) = (n + 5, 2n + 3) = (n + 5, -7)$. Thus

$$(2) \quad (P(n), P(n+2)) = \begin{cases} 7 & \text{if } n \equiv 2 \pmod{7} \\ 1 & \text{if } n \not\equiv 2 \pmod{7} \end{cases}.$$

Similarly, one can calculate that

$$(3) \quad (P(n), P(n+3)) = \begin{cases} 3 & \text{if } n \equiv 1 \pmod{3} \\ 1 & \text{if } n \not\equiv 1 \pmod{3} \end{cases}.$$

Consider $P(a+2)$. By (1), it is relatively prime to $P(a+1)$ and $P(a+3)$. Without loss of generality, we may assume $(P(a), P(a+2)) > 1$. By (2), $a \equiv 2 \pmod{7}$. Thus $a+1 \not\equiv 2 \pmod{7}$. Then by (2), $(P(a+1), P(a+3)) = 1$. In order that the set is fragrant, $(P(a), P(a+3))$ and $(P(a+1), P(a+4))$ must both be greater than 1. By (3), $a \equiv 1 \pmod{3}$ and $a+1 \equiv 1 \pmod{3}$, which is a contradiction.

5. *Solution by Tan Kieren Sheldon.* The answer is 2016. Note that if a linear factor $(x - i)$ still appears on both sides of the equation, then $x = i$ is a real solution. So each linear factor can appear at most once on the board. Thus each linear factor must be erased at least once. This implies $k \geq 2016$. Now we claim that the equation

$$\prod_{i=1}^{504} (x - (4i - 2))(x - (4i - 1)) = \prod_{j=1}^{504} (x - (4j - 3))(x - 4j) \quad (1)$$

has no real solutions. To do so, we shall prove that $LHS > RHS$ in (1).

Lemma. Let a be a positive integer. Then for all real numbers x ,

$$(x - (a + 1))(x - (a + 2)) > (x + a)(x - (a + 3)).$$

The lemma can be proved by direct simplification on both sides of the inequality.

Note that the LHS of (1) is negative if and only if $4i - 2 < x < 4i - 1$ for some $1 \leq i \leq 504$, and the RHS of (1) is negative if and only if $4j - 3 < x < 4j$ for some $1 \leq j \leq 504$. We split into the following cases.

Case 1: ($x < 1$ or $x > 2016$). By the lemma, $(x - (4i - 2))(x - (4i - 1)) > (x - (4i - 3))(x - 4i) > 0$, for all $i = 1, \dots, 504$. Thus multiplying across all i on both sides, we have $LHS > RHS$ in (1).

Case 2: (x is an integer from 1 to 2016). If $x \equiv 0, 1 \pmod{4}$, then $LHS > 0$ and $RHS = 0$. If $x \equiv 2, 3 \pmod{4}$, then $LHS = 0$ and $RHS < 0$. So $LHS > RHS$.

Case 3: ($k < x < k + 1$ for some k where $1 \leq k \leq 2015$).

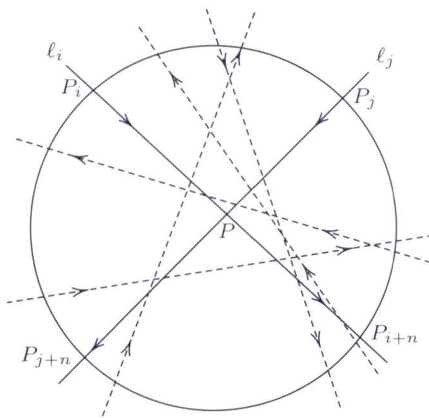
Case 3a: ($k \equiv 1$ or $3 \pmod{4}$). Then it is clear that $LHS > 0 > RHS$.

Case 3b: ($k \equiv 0 \pmod{4}$). By the lemma, $(x - (4i - 2))(x - (4i - 1)) > (x - (4i - 3))(x - 4i)$ for all $i = 1, \dots, 504$. If $i > k/4$, the terms $(x - (4i - 2))$, $(x - (4i - 1))$, $(x - (4i - 3))$ and $(x - 4i)$ are all negative as $k < x < k + 1$, and $i > k/4 \Rightarrow i \geq k/4 + 1 \Rightarrow 4i - 4 \geq k \Rightarrow 4i - 3 \geq k + 1 > x$. If $i \leq k/4$, then $x > k > 4i$ and thus the terms $(x - (4i - 2))$, $(x - (4i - 1))$, $(x - (4i - 3))$, $(x - 4i)$ are all positive. Hence $(x - (4i - 2))(x - (4i - 1)) > (x - (4i - 3))(x - 4i) > 0$ for all $i = 1, \dots, 504$. Thus multiplying across all i on both sides, we have $LHS > RHS$ in (1).

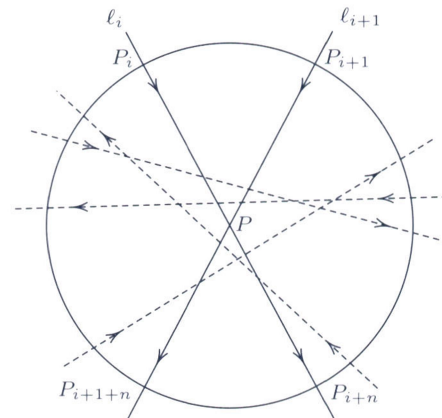
Case 3c: ($k \equiv 2 \pmod{4}$). In this case, both LHS and RHS are negative. By the lemma, $(x - (4i - 1))(x - (4i + 2)) < (x - 4i)(x - (4i + 1))$ for all $i = 1, \dots, 503$. For $i > (k - 2)/4$, the terms $(x - (4i - 1))$, $(x - (4i + 2))$, $(x - 4i)$ and $(x - (4i + 1))$ are all negative as $i > (k - 2)/4 \Rightarrow i \geq (k - 2)/4 + 1 \Rightarrow 4i - 1 \geq k + 1 > x$. For $i \leq (k - 2)/4$, these terms are all positive. Therefore, $0 < (x - (4i - 1))(x - (4i + 2)) < (x - 4i)(x - (4i + 1))$ for all $i = 1, \dots, 503$. Lastly, $0 < x - 2 < x - 1$ and $|x - 2015| < |x - 2016|$ with $x - 2015 < 0$ and $x - 2016 < 0$. Thus multiplying across all i on both sides, we have $0 > LHS > RHS$ in (1).

Equation (1) is obtained by erasing 2016 linear factors from the original equation and it has no real solutions. Thus the minimum value of k is 2016.

6. *Solution by Lim Wei An Glen.* We may enclose all segments by a circle, and extend each segment to a line so that each of them intersects the circle at two distinct points. Label these points in the clockwise sense as P_1, \dots, P_{2n} . For any $1 \leq i \leq n$, let ℓ_i be the line through P_i , and let its other intersection point with the circle be P_r . Since every pair of lines intersect inside the circle, there are exactly $n - 1$ points between P_i and P_r along the circle. Thus $r = i + n$. Therefore the lines containing the given line segments are precisely the lines $P_1P_{n+1}, P_2P_{n+2}, \dots, P_nP_{2n}$. That is ℓ_i is the line containing the points P_i and P_{n+i} .



The dotted lines intersect the segments P_iP and P_jP in an odd number of points.



The dotted lines intersect the segments P_iP and $P_{i+1}P$ in an even number of points.

(a) Since n is odd, i and $n + i$ have opposite parity. For each line segment on the line ℓ_i containing the points P_i and P_{n+i} , place a frog on the endpoint of the line segment closer to P_α , where $\alpha = i$ or $i + n$ is odd. We claim that the frogs on ℓ_i and ℓ_j do not meet for any pairs of i and j . Without loss of generality, we may suppose the frogs start at P_i and P_j respectively. Let ℓ_i and ℓ_j intersect at P . Note that there is an odd number of points between the arc P_iP_j . Each of these points belongs to a line ℓ_k . Since a line ℓ_k must intersect exactly one of the segments P_iP and P_jP , making an odd number of intersections. The other lines may intersect both segments P_iP and P_jP , or miss both of them. Hence, the total number of intersection points on the segments P_iP and P_jP , not counting P , is odd. If the frogs reach at P at the same time, then there should be the same number of intersections on P_iP and P_jP , which gives an even number of intersections. This contradiction shows that the frogs do not meet each other.

(b) Let n be even. Consider any way that Jeff places the frogs. For each pair of points P_i and P_{n+i} , color P_i black if the frog jumps away from it, and white otherwise; do the same for P_{n+i} . So out of each P_i and P_{n+i} , one is black the other is white. Since n is even, there exist 2 adjacent points of the same color. That is P_i, P_{i+1} are of the same color, and P_{n+i}, P_{n+i+1} are of the same color. Without loss of generality, suppose both points P_i and P_{i+1} are black. Let P be the intersection between the segments P_iP_{n+i} and $P_{i+1}P_{n+i+1}$. Then any other segment meeting one of the segments P_iP and $P_{i+1}P$ must also meet the other one, and so the number of intersections on P_iP and $P_{i+1}P$ are the same. This shows that the frogs jumping away from P_i and P_{i+1} will meet at P .