Caustic Envelope

Poon Zong Wei Julian, Toh Yi Sheng Eusebius, Tan Zi Hao

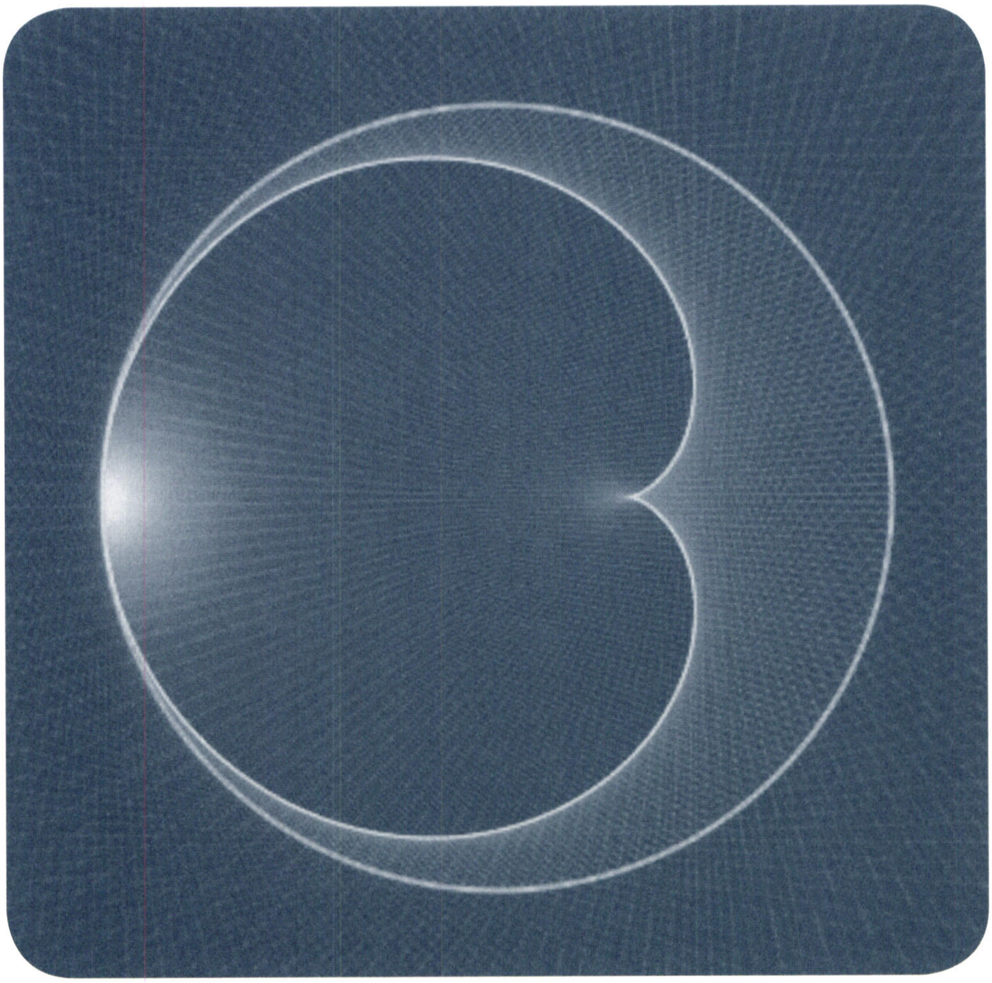


Image 1 (Caustic Envelope)

Abstract

This study investigated the best method to position individuals with respect to the arrangement of a sound source. We crafted several arrangements of the sound sources and reflected them off a wall in the shape of a function. By finding the envelope of the reflected waves, we found the points at which the sound was loud. This was done by equating the expression of the reflected waves to zero, then finding the equation of the line which most of the waves intersect. After which we identified the envelope that is most favourable to a particular situation. The end product of this study was a general case function, where any envelope could be generated based on the shape of the wall.

Introduction

In today's world, speakers are extensively used. However, in many instances, there is an issue with the clarity of the sound that the speaker is projecting. This is due to the fact that if speakers are used without the appropriate positioning of the seats or building design, sound gets reflected haphazardly and do not reach the audience well. (1995, Ambrose, J., & Ollswang, J.) To resolve this issue, we desire to discover the best way to position individuals according to the arrangement of the speakers. Assuming that sound travels in a straight line, what we plan to do is to find the spots where the lines would intersect the most, as these spots would be where the sound is the loudest. However, these intersections would vary according to the arrangement of the speakers, thus making this process of finding the intersections for every case laborious. As such, we have limited the scope of our research to speakers that are arranged in the shape of functions. We created 2 cases, the sound coming from a function, and the sound reflected off a function. Thereafter, using these known functions, we would proceed to find the intersections of their perpendicular bisectors to find the envelope of these functions.

Methodology

Envelope of a Sliding Ladder

We will be using the sliding ladder example to explain how envelope works

In this situation,
→ The y-axis is the wall, x-axis is the floor
→ The length of the ladder is 1 arbitrary unit

First, define function of $F(x, y, t) = 0$ as an animation of a falling ladder on the x- and y-axes where k is the parameter, which can be seen as time. In order to do this, we can find the coordinates of the points where the ladder will intersect the x and y axis for a given t.

$$(0, t)\ (\sqrt{1-t^2}, 0)$$

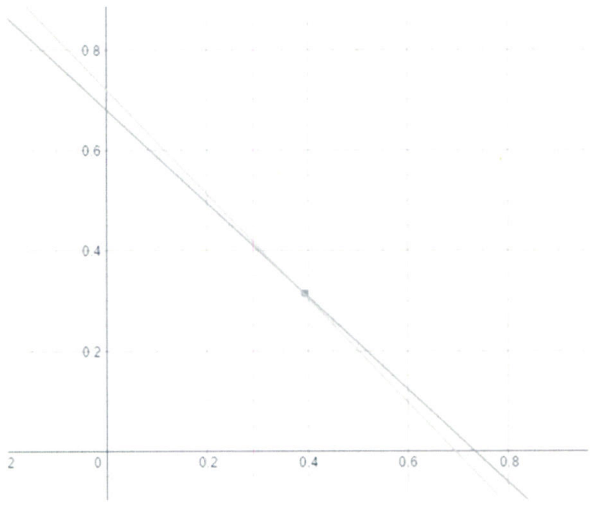
From this, we can derive the equation which describes the falling ladder.

$$y = -\frac{t}{\sqrt{1-t^2}}x + t$$

Now, since $F(x, y, t)$ has to be equal to 0, we can minus y on both sides. This is the universal way to use Envelope, since it reduces the need to differentiate both sides later on.

$$F(x, y, t) = -\frac{t}{\sqrt{1-t^2}}x + t - y = 0$$

Here is where we apply Envelope. Envelope is basically finding the intersection points of $F(x, y, t)$ and $F(x, y, t + n)$ for a very small n and for all k.



Purple: F(x,y,k)
Green: F(x,y,k+n)
Red Point: Intersection of the 2 lines when n is very small

The conventional way to do that, is to equate $F(x, y, t)$ and $F(x, y, t + n)$ and solve for the coordinates for x and y.

$$F(x, y, t) = F(x, y, t + n)$$

The problem with this method however, is that as n approaches 0, there is nothing to solve, since it becomes

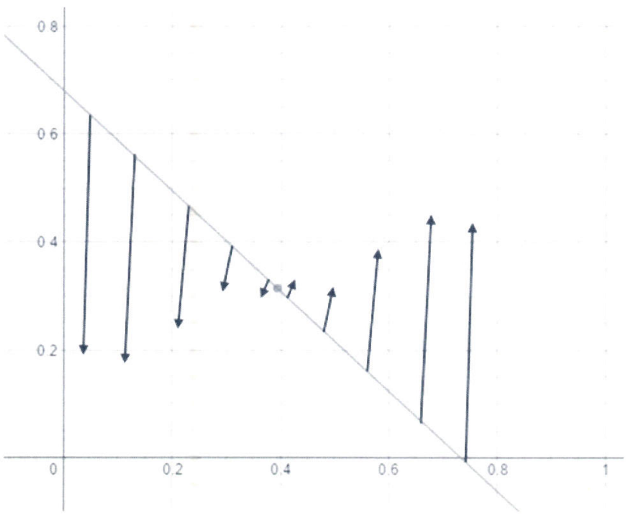
$$F(x, y, t) = F(x, y, t)$$

Therefore, in order to solve this, we can use the fact that $\frac{\partial}{\partial t} F(x, y, t) = 0$, if x and y are the coordinates of the intersection.

In order to proof that the above is the case, there are 2 methods:

Method 1:

We can think of $F(x, y, t) = 0$ as a function consisting of an infinite number of points, and t as time. So, $\frac{\partial}{\partial t} F(x, y, t)$ is describing the velocity of each of the points at a certain t.



The intersection point will be the only point that has a velocity of 0 in both the x and y direction, (as illustrated on the left). Therefore, the following is true:

$$\frac{\partial}{\partial t} F(x, y, t) = 0$$

Method 2:

This is a much less intuitive method, but it is a more conventional explanation.

In order to solve for the intersection, we need to solve this:

$$F(x, y, t) = F(x, y, t + n)$$
$$F(x, y, t) - F(x, y, t + n) = 0$$
$$\frac{F(x, y, t + n) - F(x, y, t)}{n} = 0$$

Limiting n to 0 will give us

$$\frac{d}{dt} F(x, y, t) = 0$$

Using the fact that

$$F(x,y,t) = 0 --(1)$$

$$\frac{d}{dt}F(x,y,t) = 0 --(2)$$

We can then solve for x and y in terms of t.

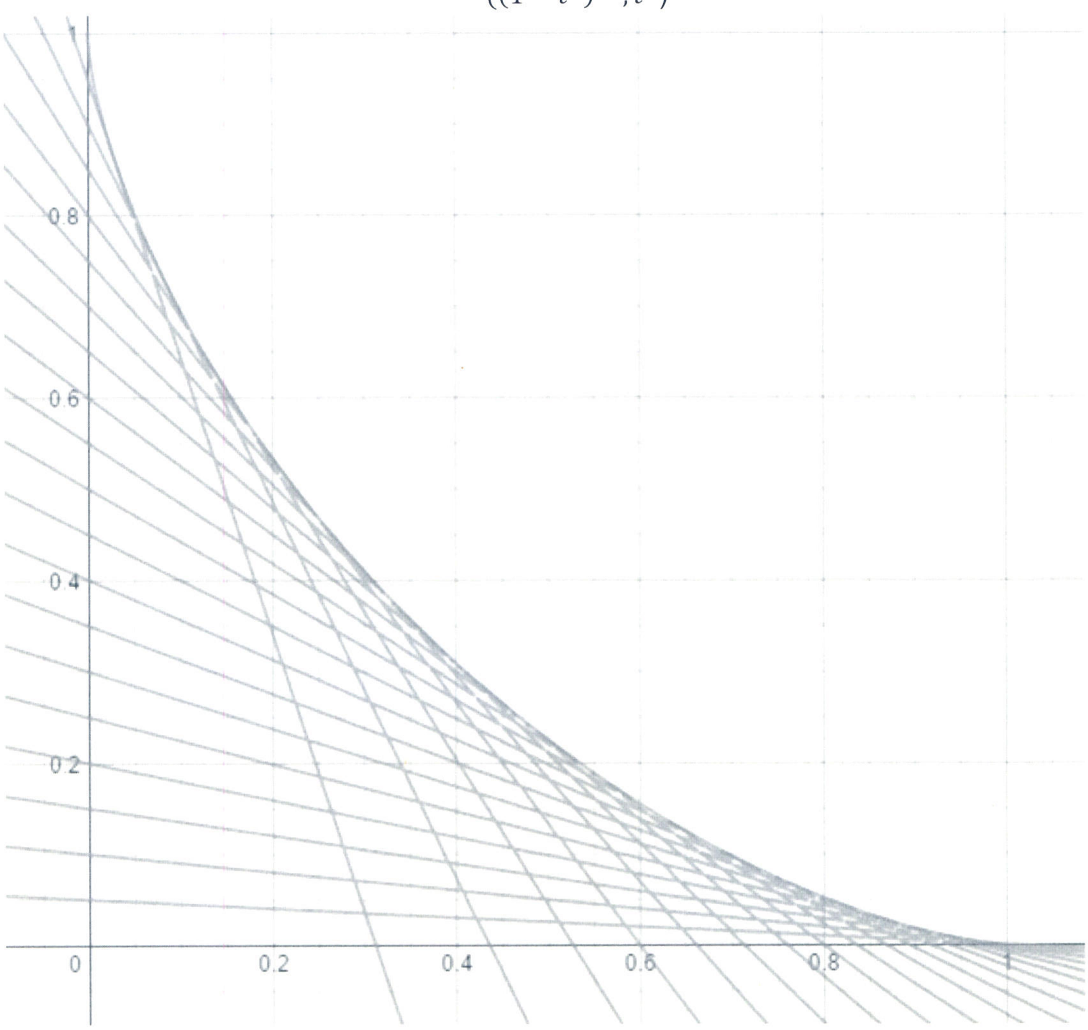
$$x = f(t), y = g(t)$$

The Envelope can thus be described in parametric form:

$$(f(t), g(t))$$

Now, back to the Ladder. Applying Envelope, we get the following parametric form:

$$((1-t^2)^{1.5}, t^3)$$



Now with envelope explained, we proceed on to the main part of our research.

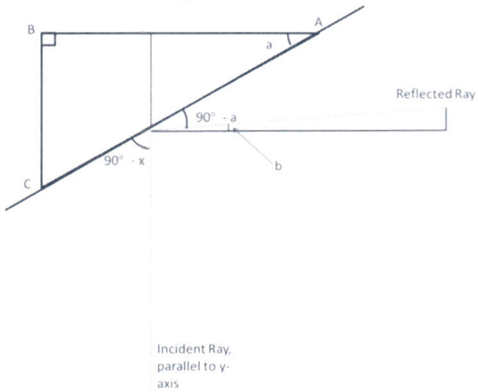
Sound Reflection

For reflection cases, we will be finding the envelope of reflected rays of sounds that hit the surface of a function in parallel.

Basic case:

For the basic case, the sound waves will be parallel to the y-axis and the function will be of the form of $y = f(x)$

Derivation of gradient for the reflected ray



AB is the tangent of a point of the function f(x).
We want to find the gradient of the reflected ray.

$$\tan a = g$$
$$\tan b = g_r$$

g is gradient of AB and g_r is the gradient of the reflected ray.
Since

$$\mathrm{b} = 2\mathrm{a} - 180°$$
$$\tan b = \frac{\tan^2 a - 1}{2\tan a} = \frac{g^2 - 1}{2g} = g_r \quad --(1)$$

Derivation of $F(x, y, t) = 0$

$F(x, y, t) = 0$ is the function describing the reflected ray when an incident ray of the form $x = k$ reflects off the function $f(x)$. Using the basic graph transformation used in the previous case, the function is.

$$F(x, y, t) = g_{r,t}(x - t) + f(t) - y = 0 \quad --(2)$$

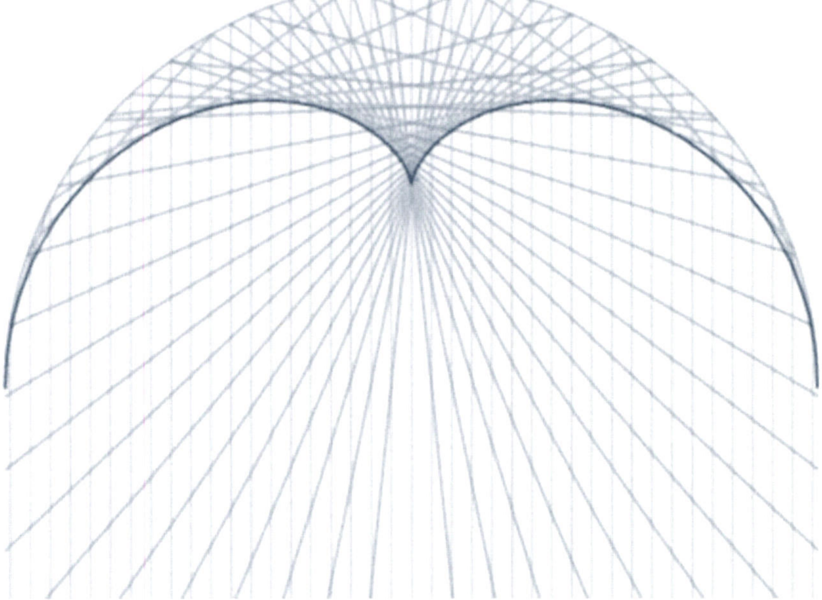
Where $g_{r,t}$ is the gradient of the reflected ray where the incident ray is hits the function $f(x)$ at $x = t$

Substituting (1) into (2):

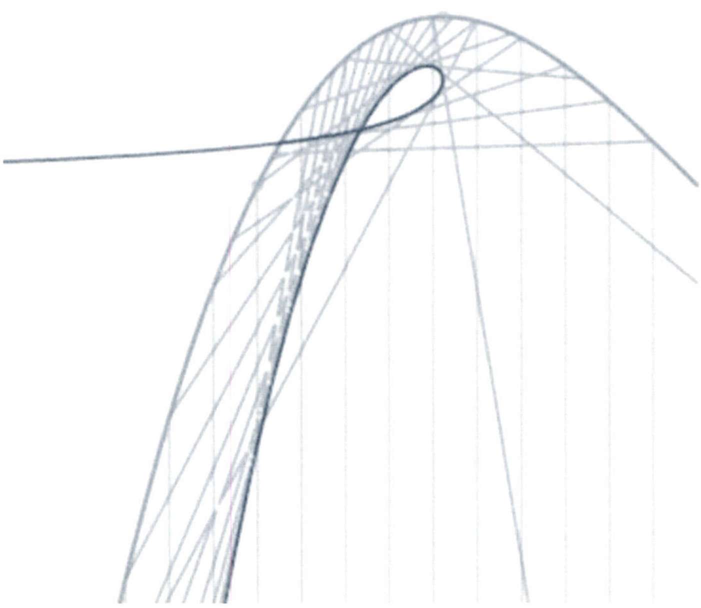
$$F(x, y, t) = \frac{f'(t)^2 - 1}{2f'(t)}(x - t) + f(t) - y = 0 \quad --(3)$$

Applying envelope to Equation (3), we get:

$$\left(t - \frac{f'(t)}{f''(t)}, \frac{2f(t)f''(t) - f'(t)^2 + 1}{2f''(t)}\right)$$



For the well-known semi-circle.

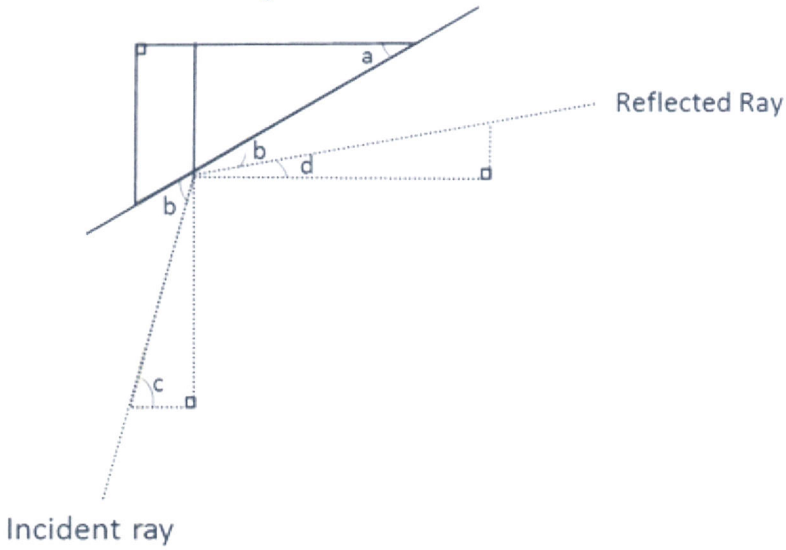


, $f(x) = x^3 - x \{x < 0\}$

Next, we will be considering the effects of we reflect off the sound reflectors at an angle. For this case, the sound wave will have a gradient of $\tan A$, where \(A\) is the angle of the rays with respect to the x-axis. This wave will hit a function $y = f(x)$, which acts as a sound reflector

Derivation:

Consider a similar image:



Now,

$$\tan(a) = g_m(t)$$
$$\tan(c) = g_i$$
$$\tan(d) = g_r$$

Where $g_m(t)$ is the gradient of the mirror at a particular point $x = t$, g_i is the gradient of the incident ray, g_r is the gradient of the reflected ray.

Where $g\_m(t)$ is the gradient of the sound reflector at a particular point, $x = t, g\_i$ is the gradient of the incident wave and g_r is the gradient of the reflected wave.
Through some angle chasing, it can be derived that:

$$d = 2a - c$$

$$g_r = \tan(d) = \tan(2a - c) = \frac{2\tan(a) - \tan(c) + \tan(a)^2 \tan(c)}{1 - \tan(a)^2 + 2\tan(a)\tan(c)} = \frac{2g_m(t) - g_i + g_m(t)^2 g_i}{1 - g_m(t)^2 + 2g_m(t)g_i}$$

Now, $g_i = \tan(A)$ and $g_m(t) = f'(t)$
Therefore, $F(x, y, t) = 0$, or the function that describes the reflected wave, can be defined as:

$$\frac{g_i(g_m(t))^2 + 2g_m(t) - g_i}{1 - (g_m(t))^2 + 2g_i g_m(t)}(x - t) + f(t) - y = 0$$

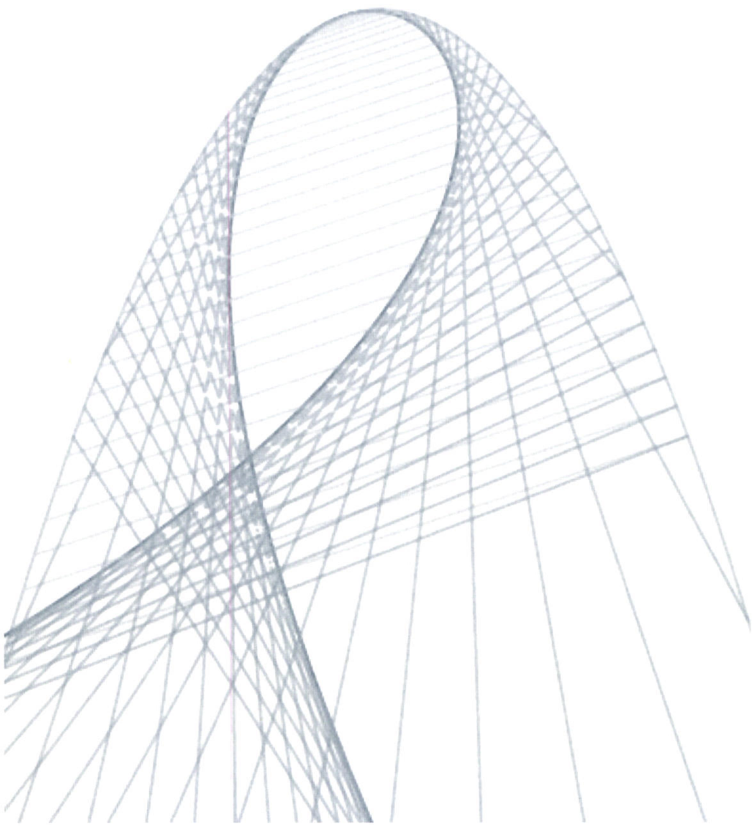
Applying envelope, we get:

$$\left(A, \frac{-g_i f'(t)^2 + 2f'(t) - g_i}{1 - f'(t)^2 + 2g_i f'(t)}(A - t) + f(t)\right)$$

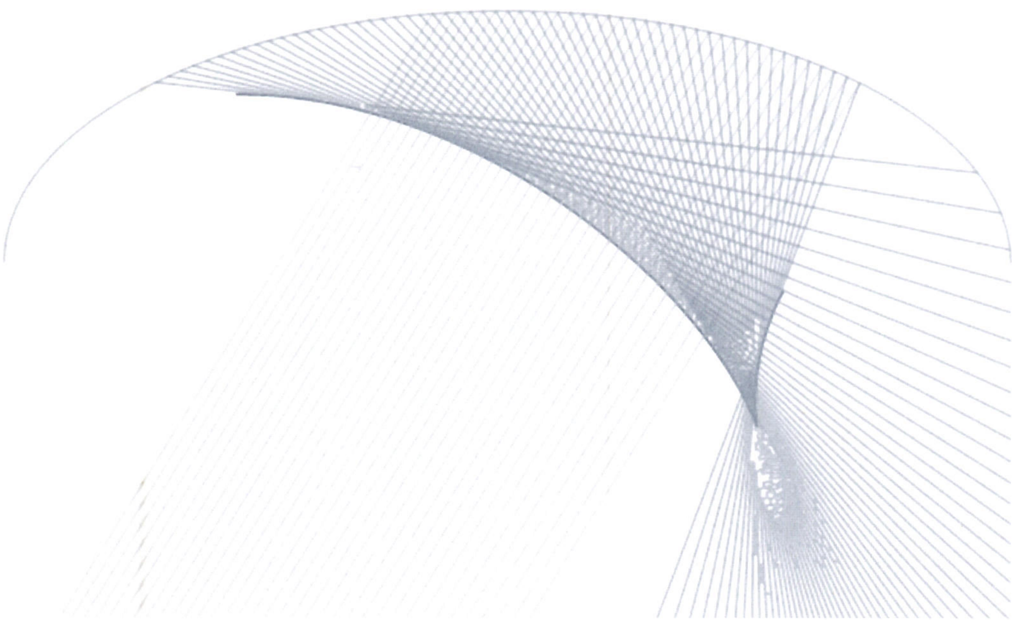
Where

$$A = \frac{-2g_i^2 f'(t) + 2g_i^2 f''(t)t + 3g_i f'(t)^2 - g_i - f'(t)^3 + f'(t) + 2f''(t)t}{2(g_i^2 + 1)f''(t)}$$

For this case, the final equation of the envelope is very big and thus not very friendly to work with.



The classic parabola $f(x) = -x^2$ with $a = 0.3$



$f(x) = \sqrt{1 - \frac{x^2}{4}}, a = 1$

Parametric forms

As seen from the previous case, the final equation of envelope became really huge and thus not easy to work with. To simplify the equation, parametric forms are explored. For this case, the incident rays are parallel to the y-axis again but this time, the function which act as the mirror is in the form of a parametric equation $(f(t), g(t))$.

We can derive $F(x, y, t) = 0$ similarly to the previous case, and

$$F(x, y, t) = \frac{g'(t)^2 - f'(t)^2}{2g'(t)f'(t)}\left(x - f(t)\right) + g(t) - y = 0$$

Applying envelope to this we get the parametric equation:

$$\left(A,\ \frac{g'(t)^2 - f'(t)^2}{2g'(t)f'(t)}(A - f(t)) + g(t)\right)$$

Where

$$A = \frac{f(t)g'(t)f''(t) - f(t)g''(t)f'(t) + g'(t)f'(t)^2}{g'(t)f''(t) - g''(t)f'(t)}$$

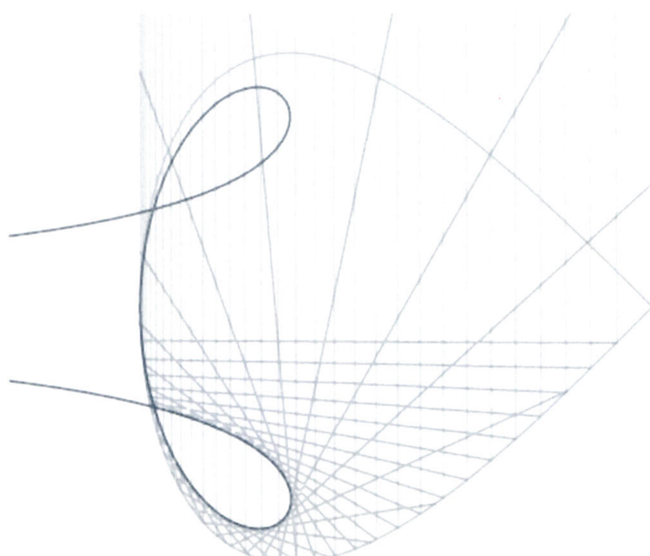
Alternative method for changing the direction of the sound wave

The idea for this is to rotate the graph instead of the sound waves. We make use of how we can rotate a graph parametrically of angle a, where

$$(g(t)cox\ (a) - f(t)\sin(a), g(t)\sin(a) + f(t)\cos(a))$$

is the rotated form of the graph

$$(f(t), g(t))$$



Mirror described as $\left(\cos(t), \frac{1}{2}\sin(2t)\right), x < 0$

General Case (Normal)

Now, assuming the speakers are arranged in $f(x)$ shape, we want to find the loudest points. This can be done by the following steps.

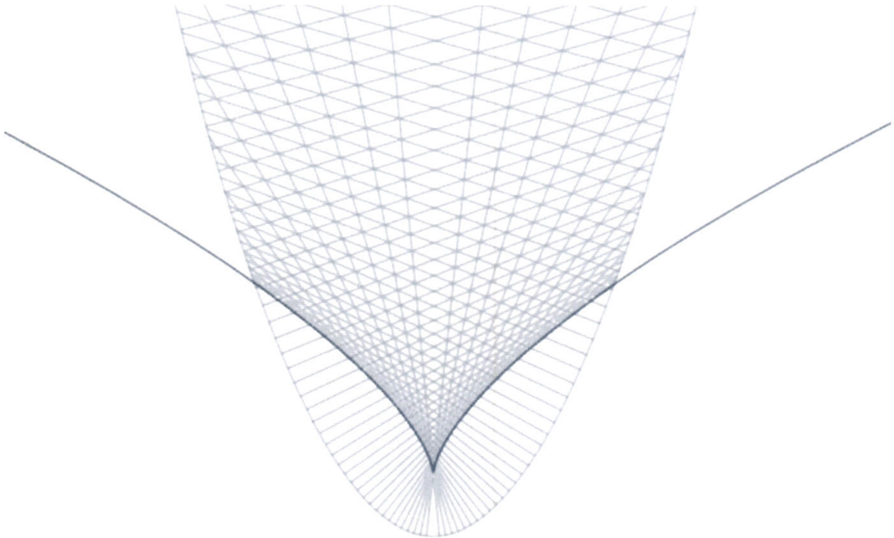
Firstly, plot the $y = f(x)$ graph. Then, differentiating this, you get the gradient each point, given by the equation $y = g(x)$. For a particular value of x, t, the gradient of the curve at that point can be calculated by $y = g(k)$.

With this information, the equation of the perpendicular bisector of the curve at point t can be found. Due to the fact that the bisector is based at point k, we have to shift the graph according to the position of t. Using simple knowledge of transformation of graphs, the equation is is given by

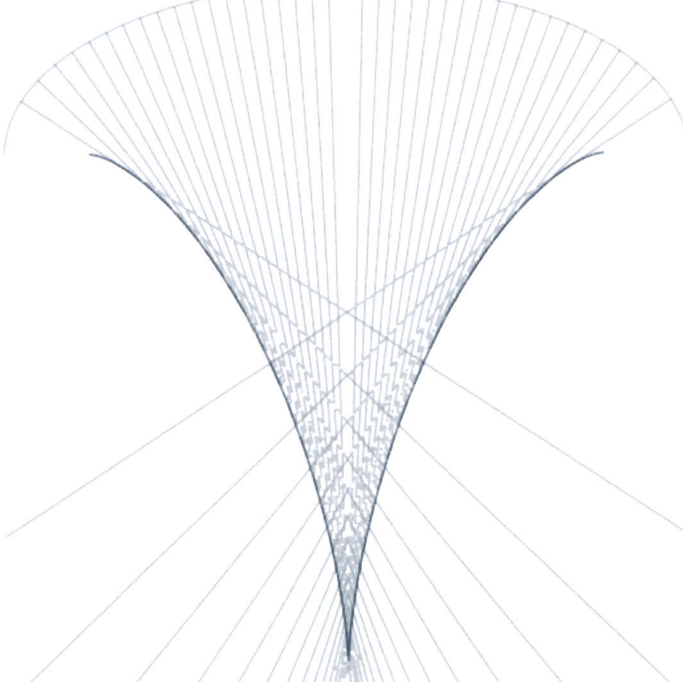
$$F(x, y, t) = -\frac{1}{f'(x)}(x - t) + f(t)$$

Thus, applying envelop to this function, we get the envelope coordinates to be

$$\left(\frac{-f'(t)^3 - f'(t) + tf''(t)}{f''(t)}, f(t) + \frac{f'(t)^2 + 1}{f'(t)}\right)$$



$f(x) = x^2$



The envelope for an ellipsoid, $f(x) = \sqrt{1 - \frac{x^2}{4}}$

Summary

In summary, the way in which speakers are arranged can be tweaked so as to allow the best performance. The loudest points vary with the arrangement of the speakers or instruments. Applying envelop allows a person to know where he should be standing to hear the best quality of sound. We did not manage to find the best envelope as there are too many ways to compare the envelopes, for example, in terms of distance the sound travels or the length of the envelope. There were too many variables to consider. However, we managed to achieve a generalised function for each of the cases.

References

Image 1: Why the Caustic of a circle is the most bright curve in the circle? (n.d.). Retrieved October 28, 2015.

Lawrence, J. D. A Catalog of Special Plane Curves. New York: Dover, p. 60, 1972.

Lockwood, E. H. "Caustic Curves." Ch. 24 in A Book of Curves. Cambridge, England: Cambridge University Press, pp. 182-185, 1967.

Ambrose, J., & Ollswang, J. (1995). Simplified design for building sound control. New York: Wiley.