

A characterization of the golden arbelos

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1 Introduction

For a point O on the line segment AB in the plane, let α , β and γ be the semicircles with diameters AO , BO and AB , respectively constructed on the same side of AB . The area bounded by the three semicircles, or more generally the configuration consisting of the three semicircles, is called an arbelos and is denoted by (α, β, γ) (see Figure 1). The word “arbelos” means shoemaker’s knife in Greek. Indeed the shape of the bounded region resembles the blade of a knife used by ancient cobblers [4] and it is believed that Archimedes of Syracuse studied the properties of this figure. In late twentieth century, the publication of the paper [3] seems to generate interest among mathematicians in this figure. In the present century, there have been many discoveries on the arbelos again and again, some of which are published in a book [6].

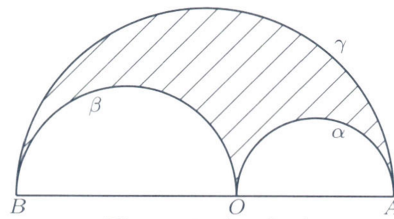


Figure 1: (α, β, γ)

In this paper we consider (α, β, γ) together with its reflection in the line AB , i.e., we suppose that α , β and γ are mutually touching circles with collinear centers, and consider a special type of the arbelos called the golden arbelos. Two positive numbers a and b ($a < b$) are said to be in golden ratio if

$$\frac{b}{a+b} = \frac{a}{b} \quad (1)$$

holds. This is equivalent to $b/a = \phi$, where ϕ is the golden number, i.e., $\phi = (1 + \sqrt{5})/2$. Golden ratio is also called golden mean or golden section [7]. It can be found not only in mathematics, art and architecture, but also in nature, since it is closely related to growth in plants and animals. Geometrically golden ratio is closely associated with the regular pentagon. Let a and b ($a < b$) be the radii of the circles α and β , respectively. The arbelos (α, β, γ) is called a golden arbelos if (1) holds. Figure 2 shows a recent discovery of four regular pentagons three of which are inscribed in one of the circles α , β and γ and a regular decagon inscribed in γ arising from the golden arbelos [5].

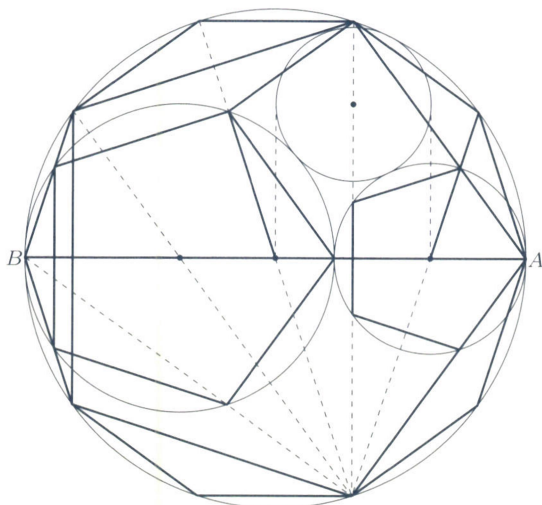


Figure 2: $b/a = \phi$

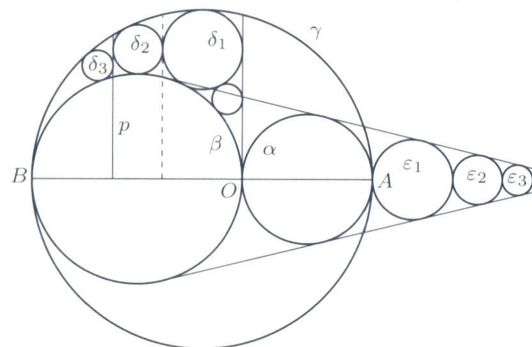


Figure 3: $b/a = \phi$

We call the radical axis of α and β , i.e., the perpendicular to AB at the point O , the axis of the arbelos. Let δ_i ($i = 1, 2, 3$) be the circles touching β externally γ internally, such that δ_1 touches the axis from the side opposite to A , δ_2 touches δ_1 externally from the side opposite to the axis, and p is the tangent of δ_2 parallel to the axis touching δ_2 from the side opposite to δ_1 , and δ_3 touches p from the side opposite to δ_2 (see Figure 3). Let ε_i ($i = 1, 2, 3$) be the circles touching the external common tangents of α and β from the same side as α such that ε_1 touches α externally at A , and ε_j ($j = 2, 3$) touches ε_{j-1} externally from the side opposite to A . Leon Bankoff seems to be the first researcher studied the golden arbelos. He discovered that the circles δ_i and ε_i ($i = 1, 2$) are congruent and ε_3 is congruent to the incircle of the curvilinear triangle made by β , δ_1 and the axis for the golden arbelos [2]. In later part of the paper, he pointed out that the incircle is also congruent to δ_3 . Therefore δ_3 and ε_3 are also congruent. But he did not refer to this fact in the paper and in a later paper [1]. While he also referred that the line joining the centers of δ_1 and δ_2 is parallel to AB . Therefore δ_1 and δ_2 share a common tangent parallel to the axis as δ_2 and δ_3 do. In this paper we generalize those facts.

2 A new property of golden arbelos

We use two lemmas (see Figures 4 and 5).

Proposition 1. If a circle of radius r touches the circles β externally and γ internally, and d is the distance from its center to the perpendicular to AB at the point B , then

$$r = \frac{a}{a + 2b}d. \tag{2}$$

Proof. By the Pythagorean theorem, we have $(a + b - r)^2 - (d - (a + b))^2 = (b + r)^2 - (d - b)^2$. Hence we have (2). \square

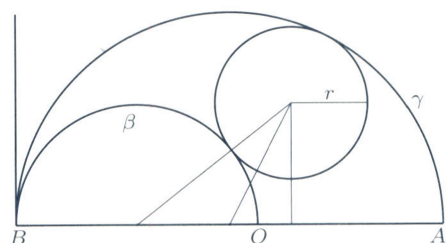


Figure 4

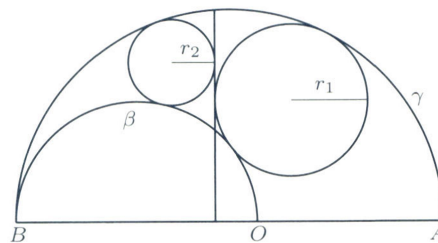


Figure 5

Proposition 2. If two circles of radii r_1 and r_2 ($r_1 > r_2$) touch β externally γ internally and a perpendicular to AB from the opposite side, then,

$$\frac{r_2}{r_1} = \frac{b}{a + b}. \tag{3}$$

Proof. If d is the distance from this perpendicular to the perpendicular to AB at B . Then

$$r_1 = \frac{a}{a + 2b}(d + r_1), \quad r_2 = \frac{a}{a + 2b}(d - r_2),$$

by Proposition 1. Solving the equations for r_1 and r_2 , we get

$$r_1 = \frac{a}{2b}d, \quad r_2 = \frac{a}{2(a + b)}d.$$

Therefore we get (3). \square

The next theorem gives a new characterization of the golden arbelos (see Figure 6).

Theorem 1. If two circles of radii d_1 and d_2 ($d_1 > d_2$) touch β externally γ internally and a perpendicular to AB from the opposite side and two circles of radii e_1 and e_2 ($e_1 > e_2$) touch the common external tangents of α and β from the same side as α and also touch externally to each other, then (α, β, γ) is a golden arbelos if and only if $d_1/d_2 = e_1/e_2$.

Proof. Since $e_2/e_1 = a/b$, while $d_2/d_1 = b/(a + b)$ by Proposition 2, the theorem is proved. \square

Let $\delta_1 = \varepsilon_1 = \alpha$ and let p_1 be the axis (see Figure 7). For a natural number $j \geq 2$, let δ_j be the circle touching β externally γ internally and p_{j-1} from the side opposite to α and let p_j be the remaining tangent of δ_j parallel to p_1 . Also let ε_j be the circle touching the external common tangents of α and β from the same side as α and touching ε_{j-1} externally from the side opposite to α . Then the radii of δ_i and the radii of ε_i form geometric progressions with common ratios $b/(a + b)$ and a/b , respectively, and (α, β, γ) is a golden arbelos if and only if δ_j and ε_j are congruent for any natural number $j \geq 2$.

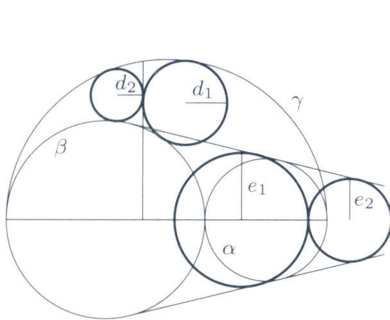


Figure 6

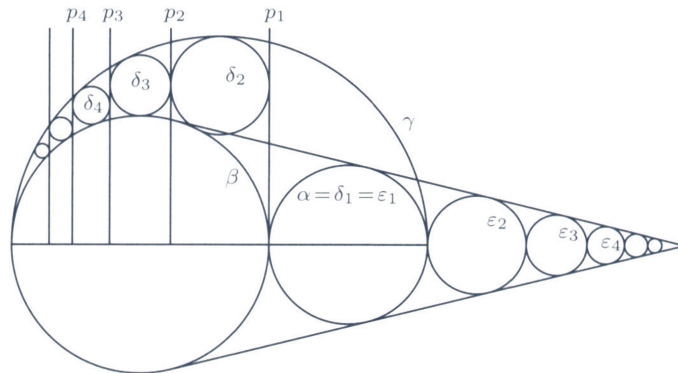


Figure 7

References

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