Evaluation of an Improper Integral

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Abstract

In this project, we will show that if $\sum_{k=1}^{\infty} b_k$ is an absolutely convergent series of real numbers, then the improper integral

$$\lim_{x\to 0^+} \int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt \right) \cot \frac{t}{2}\, dt$$

exists and

$$\lim_{x\to 0^+} \int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt \right) \cot \frac{t}{2}\, dt = \pi \sum_{k=1}^{\infty} b_k .$$

1 Introduction

It is well known that the First Fundamental Theorem of Calculus plays an important role in evaluating many elementary integrals. However, since this important theorem need not hold for improper integrals, we need new ideas to calculate such integrals. In this project, we evaluate an improper integral arising from an absolutely convergent series and the cotangent function. For any absolute convergent series $\sum_{k=1}^{\infty} b_k$ of real numbers, we will prove that the improper integral $\lim_{x\to 0^+} \int_x^{\pi} (\sum_{k=1}^{\infty} b_k \sin kt) \cot \frac{t}{2}\, dt$ exists and has a value of $\pi \sum_{k=1}^{\infty} b_k$(which is our main result).

The paper is organized as follows. In section 2, we obtain uniform bounds for a finite trigonometric series(Theorem 2.5) and a crucial integral involving trigonometric polynomials(Lemma 2.2). In section 3, we prove the main result and establish a corollary. Finally, in section 4 we will be discussing possible applications of the main result.

2 Uniform Bounds For The Sum $\sum_{k=1}^{n} \frac{\sin kx}{k}$

Let $\mathbb{N}$ denote the set of all positive integers. We use mathematical induction to establish the following crucial trigonometric identity.

Lemma 2.1. *Let $t \in (0, \pi]$. If $n \in \mathbb{N}$, then*

$$\sin nt \cot \frac{t}{2} = 1 + 2\sum_{k=1}^{n} \cos kt - \cos nt. \tag{1}$$

Proof. Let $t \in (0, \pi]$. For each $n \in \mathbb{N}$, we let $P(n)$ be the statement (1). When $n = 1$, we have

$$\begin{aligned} & \sin nt \cot \frac{t}{2} \\ = & \begin{cases} (\sin t)\frac{1+\cos t}{\sin t} & \text{if } t \in (0,\pi), \\ 0 & \text{if } t = \pi. \end{cases} \\ = & \ 1 + \cos t \\ = & \ 1 + 2\sum_{k=1}^{1} \cos kt - \cos t, \end{aligned}$$

which proves that $P(1)$ is true. Let $n \in \mathbb{N}$ and suppose $P(n)$ is true. Then the statement $P(n+1)$ is true:

$$\begin{aligned} & \sin(n+1)t \cot \frac{t}{2} \\ = & \ (\sin nt \cos t + \cos nt \sin t) \cot \frac{t}{2} \\ = & \ (\sin nt \cot \frac{t}{2}) \cos t + (\cos nt)(\sin t \cot \frac{t}{2}) \\ = & \left(1 + 2\sum_{k=1}^{n} \cos kt - \cos nt\right) \cos t + (\cos nt)(2\cos^2 \frac{t}{2}) \ \ \text{(since } P(n) \text{ is true)} \\ = & \ \cos t + \sum_{k=1}^{n} (\cos(k+1)t + \cos(k-1)t) - \cos nt \cos t + \cos nt \cos t + \cos nt \\ = & \ \cos t + \sum_{k=2}^{n+1} \cos kt + \sum_{k=1}^{n+1} \cos(k-1)t \\ = & \ \sum_{k=1}^{n+1} \cos kt + 1 + \sum_{k=1}^{n} \cos kt \\ = & \ 1 + 2\sum_{k=1}^{n+1} \cos kt - \cos(n+1)t. \end{aligned}$$

Since $P(1)$ is true and $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$, it follows from mathematical induction that (1) is true for every $n \in \mathbb{N}$ and $t \in (0, \pi]$.

$\square$

Let $n \in \mathbb{N}$ and let $x \in (0, \pi]$. Using Lemma 2.1 and the continuity of the function $t \mapsto 1 + 2\sum_{k=1}^{n} \cos kt - \cos nt$ on $[0, x]$, we see that the following right-hand limit

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{x} \sin nt \cot \frac{t}{2}\, dt$$

exists. For simplicity, we write the above right-hand limit as

$$\int_{0}^{x} \sin nt \cot \frac{t}{2}\, dt.$$

The following result shows that $\int_0^x \sin nt \cot \frac{t}{2}\, dt - x$ is a trigonometric polynomial.

Lemma 2.2. *If $n \in \mathbb{N}$ and $x \in (0, \pi]$, then*

$$\int_{0}^{x} \sin nt \cot \frac{t}{2}\, dt = x + 2\sum_{k=1}^{n} \frac{\sin kx}{k} - \frac{\sin nx}{n}.$$

Proof. We have

$$\begin{aligned}
& \int_{0}^{x} \sin nt \cot \frac{t}{2}\, dt \\
= {} & \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{x} \left(1 + 2\sum_{k=1}^{n} \cos kt - \cos nt\right) dt \quad \text{(by Lemma 1.1)} \\
= {} & \int_{0}^{x} \left(1 + 2\sum_{k=1}^{n} \cos kt - \cos nt\right) dt \\
= {} & \left[t + 2\sum_{k=1}^{n} \frac{\sin kt}{k} - \frac{\sin nt}{n}\right]_0^x \\
= {} & x + 2\sum_{k=1}^{n} \frac{\sin kx}{k} - \frac{\sin nx}{n} - 0 - 2\sum_{k=1}^{n} 0 + 0 \\
= {} & x + 2\sum_{k=1}^{n} \frac{\sin kx}{k} - \frac{\sin nx}{n}.
\end{aligned}$$

$\square$

The next Lemma shows that the function $x \mapsto \int_0^x \sin nt \cot \frac{t}{2}\, dt$ is non-negative on $(0, \pi]$.

Lemma 2.3. *If* $n \in \mathbb{N}$ *and* $x \in (0, \pi]$, *then*

$$\int_0^x \sin nt \cot \frac{t}{2}\, dt \geq 0.$$

Proof. Using integration by parts, we obtain

$$\begin{aligned} &\int_0^x \sin nt \cot \frac{t}{2}\, dt \\ =\ & \lim_{\varepsilon \to 0^+} \left(\frac{2}{n} \sin^2 \frac{nt}{2} \cot \frac{t}{2} \bigg|_{\varepsilon}^{x} + \int_{\varepsilon}^{x} \frac{1}{n} \sin^2 \frac{nt}{2} \csc^2 \frac{t}{2}\, dt \right) \\ =\ & \frac{2}{n} \sin^2 \frac{nx}{2} \cot \frac{x}{2} + \int_0^x \frac{1}{n} \sin^2 \frac{nt}{2} \csc^2 \frac{t}{2}\, dt. \end{aligned}$$

The last equality is true because we have $\lim_{\varepsilon \to 0^+} \frac{2}{n} \sin^2 \frac{n\varepsilon}{2} \cot \frac{\varepsilon}{2} = 0$ by L'Hospital's rule[1, pp.183]. The inequality now follows. □

The next inequality will also be needed.

Lemma 2.4. *If* $n \in \mathbb{N}$ *and* $t \in [0, \pi]$, *then*

$$-2\pi \leq -t + \frac{\sin nt}{n} \leq 0$$

Proof. From the inequality $|\sin nt| \leq nt$, we deduce that

$$-2\pi \leq -t - t \leq -t + \frac{\sin nt}{n} \leq -t + t \leq 0.$$

□

Using Lemmata 2.2, 2.3, and 2.4, we can now find the uniform bounds for the sum $\sum_{k=1}^{n} \frac{\sin kx}{k}$.

Theorem 2.5. *We have*

$$\sup_{x \in [0,\pi]} \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leq \pi.$$

Proof. Let $n \in \mathbb{N}$. If $x \in \{0, \pi\}$, then $\sum_{k=1}^{n} \frac{\sin kx}{k} = 0$. Henceforth, we suppose that $x \in (0, \pi)$ and observe that

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\sin kx}{k} \\
&= \frac{1}{2}\left(\int_0^x \sin nt \cot \frac{t}{2}\, dt - x + \frac{\sin nx}{n}\right) && \text{(by Lemma 2.2)} \\
&\geq \frac{1}{2}\left(-x + \frac{\sin nx}{n}\right) && \text{(by Lemma 2.3)} \\
&\geq -\pi. && \text{(by Lemma 2.4)}
\end{aligned}$$

It remains to establish the inequality $\sum_{k=1}^{n} \frac{\sin kx}{k} \leq \pi$. According to Lemma 2.2,

$$\sum_{k=1}^{n} \frac{\sin kx}{k} = \frac{1}{2}\left(\pi - x + \frac{\sin nx}{n} - \int_x^{\pi} \sin nt \cot \frac{t}{2}\, dt\right).$$

Next, we use integration by parts to write

$$\begin{aligned}
&\int_x^{\pi} \sin nt \cot \frac{t}{2}\, dt \\
&= \frac{2}{n} \sin^2 \frac{nt}{2} \cot \frac{t}{2} \bigg|_x^{\pi} + \int_x^{\pi} \frac{1}{n} \sin^2 \frac{nt}{2} \csc^2 \frac{t}{2}\, dt \\
&= -\frac{2 \sin^2 \frac{nx}{2} \cot \frac{x}{2}}{n} + \frac{1}{n} \int_x^{\pi} \sin^2 \frac{nt}{2} \csc^2 \frac{t}{2}\, dt.
\end{aligned}$$

Finally, since the inequalities $\left|\sin \frac{nx}{2}\right| \leq \min\left\{\frac{nx}{2}, 1\right\}$ and $\tan \frac{x}{2} > \frac{x}{2}$ hold for all $x \in (0, \pi)$, we have

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\sin kx}{k} \\
&= \frac{1}{2}\left(\pi - x + \frac{\sin nx}{n} + \frac{2 \sin^2 \frac{nx}{2} \cot \frac{x}{2}}{n} - \frac{1}{n} \int_x^{\pi} \sin^2 \frac{nt}{2} \csc^2 \frac{t}{2}\, dt\right) \\
&\leq \frac{1}{2}\left(\pi - x + \frac{\sin nx}{n} + x \cot \frac{x}{2}\right) \\
&< \frac{1}{2}\left(\pi - x + \frac{\sin nx}{n} + 2\right) \\
&< \pi. && \text{(by Lemma 2.4)}
\end{aligned}$$

Hence we have

$$\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| \leq \pi$$

for all $n \in \mathbb{N}$ and $x \in [0, \pi]$. The result now follows from the definition of the supremum. □

3 Main result

Let $\sum_{k=1}^{\infty} b_k$ be an absolutely convergent series of real numbers. By the Comparison Test, the series $\sum_{k=1}^{\infty} b_k \sin kt$ is absolutely convergent (in particular, uniformly convergent by the Weierstrass M-Test[1, pp.282]) on $[0, \pi]$. Thus, for each $x \in (0, \pi)$, the function $t \mapsto \sum_{k=1}^{\infty} b_k \sin kt$ is continuous and hence Riemann integrable on $[x, \pi]$. Since the function $t \mapsto \cot \frac{t}{2}$ is Riemann integrable on the interval $[x, \pi]$, we conclude that the function $t \mapsto (\sum_{k=1}^{\infty} b_k \sin kt) \cot \frac{t}{2}$ is Riemann integrable on the interval $[x, \pi]$ too. The following Lemma yields the value of the definite integral

$$\int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt \right) \cot \frac{t}{2} \, dt.$$

Lemma 3.1. *Let $\sum_{k=1}^{\infty} b_k$ be an absolutely convergent series of real numbers. If $x \in (0, \pi)$, then*

$$\int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt \right) \cot \frac{t}{2} \, dt = \sum_{k=1}^{\infty} b_k \left(\pi - x - 2 \sum_{j=1}^{k} \frac{\sin jx}{j} + \frac{\sin kx}{k} \right).$$

Proof. In view of Lemma 2.2, we need to show that

$$\sum_{k=1}^{n} \int_x^{\pi} b_k \sin kt \cot \frac{t}{2} \, dt - \int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt \right) \cot \frac{t}{2} \, dt \to 0 \text{ as } n \to \infty.$$

Let $n \in \mathbb{N}$. Then

$$\begin{aligned}
& \left| \sum_{k=1}^{n} \int_x^{\pi} b_k \sin kt \cot \frac{t}{2} \, dt - \int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt \right) \cot \frac{t}{2} \, dt \right| \\
= & \left| \int_x^{\pi} \left(\sum_{k=1}^{n} b_k \sin kt - \sum_{k=1}^{\infty} b_k \sin kt \right) \cot \frac{t}{2} \, dt \right| \\
= & \left| \int_x^{\pi} \left(\sum_{k=n+1}^{\infty} b_k \sin kt \right) \cot \frac{t}{2} \, dt \right| \\
\leq & \int_x^{\pi} \left| \sum_{k=n+1}^{\infty} b_k \sin kt \right| \cot \frac{t}{2} \, dt \\
\leq & \int_x^{\pi} \sum_{k=n+1}^{\infty} |b_k| \, |\sin kt| \cot \frac{x}{2} \, dt \\
\leq & (\pi - x) \left(\cot \frac{x}{2} \right) \sum_{k=n+1}^{\infty} |b_k| .
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} b_k$ is absolutely convergent, we conclude that

$$\lim_{n\to\infty} (\pi - x)\left(\cot\frac{x}{2}\right) \sum_{k=n+1}^{\infty} |b_k| = 0.$$

An application of the Squeeze Theorem[1, pp.66] completes the argument. □

Theorem 3.2 (Main Result). *If $\sum_{k=1}^{\infty} b_k$ is an absolutely convergent series of real numbers, then the improper integral*

$$\lim_{x\to 0^+} \int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt\right) \cot\frac{t}{2}\, dt \tag{2}$$

exists and

$$\lim_{x\to 0^+} \int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt\right) \cot\frac{t}{2}\, dt = \pi \sum_{k=1}^{\infty} b_k.$$

Proof. According to Theorem 2.5 and the triangle inequality, for each $k \in \mathbb{N}$ and $x \in [0,\pi]$ we have

$$\left|\left\{\pi - x - 2\sum_{j=1}^{k} \frac{\sin jx}{j} + \frac{\sin kx}{k}\right\}\right| \leq 5\pi.$$

Hence, by Weierstrass M-Test, the series

$$\sum_{k=1}^{\infty} b_k \left(\pi - x - 2\sum_{j=1}^{k} \frac{\sin jx}{j} + \frac{\sin kx}{k}\right)$$

is uniformly convergent on $[0,\pi]$. Thus the function

$$x \longmapsto \sum_{k=1}^{\infty} b_k \left(\pi - x - 2\sum_{j=1}^{k} \frac{\sin jx}{j} + \frac{\sin kx}{k}\right)$$

is continuous on $[0,\pi]$. Since Lemma 3.1 holds and the above function is continuous on $[0,\pi]$, we conclude that the improper integral (2) exists and

$$\begin{aligned}
&\lim_{x\to 0^+} \int_x^{\pi} \left(\sum_{k=1}^{\infty} b_k \sin kt\right) \cot\frac{t}{2}\, dt \\
= &\lim_{x\to 0^+} \sum_{k=1}^{\infty} b_k \left(\pi - x - 2\sum_{j=1}^{k} \frac{\sin jx}{j} + \frac{\sin kx}{k}\right) \\
= &\ \pi \sum_{k=1}^{\infty} b_k.
\end{aligned}$$

□

Thus we have shown that the absolute convergence of the series $\sum_{k=1}^{\infty} b_k$ is a sufficient condition for the improper integral (2) to exist. Now, we ask the question: What conditions imposed on the series $\sum_{k=1}^{\infty} b_k$ will allow the improper integral

$$\lim_{x\to 0^+}\int_x^{\pi}\frac{1}{t}\sum_{k=1}^{\infty} b_k \sin kt\, dt$$

to converge? That question can be answered by the following result of Boas[2, pp.1], which can be deduced from Theorem 3.2.

Corollary 3.3 (Boas 1951). *If $\sum_{k=1}^{\infty} b_k$ is an absolutely convergent series of real numbers, then the improper integral*

$$\lim_{x\to 0^+}\int_x^{\pi}\frac{1}{t}\sum_{k=1}^{\infty} b_k \sin kt\, dt \tag{3}$$

exists.

Proof. Let $g(0)=0$ and $g(t)=\frac{2}{t}-\cot\frac{t}{2}$ for $t\in(0,\pi]$. Then g is continuous on $[0,\pi]$. Since the function $t\mapsto\sum_{k=1}^{\infty} b_k\sin kt$ is continuous on $[0,\pi]$, we conclude that the function $t\mapsto g(t)\sum_{k=1}^{\infty} b_k \sin kt$ is continuous and hence Riemann integrable on $[0,\pi]$. Thus, the improper integral (3) exists and

$$\begin{aligned}
&\lim_{x\to 0^+}\int_x^{\pi}\frac{1}{t}\left(\sum_{k=1}^{\infty} b_k\sin kt\right)\, dt\\
=\;&\lim_{x\to 0^+}\int_x^{\pi}\frac{1}{2}g(t)\left(\sum_{k=1}^{\infty} b_k\sin kt\right)\, dt+\lim_{x\to 0^+}\int_x^{\pi}\frac{1}{2}\left(\sum_{k=1}^{\infty} b_k\sin kt\right)\cot\frac{t}{2}\, dt\\
=\;&\int_0^{\pi}\frac{1}{2}g(t)\left(\sum_{k=1}^{\infty} b_k\sin kt\right)\, dt+\frac{\pi}{2}\sum_{k=1}^{\infty} b_k.
\end{aligned}$$

□

Remark 3.4. The absolute convergence of the series $\sum_{k=1}^{\infty} b_k$ is a sufficient (but not necessary) condition for the existence of the improper integral (3). On the other hand, the following improper integral

$$\lim_{x\to 0^+}\int_x^{\pi}\left|\frac{1}{t}\sum_{k=1}^{\infty} b_k\sin kt\right|\, dt.$$

may not exist. For an example, see[3, pp.221].

All in all, in this paper, we first established several lemmata involving trigonometric functions, then we proceeded to prove the main result. We also managed to established Boas' result as a corollary.

4 Future Directions

The main result has some rather suprising applications. From the main result, it is possible to obtain a new proof of Fejer Jackson inequality: For all $n \in \mathbb{N}$ and $x \in [0, \pi]$, we have

$$\sum_{k=1}^{n} \frac{\sin kx}{k} \geq 0.$$

It is also possible to deduce the value of certain infinite trigonometric series from the main result. For example, the value of the series

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k(k^2\pi^2 + \theta^2)},$$

where $\theta \in \mathbb{C} \backslash \pi\mathbb{Z}$, can be found. We can also use the main result to evaluate certain mproper integrals. One such example is the improper integral

$$\int_0^{\pi} t^2 \left(\ln \left(2 \cos \frac{t}{2} \right) \right)^2 \, dt,$$

which can be proved to have a value of $\frac{11\pi^5}{180}$. Another example is the integral

$$\int_0^{\pi} t \cot \frac{t}{2} \, dt,$$

where we can prove from the main result that it has a value of $2\pi \ln 2$.

References

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