Minimum Distance Problem

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1. Introduction

1.1 Overview of Minimum Distance Problem

The Steiner Tree problem was first proposed by Jakob Steiner. Given *n* points on a plane, we have to connect them with lines either directly or through other points or line segments, such that the total length of lines are minimal. A simpler version of this problem is the Fermat's point, which is just one point instead of a multiple points spanning into a tree.

1.2 Rationale

We decided to take on this project as we wanted to find out how the Steiner Tree Problem can be applied in real-life. Our inspiration for the project came from simple everyday life activities such as planning the placement of various buildings and the layout of roads that interconnect them. We became motivated to figure out the solution to connecting a given set of locations with roads such that the total length is minimised. This would be a beneficial breakthrough. Not only will it reduce the amount of resources spent, it can also save people's travelling time. However, we understand the constraints in real-life situations, such as the fact that we cannot always connect places with straight roads. Therefore, we aim to solve other variations of the Steiner Tree Problem.

1.3 Objectives

- To find a general construction to minimise total length of segments connecting a number of points

- To find out how to interconnect a number of points with minimum total length of segments under certain restrictions

1.4 Research Problems

1. How to construct the minimum Steiner trees/vertex in a triangle or quadrilateral? (with internal angles < 120°)
2. What is the total length in the minimum Steiner tree of a triangle or quadrilateral? (with internal angles < 120°)
3. How is a minimum Rectilinear Steiner Tree of a triangle or quadrilateral constructed and what is the length of it?

1.5 Field of Mathematics

1. Geometry
2. Trigonometry
3. Algebra

1.6 Terminologies

1. Steiner Tree

The Steiner Tree of n points is a tree with all n points interconnected directly or indirectly through more points, such that the total length of the connecting lines are minimal.

2. Rectilinear Steiner Tree

Rectilinear Steiner Trees are Steiner Trees with all the connecting lines perpendicular to one another.

3. Steiner Point

A point in the Steiner Tree of a given set of points, but which is not part of the given set of points which have to be connected.

2. Literature Review

Jakob Steiner (18 March 1796-1 April 1863) first proposed Steiner tree problem. It is not yet known if there exists a definite solution, but approximations extremely close to the minimum are currently possible. The extra intermediate points added to reduce length of connection are known as Steiner points.

Fermat's point, named after Pierre de Fermat, is a point P in ABC such that PA+PB+PC is minimum. Much research has been done on Fermat's point, and many proofs and properties and methods to construct Fermat's point has been found. The difference between Fermat's point and Steiner tree is that in Steiner tree multiple Steiner points may be added.

Hermann Minkowski (22 June 1864 – 12 January 1909) considered Taxicab Geometry in Germany in 19th Century. It is a form of geometry in which the usual Euclidean distance function is replaced with rectilinear distance, i.e. distance can only travel in right angles. Thus the shortest possible distance between 2 points is the sum of absolute differences of their coordinates on Cartesian plane. E.g Rectilinear Distance between points (a,b) and (c,d) = $|a-c| + |b-d|$. It is also known as Manhattan Distance as the grid layout of most streets in Manhattan makes Rectilinear Distance the shortest possible way a taxicab could take between 2 locations.

3. Methodology

We conduct our research by following the steps below:

1. First find the Fermat point of a triangle, then try to apply similar techniques to find how to construct the Minimum Steiner Tree for a quadrilateral.

2. Find out how to construct the Fermat point of a triangle/Minimum Steiner Tree for a quadrilateral with geometry, then find the total length of the segments in the Steiner Tree with trigonometry and algebra.

4. Results

4.1.1 Finding Fermat Point of Triangle

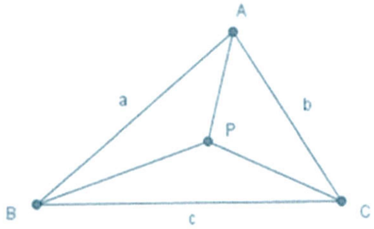
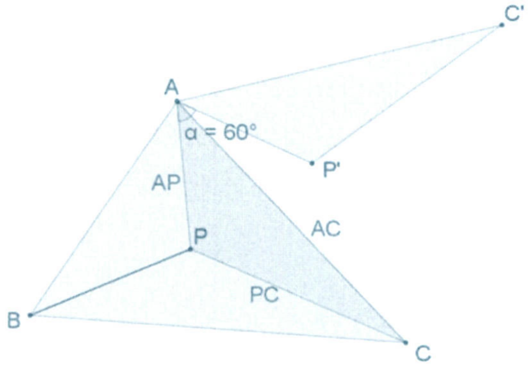
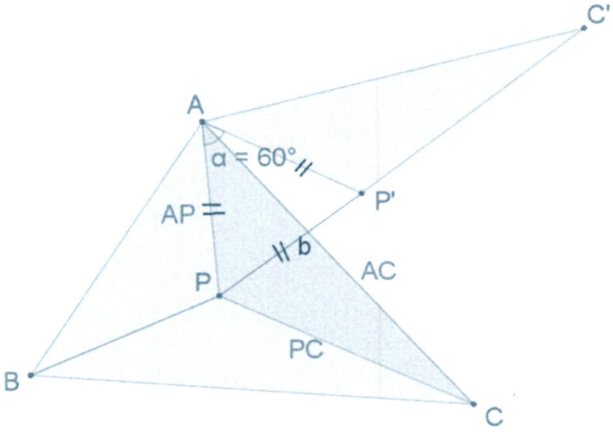


Diagram 1

Definition of Fermat point: in a triangle *ABC*, Fermat point is Point *P* such that *PA+PB+PC* is minimised.



Rotate ΔABC by 60° counterclockwise to get $\Delta AP'C'$.

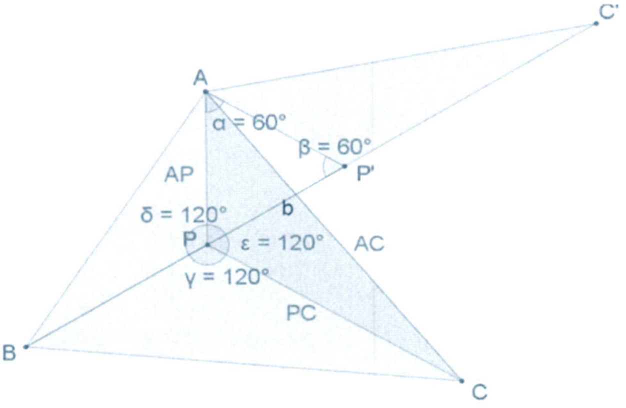


$AP = AP' = PP'$

$P'C' = PC$

$AP + BP + CP = BP + PP' + P'C'$

Since C' is a fixed point obtained from rotating C by 60° counterclockwise around A, and B is also a fixed point, the sum is minimum when $BPP'C'$ is a straight line.



When BC' is a straight line, since $\Delta APP'$ is equilateral, $\angle APB = 120°$ and $\angle AP'C' = \angle APC = 120°$, so $\angle BPC = 120°$

4.1.2 Finding Steiner Tree of Quadrilateral

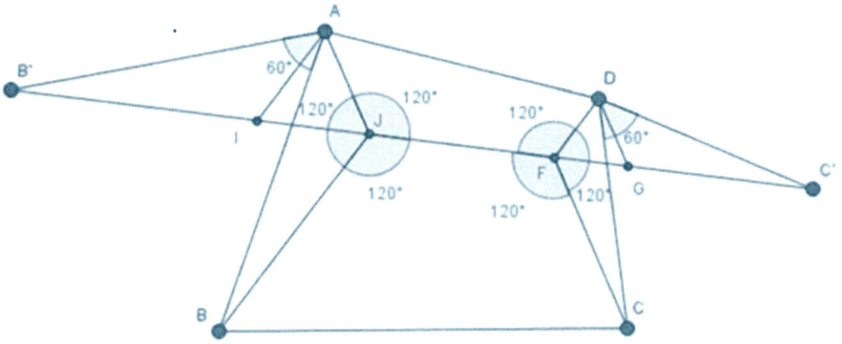


Diagram 3

Diagram 3 shows one of the possible Steiner Trees in a quadrilateral *ABCD*, as well as the construction method to find Steiner Points *J* and *F*.

First assume 2 random points *J* and *F* in quadrilateral *ABCD*.

Rotate triangles *AJB* and *DFC* by 60° clockwise and counterclockwise respectively to obtain triangles *AIB'* and *DGC'* respectively.

Since $\angle IAJ = 60°$ and $IA = AJ$, ΔIAJ is equilateral, $AJ = IJ$.

Similarly, ΔDFG is equilateral, $DF = FG$.

Since $B'I = BJ$ and $GC' = FC$,

$BJ + AJ + JF + DF + FC = B'I + IJ + JF + FG + GC'$.

Thus the minimum total length in the Steiner Tree is minimised when $B'IJFGC'$ is a straight line.

For this to be true, similar to the argument we used in finding the Fermat Point,

the angles around the Steiner Points J and F should be equal to 120°.

4.2 Total length of Steiner Tree

4.2.1 Rectangle

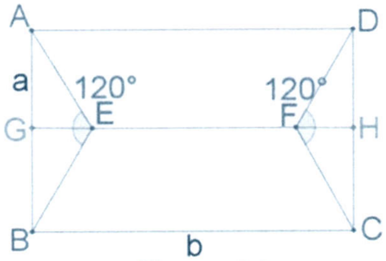


Diagram 4.1

Diagram 4.1 shows case 1 of a Steiner Tree for a rectangle $ABCD$. Let the length of AB be a and BC be b, and $b > a$. E and F are points in the rectangle such that $\angle AEB$ and $\angle CFD$ are 120° each. Construct EG and FH, with G on line AB and H on line CD, such that AB is perpendicular to GE and HF is perpendicular to DC. Triangles AGE, BGE, DHF and CHF are all 30-60-90 triangles.

$\because AG = \frac{a}{2} \therefore AE = \frac{a}{\sqrt{3}}$

Similarly, $BE = CF = DF = \frac{a}{\sqrt{3}}$

$EF = b - GE - FH = b - \frac{a}{\sqrt{3}}$

Thus the total length is $b + a\sqrt{3}$.

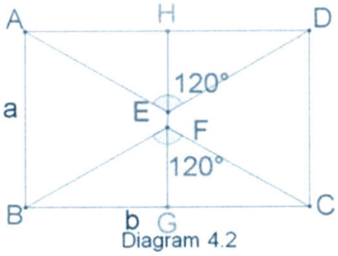


Diagram 4.2

Diagram 4.2 shows case 2 of Steiner Tree for rectangle $ABCD$. In this diagram, $\angle AED$ and $\angle BFC$ are 120°, with EH perpendicular to AD and FG perpendicular to BC. Triangle AEH, DEH, BFG and CFG are all 30-60-90 triangles.

Since $AH = \frac{b}{2}$, $AE = \frac{b}{\sqrt{3}}$

Similarly, $DE = BF = CF = \frac{b}{\sqrt{3}}$

$EF = a - EH - FG = a - \frac{b}{\sqrt{3}}$

Thus, the total length is $a + b\sqrt{3}$

Since $b>a$, it is obvious that Case 1 is more ideal as the length is shorter.

4.2.2 Isosceles Trapezium

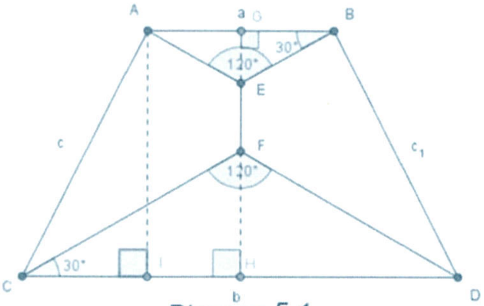


Diagram 5.1

In an isosceles trapezium $ABCD$, let $AB = a$, $CD = b$, $AC = BD = c$, where $b > a$, with Steiner points E and F, such that $\angle AEB = \angle CFD = 120°$, with isosceles triangles AEB and CFD.

Extend EF to intersect AB at G, CD at H, and we know $EF \perp AB$, $EF \perp CD$.

Point I is on CD such that $AI \perp CD$.

$$\because GH = AI,\ CI = \frac{b-a}{2}$$

$$\therefore GH = \sqrt{c^2 - (\frac{b-a}{2})^2}$$

$\because \Delta AGE$, ΔEGB, ΔCFH, ΔHFD are 30-60-90 triangles

$$\therefore AE = EB = \frac{a}{\sqrt{3}},\ CF = FD = \frac{b}{\sqrt{3}},\ GE = \frac{a}{2\sqrt{3}},\ FH = \frac{b}{2\sqrt{3}}$$

$\therefore$ Total length of Steiner Tree

$$= \frac{2a}{\sqrt{3}} + \frac{2b}{\sqrt{3}} + \sqrt{c^2 - (\frac{b-a}{2})^2} - \frac{a}{2\sqrt{3}} - \frac{b}{2\sqrt{3}}$$

$$= \sqrt{c^2 - (\frac{b-a}{2})^2} + \frac{\sqrt{3}}{2}(a+b)$$

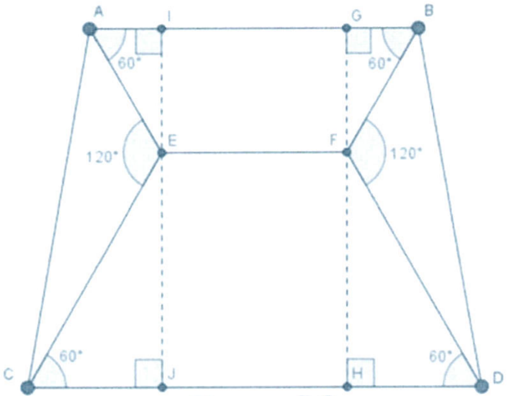


Diagram 5.2

Case 2 for Steiner Tree in isosceles trapezium is shown in Diagram 5.2. $AB = a$, $CD = b$, $AC = BD = c$. $\angle AEC = \angle BFD = 120°$ and $AB \parallel EF \parallel CD$. Drop perpendiculars from points E, F to points J, H on line CD and points I, G on line AB respectively, resulting in straight lines IEJ and GFH. Triangles GBF, HDF, CJE and AIE are 30-60-90 triangles.

$$\text{By trigonometric ratios, } AE + CE = \frac{2IE}{\sqrt{3}} + \frac{2EJ}{\sqrt{3}} = \frac{2IJ}{\sqrt{3}}$$

$$\text{Similarly, } BF + DF = \frac{2GH}{\sqrt{3}} = \frac{2IJ}{\sqrt{3}}$$

$$EF = \frac{a+b-AI-CJ-BG-DH}{2} = \frac{a+b-2(AI+CJ)}{2}$$

$$\text{By trigonometric ratios, } AI + CJ = \frac{IE}{\sqrt{3}} + \frac{EJ}{\sqrt{3}} = \frac{IJ}{\sqrt{3}}$$

$$\text{Total Length} = \frac{4IJ}{\sqrt{3}} + \frac{a+b}{2} - \frac{IJ}{\sqrt{3}} = \frac{3IJ}{\sqrt{3}} + \frac{a+b}{2}$$

$$\text{By Pythagoras Theorem, } IJ = \sqrt{c^2 + \left(\frac{b-a}{2}\right)^2}$$

$$\therefore \text{Total Length} = \frac{a+b}{2} + \sqrt{3\left[c^2 + \left(\frac{b-a}{2}\right)^2\right]}$$

4.2.3 Parallelogram

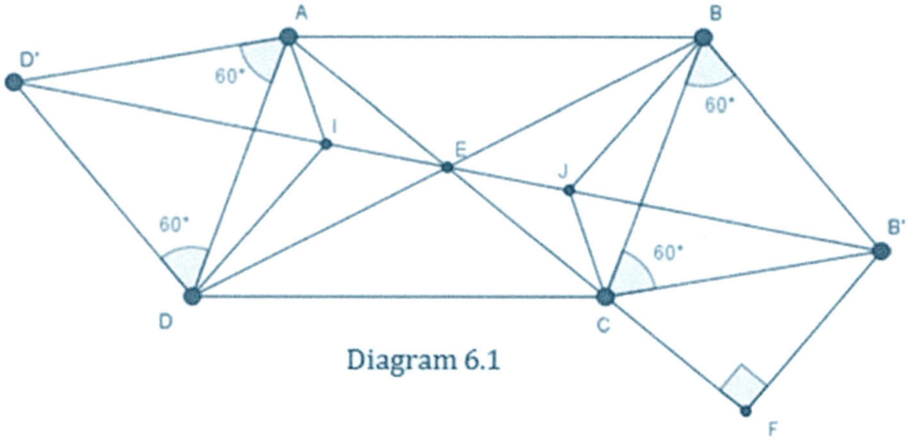


Diagram 6.1

Diagram 6.1 shows one of the possible Steiner Trees of a parallelogram *ABCD*, which is defined as shown below.

Let $AB = CD =$ a, $BC = AD =$ b, *AC*, *BD* intersect at *E*, and $CE = AE =$ c.

We see that this is sufficient to define a unique parallelogram *ABCD*, as from the congruent triangles concept, we can show triangles *ACB* and *ADC* are uniquely defined.

We get the points *B'*, and *D'* from rotating *B* and *D* clockwise by 60° around *C* and *A* respectively.

Without loss of generality, we may assume $a > b$.

With the above definitions, Diagram 6.1 shows Case 1 of the Steiner Tree of a parallelogram.

We recall that the total length of segments in this case of the Steiner Tree is equal to the length of *B'D'*. Thus we just have to find the length of *B'D'*.

We first show that *AC*, *BD*, *B'D'* are concurrent.

We can show triangles *AEB*, *CED* are congruent, giving $AE = CE$.

We let *B'D'* intersect AC at point H (not shown in diagram), then we can easily show that triangles *AHD'* and *CHB'* are congruent, giving $AH = CH$.

Combining the above two results, we obtain $E = H$. Thus *AC*, *BD*, *B'D'* are concurrent.

Thus $B'D' = 2B'E$.

We extend AC to F such that B'F is perpendicular to AC.

By Pythagoras Theorem, $B'E^2$

$$= EF^2 + B'F^2$$

$$= EC^2 + 2 \cdot EC \cdot CF + CF^2 + B'F^2$$

$$= b^2 + c^2 + 2bc\cos\angle B'CF$$

$$= b^2 + c^2 + 2bc\cos(120° - \angle ACB)$$

From the area formula of a triangle, we get:

$$\frac{1}{2}\sin\angle ACB \cdot AC \cdot BC$$

$$= \sin\angle ACB \cdot bc$$

$$= \text{Area of } \Delta ACB$$

By Heron's Formula,

Area of ΔACB

$$= \frac{\sqrt{(a+b+2c)(a+2c-b)(a+b-2c)(b+2c-a)}}{4}$$

$$\therefore \angle ACB = \sin^{-1}\left(\frac{\sqrt{(a+b+2c)(a+2c-b)(a+b-2c)(b+2c-a)}}{4bc}\right)$$

$$\therefore B'D' = 2B'E = \sqrt{b^2 + c^2 + 2bc\cos\left[120° - \sin^{-1}\left(\frac{\sqrt{(a+b+2c)(a+2c-b)(a+b-2c)(b+2c-a)}}{4bc}\right)\right]}$$

And the above result is the total length of the Steiner Tree for Case 1.

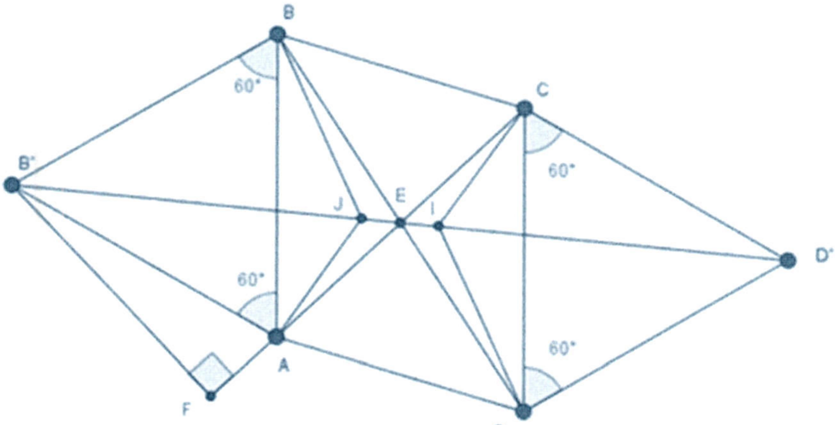


Diagram 6.2

Diagram 6.2 shows Case 2 of the Steiner Tree of a parallelogram *ABCD*, where *I*, *J* are the Steiner Points.

Let $AB = CD = a$, $BC = AD = b$, *AC*, *BD* intersect at *E*, and $CE = AE = c$.

Case 2 applies when $\angle BAC > 30°$. To get *B'* and *D'*, we rotate points *B* and *D* 60° anti-clockwise around *A* and *C* respectively.

We have *B'D'*, *AC*, *BD* concurrent as shown in Case 1.

Let *F* be on the extension of *CA* such that *BF* is perpendicular to *CA*.

We can find the total length of connecting segments in the Steiner Tree with a similar method as in Case 1.

As a final result, we get:

$$\therefore B'D' = 2B'E = \sqrt{a^2 + c^2 + 2ac\cos\left[120° - \sin^{-1}\left(\frac{\sqrt{(a+b+2c)(a+2c-b)(a+b-2c)(b+2c-a)}}{4ac}\right)\right]}$$

There would be another possible total length of the Steiner Tree in Case 2 if $\angle BAC \leq 30°$, but if that is so, one of the internal angles of the parallelogram *ABCD* would be greater than or equal to 120°. However, Research Question 2 states that we only consider quadrilaterals with internal angles all less than 120°, thus we do not consider this case.

4.3 Rectilinear Steiner Tree

In a Rectilinear Steiner Tree, all lines must be perpendicular to one another. However, they can still be pointing in infinitely many directions, so we must find the "baseline" that all other lines are either parallel or perpendicular to. To draw the Rectilinear Steiner Tree, we draw a line through one of the vertices, which for convenience we shall call a baseline, then drop perpendiculars to the other two points from that line. For any baseline we assign, there are 6 possible cases where the total length can be minimised for that particular baseline, so we can compare those to find the ideal case.

Case 1

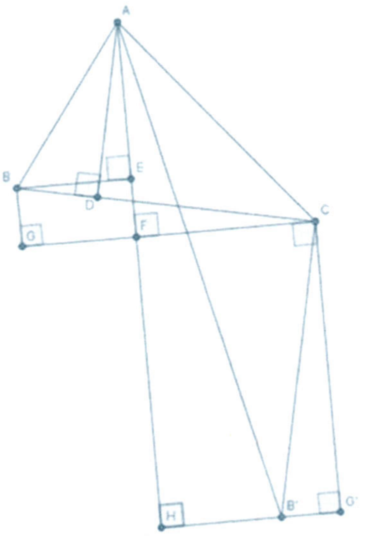


Diagram 7.1

In triangle ABC as shown in *Diagram* 7.1, we draw a line from A that is between AC and the altitude AD. Then, construct lines BE and CF such that E and F are on the line, and BE, CF are perpendicular to the line, which is now AF. Extend CF to point G such that $CG \perp BG$. $\because BEFG$ is a rectangle $\therefore GF = BE$.

that $CG \perp BG$. $\because BEFG$ is a rectangle $\therefore GF = BE$.

Total length of Steiner Tree $= AF + BE + FC = AF + GC$.

Rotate triangle CBG counter-clockwise by 90° around point C to form triangle $CB'G'$.

Extend $G'B'$ and AF to meet at point H, forming rectangle $CFHG'$, with $FH = CG'$.

Total length of Steiner Tree $= AF + FH = AH$.

$AH = \cos B'AH \cdot AB'$

As $\angle B'AH$ increases, $\cos\angle B'AH$ decreases, so the length of AH decreases.

Hence, the larger $\angle B'AH$ is, the shorter the length of the Steiner Tree.

Case 2

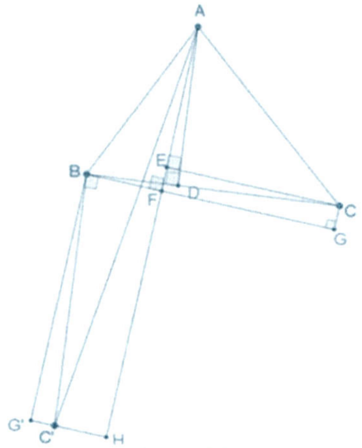


Diagram 7.2

However, if ∠*B'AH* increases to the point that *AF* is drawn between *AB* and altitude *AD*, the total length cannot be calculated in the same way. We can still use a similar method as shown in Diagram 7.2. We then obtain a similar result of

$$\text{Total length of Steiner Tree} = AH = \cos\angle C'AH \cdot AC'.$$

Since AC' is fixed for every triangle, the smaller cos∠*C'AH is*, the smaller the length of the Steiner Tree, and we can decrease the value of cos∠*C'AH* until AF is drawn between *AC* and altitude *AD*.

Since Cases 1 and 2 interchange along altitude *AD*, the optimal Steiner Tree for these two cases would be when *F* coincides with *D*, forming a Steiner Tree of length *AD* + *BC*.

Case 3

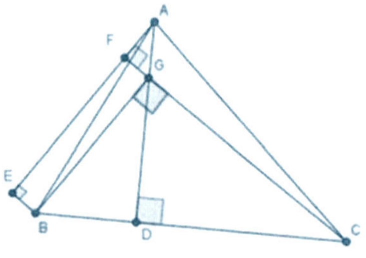


Diagram 7.3

Case 3 is when the line drawn is exterior to triangle *ABC*, as shown in Diagram 7.3. In this case, the total length of the Steiner Tree is *AE* + *FC* + *EB*.

However, we can drop a perpendicular BG onto CF, then $BG = EF$. The total length is then $AF + FC + BG < AF + FC + FE + EB$. Hence, this case does not minimise the length of the Steiner Tree.

Case 4

Case 4 considers when AE and AF do not overlap besides at point A. The total length is $AE + AF + BE + CF$, as shown in diagram 7.4. However, we can draw a line AH perpendicular to EF, such that AG and AH are perpendicular to BG and CH respectively. $BEAG$ and $CFAH$ are both rectangles, so $BG = AE$, $AG = EB$, $AH = FC$, $HC = AF$. If we change point E to G and point F to H, then the total length of the Steiner Tree $= AH + BG + HC < FC + AE + AF + BE$. Therefore, this case is not the minimum Steiner Tree.

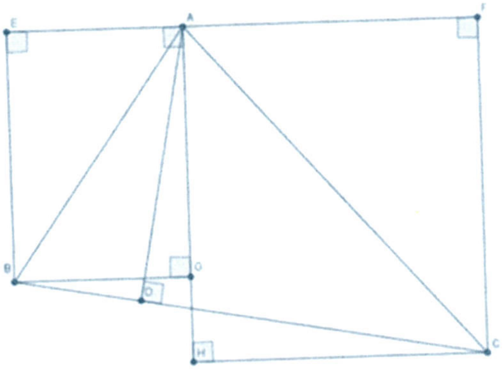


Diagram 7.4

Construction and length of optimal Tree

Therefore, the rectilinear Steiner Tree of a triangle would just be the altitude + base.

To find the length of the Steiner Tree, we can let the sides opposite points A, B and C be of length a, b and c respectively. We can then use Heron's formula to find the length of the three possible Steiner Trees,

$$\frac{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}{4a}+a$$

$$\frac{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}{4b}+b$$

$$\frac{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}{4c}+c$$

and compare their length to find the optimal case.

5. Further Extensions

In the course of our project, we were unable to find out how to construct the Rectilinear Steiner Tree for quadrilaterals and a general formula for the total length of a Steiner Tree in the general case of any convex quadrilateral. Thus we will consider these areas as further extensions for this project.

6. References

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