

Singapore International Mathematical Olympiad 2016

National Team Selection Test

Day 0

- 20 soccer teams participated in a round-robin tournament where every two teams played exactly one match. It was known that:
 - There are at least two matches which end in a draw.
 - If two teams A and B played out a draw, every other team must defeat at least one of A and B .

Show that one may write down these 20 teams in such an order that every team defeats the team immediately behind it.

- Suppose that a sequence a_1, a_2, \dots of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k . Prove that $a_1 + a_2 + \dots + a_n \geq n$ for every $n \geq 2$.

- Given an acute-angled triangle ABC , points D, E, F are on BC, CA, AB respectively such that AEF, BDF, CDE are acute-angled triangles whose orthocenters are P, Q, R respectively. If the lines AP, BQ, CR are concurrent, show that the lines DP, EQ, FR are also concurrent.
- Determine all positive integers M for which the sequence a_0, a_1, a_2, \dots , defined by $a_0 = \frac{2M+1}{2}$ and $a_{k+1} = a_k \lfloor a_k \rfloor$ for $k = 0, 1, 2, \dots$, contains at least one integer term.

Time allowed: 4 hours

Day 1

5. Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C . A point D is chosen inside the triangle CBH so that CH bisects AD . Let P be the intersection point of the lines BD and CH . Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q . Prove that the lines CQ and AD meet at ω .
6. Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:
- A player cannot choose a number that has been chosen by either player on any previous turn.
 - A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
 - The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

7. Suppose that a_0, a_1, \dots and b_0, b_1, \dots are two sequences of positive integers satisfying $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \quad b_{n+1} = \text{lcm}(a_n, b_n) - 1$$

for all $n \geq 0$. Prove that the sequence (a_n) is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

Time allowed: 4.5 hours

Day 2

8. Let a and b be positive integers such that $a!b!$ is a multiple of $a! + b!$. Prove that $3a \geq 2b + 2$.
9. Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leq r < s \leq 2n} (s - r - n)x_r x_s,$$

where $-1 \leq x_i \leq 1$ for all $i = 1, 2, \dots, 2n$.

10. Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S . Prove that there exist infinitely many positive integers that are *not* clean.

Time allowed: 4.5 hours

Day 0 (4 Hours)

1. 20 soccer teams participated in a round-robin tournament where every two teams played exactly one match. It was known that:
- (i) There are at least two matches which end in a draw.
 - (ii) If two teams A and B played out a draw, every other team must defeat at least one of A and B .

Show that one may write down these 20 teams in such an order that every team defeats the team immediately behind it.

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Solution. We first show that no team could have more than one draw. Suppose otherwise, say A drew with B and C . By (ii), C must defeat either A or B . Hence, C defeats B . Notice that this argument applies for B as well, i.e., B defeats C . This is impossible.

We call a sequence of teams A_1, A_2, \dots, A_k a winning sequence if A_1 is defeated by A_2 , A_2 is defeated by A_3 , ..., and A_{k-1} is defeated by A_k . Let A_1, A_2, \dots, A_n be the longest possible winning sequence one may have among all orderings of the teams.

If $n = 20$, nothing to prove. Now we assume $n \leq 19$. Consider team B which is not in this longest winning sequence.

Case I: A_1 defeated B .

This is impossible because B, A_1, A_2, \dots, A_n gives an even longer winning sequence, contradicting our assumption on the maximality of n .

Case II: B defeated A_1 .

One sees that B cannot be defeated by A_2 . Otherwise, A_1, B, A_2, \dots, A_n gives an even longer winning sequence. B cannot draw with A_2 either. Otherwise, A_1 was defeated by both B and A_2 , contradicting (ii). Hence, B must have defeated A_2 . By applying similar arguments repeatedly, one sees that B must have defeated A_3, A_4, \dots, A_n . Now A_1, A_2, \dots, A_n, B gives a longer winning sequence, a contradiction.

In conclusion, B and A_1 played out a draw. This argument applies for any player B not in the winning sequence A_1, A_2, \dots, A_n . Since A_1 cannot have more than one draw, $n = 19$.

Consider the winning sequence A_1, A_2, \dots, A_{19} where B played out a draw with A_1 . Notice that B cannot have any more draw.

If B defeated A_2 , he cannot be defeated by A_3 . Otherwise, $A_1, A_2, B, A_3, \dots, A_n$ gives a longer winning sequence. By applying similar arguments repeatedly, B defeated A_3, A_4, \dots, A_{19} . Now $A_1, A_2, \dots, A_{19}, B$ gives a longer winning sequence, a contradiction. Hence, A_2 must have defeated B . Similarly, A_3, A_4, \dots, A_{19} must have defeated B .

By (i), there exists another match where A_i, A_j played out a draw. Now B was defeated by both A_i and A_j , a contradiction to (ii). In conclusion, $n \leq 19$ is impossible. This completes the proof.

2. Suppose that a sequence a_1, a_2, \dots of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k . Prove that $a_1 + a_2 + \dots + a_n \geq n$ for every $n \geq 2$.

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Solution. From the given inequality

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}, \quad (1)$$

it can be seen that

$$\frac{k}{a_{k+1}} \leq \frac{a_k^2 + (k-1)}{a_k} = a_k + \frac{k-1}{a_k},$$

and so

$$a_k \geq \frac{k}{a_{k+1}} - \frac{k-1}{a_k}.$$

Summing up the above inequality for $k = 1, \dots, m$, we obtain

$$a_1 + a_2 + \dots + a_m \geq \left(\frac{1}{a_2} - \frac{0}{a_1}\right) + \left(\frac{2}{a_3} - \frac{1}{a_2}\right) + \dots + \left(\frac{m}{a_{m+1}} - \frac{m-1}{a_m}\right) = \frac{m}{a_{m+1}}. \quad (2)$$

Now we prove the problem statement by induction on n . The case $n = 2$ can be done by applying (1) to $k = 1$:

$$a_1 + a_2 \geq a_1 + \frac{1}{a_1} \geq 2.$$

For the induction step, assume that the statement is true for some $n \geq 2$. If $a_{n+1} \geq 1$, then the induction hypothesis yields

$$(a_1 + \dots + a_n) + a_{n+1} \geq n + 1.$$

Otherwise, if $a_{n+1} < 1$ then apply (2) as

$$(a_1 + \dots + a_n) + a_{n+1} \geq \frac{n}{a_{n+1}} + a_{n+1} = \frac{n-1}{a_{n+1}} + \left(\frac{1}{a_{n+1}} + a_{n+1}\right) > (n-1) + 2.$$

This completes the solution.

3. Given an acute-angled triangle ABC , points D, E, F are on BC, CA, AB respectively such that AEF, BDF, CDE are acute-angled triangles whose orthocenters are P, Q, R respectively. If the lines AP, BQ, CR are concurrent, show that the lines DP, EQ, FR are also concurrent.

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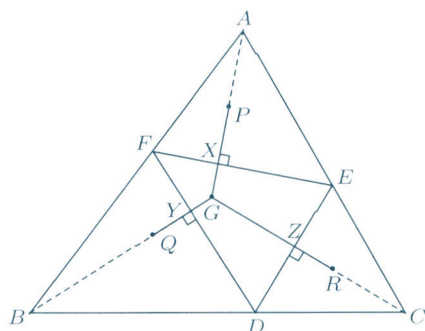


Figure: 3a

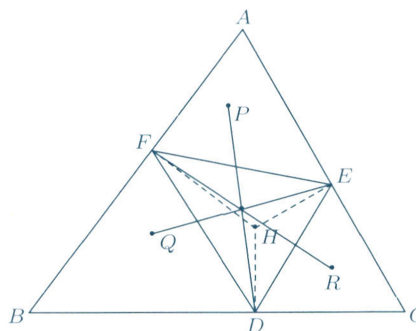


Figure: 3b

Solution. Let the lines AP, BQ, CR intersect at G . Suppose AG intersects EF at X , BG intersects DF at Y and CG intersects DE at Z . Referring to the figure 3a, it is easy to see that

$$EX^2 + FY^2 + DZ^2 = EG^2 + FG^2 + DG^2 - GX^2 - GY^2 - GZ^2 = FX^2 + DY^2 + EZ^2. \quad (1)$$

Hence,

$$\begin{aligned} AE^2 + BF^2 + CD^2 &= AX^2 + BY^2 + CZ^2 - EX^2 - FY^2 - DZ^2 \\ &= AX^2 + BY^2 + CZ^2 - FX^2 - DY^2 - EZ^2 && \text{by (1)} \\ &= AF^2 + BD^2 + CE^2. \end{aligned} \quad (2)$$

Choose H such that DH is perpendicular to BC at D , and EH perpendicular to AC at E . See figure 3b. We claim that FH is perpendicular AB at F . In fact, if we draw HF' perpendicular AB at F' , we must have $AE^2 + BF'^2 + CD^2 = AH^2 + BH^2 + CH^2 - EH^2 - FH^2 - DH^2 = AF'^2 + BD^2 + CE^2$. Now (2) implies that F and F' coincide.

Now it is easy to see that $DHER$ and $DHFQ$ are both parallelograms. Hence $ER = DH = FQ$ and $ER \parallel DH \parallel FQ$, which implies that $EFQR$ is also a parallelogram. It follows that EQ and FR bisect each other. Similarly, DP also passes through the midpoint of FR and hence the conclusion.

Remark. This essentially an application of Carnot's lemma: Let A_1, B_1 and C_1 be points in the interiors of the sides BC, CA and AB of a triangle ABC respectively. Then the perpendiculars at the points A_1, B_1, C_1 are concurrent if and only if $BA_1^2 - A_1C^2 + CB_1^2 - B_1A^2 + AC_1^2 - C_1B^2 = 0$.

4. Determine all positive integers M for which the sequence a_0, a_1, a_2, \dots , defined by $a_0 = \frac{2M+1}{2}$ and $a_{k+1} = a_k \lfloor a_k \rfloor$ for $k = 0, 1, 2, \dots$, contains at least one integer term.

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Solution. The answer is all integers $M \geq 2$. Let's define $b_k = 2a_k$ for all $k \geq 0$. Then

$$b_{k+1} = 2a_{k+1} = 2a_k \lfloor a_k \rfloor = b_k \lfloor \frac{b_k}{2} \rfloor.$$

Since b_0 is an integer, it follows that b_k is an integer for all $k \geq 0$.

Suppose the sequence a_0, a_1, a_2, \dots does not contain any integer term. Then b_k must be an odd integer for all $k \geq 0$, so that

$$b_{k+1} = b_k \lfloor \frac{b_k}{2} \rfloor = \frac{b_k(b_k - 1)}{2}. \quad (1)$$

Hence

$$b_{k+1} - 3 = \frac{b_k(b_k - 1)}{2} - 3 = \frac{(b_k - 3)(b_k + 2)}{2}, \quad (2)$$

for all $k \geq 0$.

Suppose that $b_0 - 3 > 0$. Then equation (2) yields $b_k - 3 > 0$ for all $k \geq 0$. For each $k \geq 0$, define c_k to be the highest power of 2 that divides $b_k - 3$. Since $b_k - 3$ is even for all $k \geq 0$, the number c_k is positive for every $k \geq 0$.

Note that $b_k + 2$ is an odd integer. Therefore, from equation (2), we have that $c_{k+1} = c_k - 1$. Thus the sequence c_0, c_1, c_2, \dots of positive integers is strictly decreasing, a contradiction. So, $b_0 - 3 \leq 0$, which implies that $M = 1$.

For $M = 1$, we can check that the sequence is constant with $a_k = \frac{3}{2}$ for all $k \geq 0$. Therefore, the answer is $M \geq 2$.

Day 1 (4.5 Hours)

5. Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C . A point D is chosen inside the triangle CBH so that CH bisects AD . Let P be the intersection point of the lines BD and CH . Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q . Prove that the lines CQ and AD meet at ω .

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Solution.

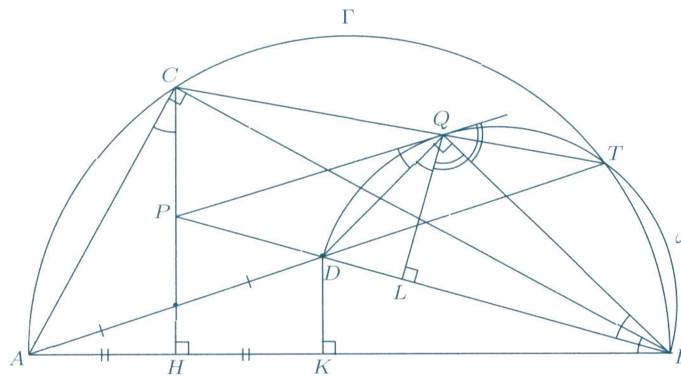


Figure: 5

Let K be the projection of D onto AB ; then $AH = HK$ as in figure 5. Since PH is parallel to DK , we have

$$\frac{PD}{PB} = \frac{HK}{HB} = \frac{AH}{HB}. \quad (1)$$

Let L be the projection of Q onto DB . Since PQ is tangent to ω and $\angle DQB = \angle BLQ = 90^\circ$, we have $\angle PQD = \angle QBP = \angle DGL$. Therefore, QD and QB are respectively the internal and the external bisectors of $\angle PQL$. By the angle bisector theorem, we obtain

$$\frac{PD}{DL} = \frac{PQ}{QL} = \frac{PB}{BL}. \quad (2)$$

The relations (1) and (2) yield $\frac{AH}{HB} = \frac{PD}{PB} = \frac{DL}{LB}$. So the spiral similarity τ centered at B and sending A to D maps H to L . Moreover, τ sends the semicircle Γ with diameter AB passing through C to ω . Due to $CH \perp AB$ and $QL \perp DB$, it follows that $\tau(C) = Q$.

Hence the triangles ABD and CBQ are similar, so $\angle ADB = \angle CQB$. This means that the lines AD and CQ meet at some point T , and this point satisfies $\angle BDT = \angle BQT$. Therefore, T lies on ω , as needed.

Remark. Since $\angle BAD = \angle BCQ$, the point T lies on the circumcircle Γ of the triangle ABC .

6. Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:
- (i) A player cannot choose a number that has been chosen by either player on any previous turn.
 - (ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.

- (iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

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Solution. The game ends in a draw when $n = 1, 2, 4, 6$; otherwise B wins.

For brevity, we denote by $[n]$ the set $\{1, 2, \dots, n\}$.

Firstly, we show that B wins whenever $n \neq 1, 2, 4, 6$. For this purpose, we provide a strategy which guarantees that B can always make a move after A 's move, and also guarantees that the game does not end in a draw. We begin with an important observation.

Lemma. Suppose that B 's first pick is n and that A has made the k^{th} move where $k \geq 2$. Then B can also make the k^{th} move.

Proof. Let \mathcal{S} be the set of the first k numbers chosen by A . Note that $n \notin \mathcal{S}$. Since \mathcal{S} does not contain consecutive integers, we see that the set $[n] \setminus \mathcal{S}$ consists of k "contiguous components" if $1 \in \mathcal{S}$, and $k + 1$ components otherwise. Since B has chosen only $k - 1$ numbers, there is at least one component of $[n] \setminus \mathcal{S}$ consisting of numbers not yet picked by B . Hence, B can choose a number from this component.

We will now describe a winning strategy for B , when $n \neq 1, 2, 4, 6$. By symmetry, we may assume that A 's first choice is a number not exceeding $\frac{n+1}{2}$. So B can pick the number n in B 's first turn. We now have two cases.

Case 1. n is odd and $n \geq 3$. The only way the game ends in a draw is that A eventually picks all the odd numbers from the set $[n]$. However, B has already chosen n , so this cannot happen. Thus B can continue to apply the lemma until A cannot make a move.

Case 2. n is even and $n \geq 8$. Since B has picked n , the game is a draw only if A can eventually choose all the odd numbers from the set $[n - 1]$. So B picks a number from the set $\{1, 3, 5, \dots, n - 3\}$ not already chosen by A , on B 's second move. This is possible since the set consists of $\frac{n-2}{2} \geq 3$ numbers and A has chosen only 2 numbers. Hereafter B can apply the lemma until A cannot make a move.

Hence in both cases, A loses.

We are left with the cases $n = 1, 2, 4, 6$. The game is trivially a draw when $n = 1, 2$. When $n = 4$, A has to first pick 1 to avoid losing. Similarly, B has to choose 4 as well. It then follows that the game ends in a draw.

When $n = 6$, B gets at least a draw by the lemma or by using a mirror strategy. On the other hand, A may also get at least a draw in the following way. In the first turn, A chooses 1. After B 's response by a number b , A finds a neighbor c of b which differs from 1 and 2, and reserves c for A 's third move. Now A can make the second move by choosing a number different from $1, 2, c - 1, c, c + 1$. Therefore A will not lose.

7. Suppose that a_0, a_1, \dots and b_0, b_1, \dots are two sequences of positive integers satisfying $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \quad b_{n+1} = \text{lcm}(a_n, b_n) - 1$$

for all $n \geq 0$. Prove that the sequence (a_n) is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

Solution. Let $s_n = a_n + b_n$. Notice that if $a_n \mid b_n$, then $a_{n+1} = a_n + 1$, $b_{n+1} = b_n - 1$ and $s_{n+1} = s_n$. So, a_n increases by 1 and s_n does not change until the first index is reached with $a_n \nmid s_n$. Define

$$W_n = \{m \in \mathbb{Z}_{>0} \mid m \geq a_n \text{ and } m \nmid s_n\} \text{ and } w_n = \min W_n.$$

Claim 1. The sequence (w_n) is non-increasing.

Proof. If $a_n \mid b_n$, then $a_{n+1} = a_n + 1$. Due to $a_n \mid s_n$, we have $a_n \notin W_n$. Moreover, $s_{n+1} = s_n$; therefore, $W_{n+1} = W_n$ and $w_{n+1} = w_n$.

Otherwise, if $a_n \nmid b_n$, then $a_n \nmid s_n$, so $a_n \in W_n$ and thus $w_n = a_n$. We show that $a_n \in W_{n+1}$; this implies that $w_{n+1} \leq a_n = w_n$. By the definition of W_{n+1} , we need that $a_n \geq a_{n+1}$ and $a_n \nmid s_{n+1}$. The first relation holds because of $\gcd(a_n, b_n) < a_n$. For the second relation, observe that in $s_{n+1} = \gcd(a_n, b_n) + \text{lcm}(a_n, b_n)$, the second term is divisible by a_n , but the first term is not. So $a_n \nmid s_{n+1}$; that completes the proof of the claim.

Let $w = \min_n w_n$ and let N be an index with $w = w_N$. Due to Claim 1, we have $w_n = w$ for all $n \geq N$.

Let $g_n = \gcd(w, s_n)$. As we have seen, starting from an arbitrary index $n \geq N$, the sequence a_n, a_{n+1}, \dots increases by 1 until it reaches w , which is the first value not dividing s_n ; then it drops to $\gcd(w, s_n) + 1 = g_n + 1$.

Claim 2. The sequence (g_n) is constant for $n \geq N$.

Proof. If $a_n \mid b_n$, then $s_{n+1} = s_n$ and hence $g_{n+1} = g_n$. Otherwise we have $a_n = w_n = w$,

$$\begin{aligned} \gcd(a_n, b_n) &= \gcd(a_n, s_n) = \gcd(w, s_n) = g_n, \\ s_{n+1} &= \gcd(a_n, b_n) + \text{lcm}(a_n, b_n) = g_n + \frac{a_n b_n}{g_n} = g_n + \frac{w(s_n - w)}{g_n}, \\ \text{and } g_{n+1} &= \gcd(w, s_{n+1}) = \gcd\left(w, g_n + \frac{s_n - w}{g_n} w\right) = \gcd(w, g_n) = g_n. \end{aligned}$$

Thus the sequence (g_n) is constant for $n \geq N$.

Let $g = g_N$. We have proved that the sequence (a_n) eventually repeats the following cycle:

$$g + 1 \mapsto g + 2 \mapsto \dots \mapsto w \mapsto g + 1.$$

Day 2 (4.5 Hours)

8. Let a and b be positive integers such that $a!b!$ is a multiple of $a! + b!$. Prove that $3a \geq 2b + 2$.

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Solution. If $a > b$, we immediately get $3a \geq 2b + 2$. In the case $a = b$, the required inequality is equivalent to $a \geq 2$, which can be checked easily since $(a, b) = (1, 1)$ does not satisfy $a! + b! \mid a!b!$. We now assume $a < b$ and denote $c = b - a$. The required inequality becomes $a \geq 2c + 2$.

Suppose, to the contrary that, that $a \leq 2c + 1$. Define $M = \frac{b!}{a!} = (a + 1)(a + 2) \cdots (a + c)$. Since $a! + b! \mid a!b!$ implies that $1 + M \mid a!M$, we obtain $1 + M \mid a!$. Note that we must have $c < a$; otherwise $1 + M > a!$, which is impossible. We observe that $c! \mid M$ since M is a product of consecutive integers. Thus $\gcd(1 + M, c!) = 1$, which implies

$$1 + M \mid \frac{a!}{c!} = (c + 1)(c + 2) \cdots a. \quad (1)$$

If $a \leq 2c$, then $\frac{a!}{c!}$ is a product of $a - c \leq c$ integers not exceeding a whereas M is a product of c integers exceeding a . Therefore, $1 + M > \frac{a!}{c!}$ which is a contradiction.

It remains to exclude the case $a = 2c + 1$. Since $a + 1 = 2(c + 1)$, we have $c + 1 \mid M$. Hence, we can deduce from (1) that $1 + M \mid (c + 2)(c + 3) \cdots a$. Now $(c + 2)(c + 3) \cdots a$ is a product of $a - c - 1 = c$ integers not exceeding a ; thus it is smaller than $1 + M$. Again, we arrive at a contradiction.

9. Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leq r < s \leq 2n} (s - r - n)x_r x_s,$$

where $-1 \leq x_i \leq 1$ for all $i = 1, 2, \dots, 2n$.

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Solution. Let Z be the expression to be maximized. Since this expression is linear in every variable x_i , and $-1 \leq x_i \leq 1$, the maximum of Z will be achieved when $x_i = -1$ or 1 . Therefore, it suffices to consider only the case when $x_i \in \{-1, 1\}$ for all $i = 1, 2, \dots, 2n$.

For $i = 1, 2, \dots, 2n$, we introduce auxiliary variables

$$y_i = \sum_{r=1}^i x_r - \sum_{r=i+1}^{2n} x_r.$$

Taking squares of both sides, we have

$$\begin{aligned} y_i^2 &= \sum_{r=1}^{2n} x_r^2 + \sum_{r < s \leq i} 2x_r x_s + \sum_{i < r < s} 2x_r x_s - \sum_{r \leq i < s} 2x_r x_s \\ &= 2n + \sum_{r < s \leq i} 2x_r x_s + \sum_{i < r < s} 2x_r x_s - \sum_{r \leq i < s} 2x_r x_s, \end{aligned} \quad (1)$$

where the last equality follows from the fact that $x_i \in \{-1, 1\}$. Notice that for every $r < s$, the coefficient of $x_r x_s$ in (1) is 2 for each $i = 1, \dots, r - 1, s, \dots, 2n$, and this coefficient

is -2 for each $i = r, \dots, s-1$. This implies that the coefficient of $x_r x_s$ in $\sum_{i=1}^{2n} y_i^2$ is $2(2n - s + r) - 2(s - r) = 4(n - s + r)$. Therefore, summing (1) for $i = 1, 2, \dots, 2n$ yields

$$\sum_{i=1}^{2n} y_i^2 = 4n^2 + \sum_{1 \leq r < s \leq 2n} 4(n - s + r)x_r x_s = 4n^2 - 4Z. \quad (2)$$

Hence it suffices to find the minimum of $\sum_{i=1}^{2n} y_i^2$.

Since $x_r \in \{-1, 1\}$, we see that y_i is an even integer. In addition, $y_i - y_{i-1} = 2x_i = \pm 2$, and so y_{i-1} and y_i are consecutive even integers for every $i = 2, 3, \dots, 2n$. It follows that $y_{i-1}^2 + y_i^2 \geq 4$, which implies that

$$\sum_{i=1}^{2n} y_i^2 = \sum_{j=1}^n (y_{2j-1}^2 + y_{2j}^2) \geq 4n. \quad (3)$$

Combining (2) and (3), we get

$$4n \leq \sum_{i=1}^{2n} y_i^2 = 4n^2 - 4Z. \quad (4)$$

Hence $Z \leq n(n-1)$.

If we set $x_i = 1$ for odd indices i and $x_i = -1$ for even indices i , then we obtain equality in (3) (and thus in (4)). Therefore, the maximum possible value of Z is $n(n-1)$, as desired.

10. Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S . Prove that there exist infinitely many positive integers that are *not* clean.

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Solution. Define an *odd* (respectively, *even*) *representation* of n to be a representation of n as a sum of an odd (respectively, even) number of distinct elements of S . Let $\mathbb{Z}_{>0}$ denote the set of all positive integers.

Suppose, to the contrary, that there exists only finitely many positive integers that are *not* clean. Therefore, there exists a positive integer N such that every integer $n > N$ has exactly one odd representation. Clearly, S is infinite. We now claim the following properties of odd and even representations.

Property 1. Any positive integer n has at most one odd and at most one even representations.

Proof. We first show that *every* integer n has at most one even representation. Since S is infinite, there exists $x \in S$ such that $x > \max\{n, N\}$. Then the number $n + x$ must be clean, and x does not appear in any even representation of n as $x > n$. If n has more than one even representations, then we obtain two distinct odd representations of $n + x$ by adding x to the even representations of n , which is impossible. Therefore, n can have at most one even representation. Similarly, there exist two distinct elements $y, z \in S$ such that $y, z > \max\{n, N\}$. If n has more than one odd representations, then we obtain two distinct odd representations of $n + y + z$ by adding y and z to the odd representations of n , which is again a contradiction.

Property 2. Fix $s \in S$. Suppose that a number $n > N$ has no even representation. Then $n + 2as$ has an even representation containing s for all integers $a \geq 1$.

Proof. It suffices to prove the following statement: If n has no even representation without s , then $n + 2s$ has an even representation containing s (and hence no even representation without s by Property 1).

Note that the odd representation of $n + s$ does not contain s ; otherwise, we have an even representation of n without s . Then, adding s to this odd representation of $n + s$, we get that $n + 2s$ has an even representation containing s , as desired.

Property 3. Every sufficiently large integer has an even representation.

Proof. Fix any $s \in S$, and let r be an arbitrary element in $\{1, 2, \dots, 2s\}$. Then, Property 2 implies that the set $\mathbb{Z}_r = \{r + 2as \mid a \geq 0\}$ contains at most one number exceeding N with no even representation. (If both $r + 2a_1s$ and $r + 2a_2s > N$ with $a_2 > a_1 \geq 0$ have no even representation, then by Property 2, $r + 2a_2s = (r + 2a_1s) + 2(a_2 - a_1)s$ has an even representation containing s , a contradiction). Therefore, \mathbb{Z}_r contains finitely many positive integers with no even representation, and so does $\mathbb{Z}_{>0} = \bigcup_{r=1}^{2s} \mathbb{Z}_r$.

In view of Properties 1 and 3, we may assume that N is chosen such that every $n > N$ has *exactly* one odd and *exactly* one even representation. In particular, each element $s > N$ of S has an even representation.

Property 4. For any $s, t \in S$ with $N < s < t$, the even representation of t contains s .

Proof. Suppose the contrary. Then $s + t$ has at least two odd representations: one obtained by adding s to the even representation of t and one obtained by adding t to the even representation of s . Since the latter does not contain s , these two odd representations of $s + t$ are distinct, a contradiction.

Let $s_1 < s_2 < \dots$ be all the elements of S , and set $\sigma_n = \sum_{i=1}^n s_i$ for each positive integer n . Fix an integer k such that $s_k > N$. Then, Property 4 implies that for every $i > k$, the even representation of s_i contain all the numbers $s_k, s_{k+1}, \dots, s_{i-1}$. Therefore,

$$s_i = s_k + s_{k+1} + \dots + s_{i-1} + R_i = \sigma_{i-1} - \sigma_{k-1} + R_i, \quad (1)$$

where R_i is a sum of some s_1, \dots, s_{k-1} . In particular, $0 \leq R_i \leq s_1 + \dots + s_{k-1} = \sigma_{k-1}$.

Let j_0 be an integer satisfying $j_0 > k$ and $\sigma_{j_0} > 2\sigma_{k-1}$. Then (1) shows that, for every $j > j_0$,

$$s_{j+1} \geq \sigma_j - \sigma_{k-1} > \sigma_j/2. \quad (2)$$

Next, let $p > j_0$ be an index such that $R_p = \min_{i > j_0} R_i$. Then using (1), we have

$$s_{p+1} = s_k + s_{k+1} + \dots + s_{p-1} + s_p + R_{p+1} = (s_p - R_p) + s_p + R_{p+1} \geq 2s_p.$$

Therefore, there is no element of S larger than s_p but smaller than $2s_p$. It follows that the even representation τ of $2s_p$ does not contain any element larger than s_p . On the other hand, inequality (2) yields $2s_p > s_1 + \dots + s_{p-1}$, so τ must contain a term larger than s_{p-1} . Thus, it must contain s_p . After removing s_p from τ , we have that s_p has an odd representation not containing s_p , which contradicts Property 1 since s_p itself also forms an odd representation of s_p .