Classic Geometric Puzzle

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ABSTRACT

We present the solutions of a classic geometric puzzle that was first encountered by the first author in the 1980s, but has since sunk into obscurity. In the puzzle, the top and front views of a three-dimensional wooden structure are given: both views are identical, consisting of a small square in the exact center of a larger square. The problem is to draw the structure or its side view. The original solution consisted of only one structure with flat surfaces. But in recent years, more solutions have been discovered: two more structures with flat surfaces, and many more structures that include curved surfaces as well. We also include three hypothetical complicated solutions for the readers to discuss whether the structures are possible to construct.

1. BACKGROUND.

Figure 1 shows a paraphrase of an interesting geometric puzzle encountered by the first author during a competition[1] in Singapore in the 1980s. An online check in early 2015 revealed absolutely nothing on this puzzle, so the first author decided to resurface it. First, he set a version of this puzzle, with the restriction that the structure must have flat surfaces only, in the Singapore International Math Olympiad Challenge (SIMOC) held on 15 August 2015. Then he contributed the more general puzzle without this restriction to the Guardian's online puzzle column run by the third author. When the puzzle appeared in the puzzle column on 28 September 2015 (Guardian, 2015), it generated a lot of interest. The purpose of this article is to discuss some of these fascinating solutions, most of which were contributed by the second author, and some by readers of the Guardian's puzzle column.

The following shows the top view and the front view of a three-dimensional wooden structure. The structure does not have any painted lines on it. Notice also that there are no dotted lines in both the top and front views (dotted lines are used to show hidden lines, i.e. lines hidden from view). Draw as many structures, or their left (or right) side view, as possible.

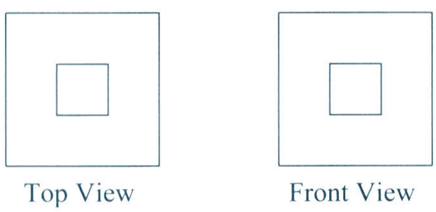

Figure 1. Classic Geometric Puzzle

[1] The first author could not recall which competition the puzzle appeared in. Being a participant of that competition, he did not get to keep the question paper, so he did not have the exact phrasing of the puzzle. He only remembered that the question did not mention anything about dotted lines or hidden lines, and that only one solution was given at the end of the competition.

2. STRUCTURES WITH FLAT SURFACES ONLY.

During the competition in the 1980s when this puzzle first appeared, it did not mention anything about dotted lines or hidden lines. As a result, many of the participants thought of a solid box with a square hole going through the entire structure from the top surface to the bottom surface, and another square hole from the front surface to the back surface. However, these holes will appear as dotted lines in both the top and front views. (In technical drawings, dotted lines are drawn in the top, front or side view to indicate edges or certain types of curved surfaces[2] that are hidden from a person's view of the object.) Even if they are not holes but square indentations on the top and front surfaces, they will also appear as dotted lines in both the top and front views. In order to save space by drawing only one solid, both these features are combined in the top diagram in Figure 2, where a square hole goes through the solid from the top surface to the bottom surface, and there is a square indentation on the front surface. The bottom diagram in Figure 2 shows another incorrect solution because the physical structure must have a certain thickness in order for it to stand, thus resulting in dotted lines in both the top and front views.

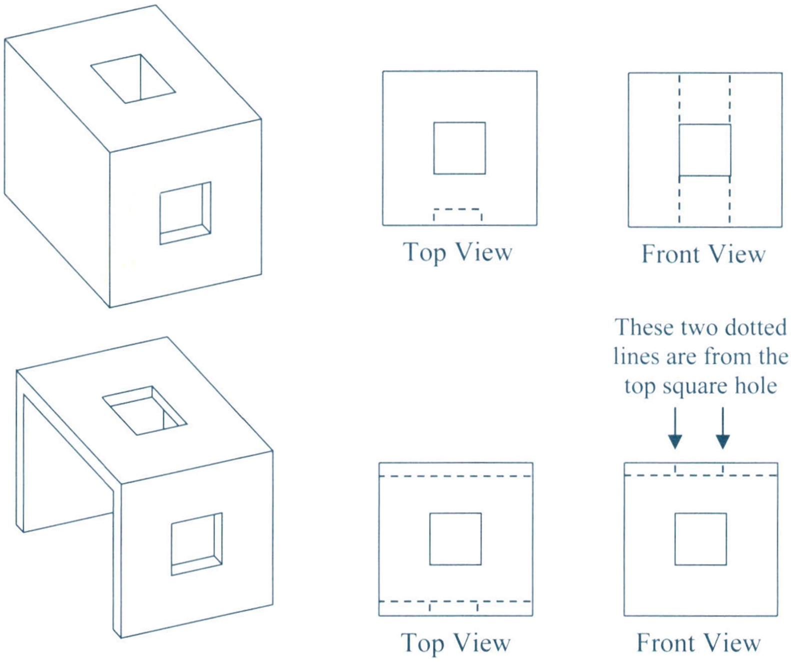

Figure 2. Incorrect Solutions with Flat Surfaces Only

The original puzzle in the 1980s only asked the participants to draw the 3D structure, so it seems that the setter was only expecting one solution. Figure 3 shows the original solution, together with a side view of the structure. The architecture community was excited about this solution when the puzzle appeared in the Guardian's puzzle column on 28 September 2015 because this was what they often encountered when they designed flat-top dormer windows that protruded from the sloping surface of a roof. (Not all dormer windows have a flat top: some may have two slanting surfaces on top.)

[2] See Figure 14 later for this kind of curved surfaces.

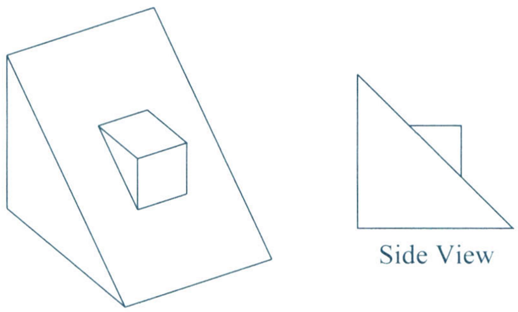

Figure 3. Original Solution

The second author was a reviewer of the questions for SIMOC 2015 mentioned in Section 1, and he came up with a second solution as shown in Figure 4.

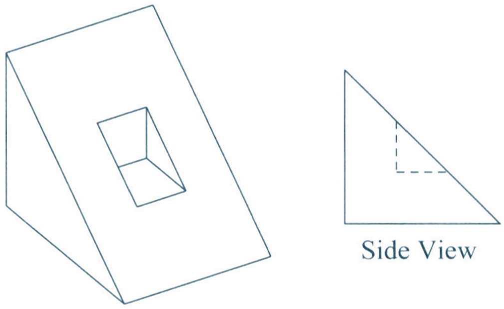

Figure 4. Second Solution with Flat Surfaces Only

This seems to be the end of the story for structures with flat surfaces only, until Anoop Puthur, an Indian from Australia, provided the third solution in the Guardian's puzzle column (see Figure 5). When asked how he came up with this intriguing structure, he said that he was a 3D designer and tool maker by profession, but this required what he called 'a reverse engineered solution'. It was obvious to him that the outer square must be 'an angular plane', so he just worked on the inner square in the top view by 'pulling each individual line up' such that the front view would still consist of an inner square. It took him less than five minutes to come up with this structure, which was his first solution. He was bewildered as to why he would think of such a complex structure instead of the other two simpler solutions.

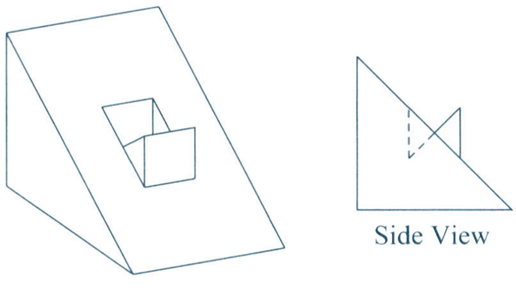

Figure 5. Intriguing Third Solution with Flat Surfaces Only

We do not know whether there are other solutions with flat surfaces only and so we invite the readers to try. You can contact the authors if you have found another one, or you can write directly to the journal to see if they wish to publish your solution as a reaction to this article. Please note that the structures given in Figure 6 are incorrect because the top view and/or front view are different from those given in the puzzle. In fact, the left diagram in Figure 6 is a common mistake made by participants in SIMOC 2015.

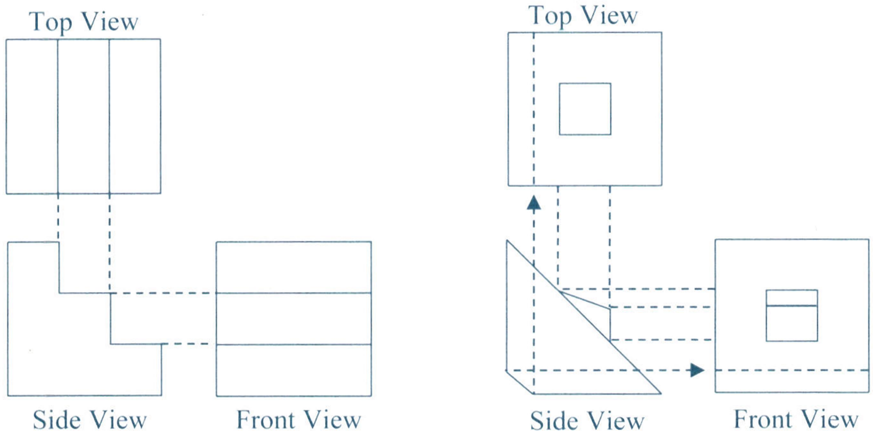

Figure 6. More Incorrect Solutions with Flat Surfaces Only

If the structure is not made of wood, which the original puzzle actually did not specify, then there are other solutions: a transparent glass cube with a smaller opaque cube or triangular prism in the exact center of the glass cube (see Figure 7). But because the left diagram in Figure 7 is an easy solution, we added in the condition that the structure must be made of wood.

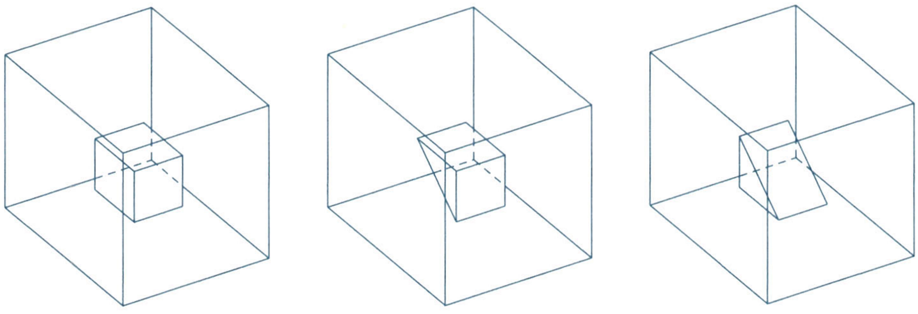

Figure 7. Some solutions if the structure is not made of wood

3. STRUCTURES WITH FLAT AND CURVED SURFACES.

The second author noticed that the side view of a cylinder is a rectangle, or a square if its height is equal to the diameter of its circular cross section. So if the structure can have curved surfaces, then Figure 8 shows two possibilities, where the curved surface in the side view is part of a cylinder, with one concaving outwards and the other inwards.

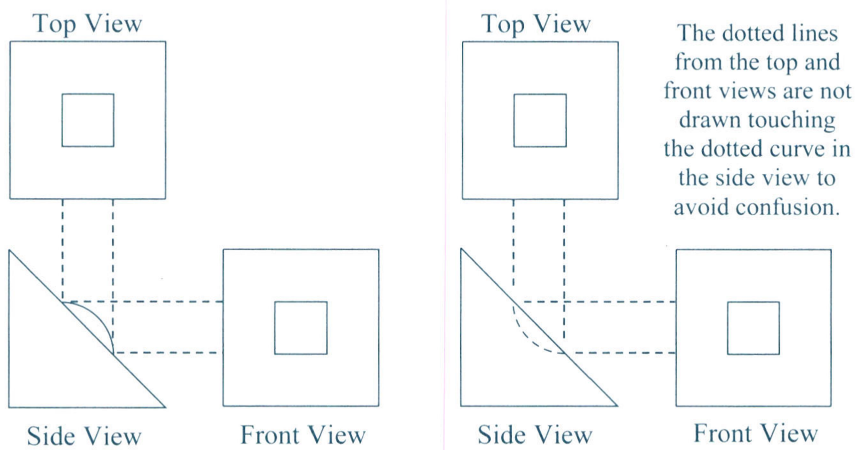

Figure 8. Two solutions if the inner square is part of a cylinder

The next question is whether there is another way to position the part-cylinder in the left diagram in Figure 8 such that the top and front views remain the same. Figure 9 shows that this will result in a hidden line and a non-square rectangle in both the top and front views. But can we shift the part-cylinder to make the non-square rectangle into a square? In order to do this for the top view, we need to shift the circular arc to the left, but this will cause the rectangle in the front view to shift even further up, resulting in an even longer rectangle in the front view. So how do we shift the circular arc to the left and downwards at the same time?

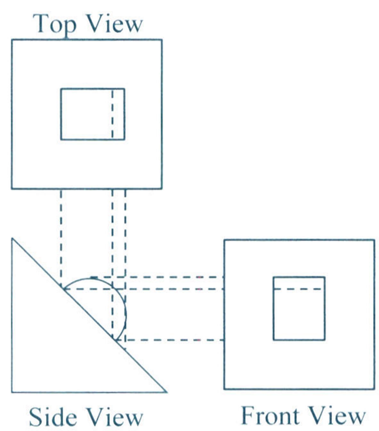

Figure 9. Incorrect solution if the inner square is part of a cylinder

It seems that there are only two ways to do it. The first structure is shown in the left diagram in Figure 8. The top left diagram in Figure 10 shows another possible structure. The first objection to the latter is whether it is possible to glue the cylinder to the curved surface of the outer structure. But if we try to overlap the cylinder and the bigger structure so that the partcylinder will stick to it (see the top right diagram in Figure 10), there will be two dotted lines in each of the top and front views, which are from the joints between the part-cylinder and the bigger structure. The second objection to the top left diagram in Figure 10 is whether the line of contact between the cylinder and the curved surface of the outer structure will appear as a dotted line in both the top and front views. In technical drawings, a dotted line is used to indicate a hidden line, but there is no proper definition of a hidden line: it all depends on whether or not one views the line of contact behind the cylinder and the curved surface of the outer structure as a hidden line. Technical drawings do have other types of lines (Coban Engineering, 2016), e.g. 'alternating dashed line and dot' is used to indicate an axis of symmetry, but there does not seem to have any other types of lines to indicate the line of contact behind the cylinder and the curved surface of the outer structure. Since this is arguable, we will leave this structure as a hypothetical solution for the readers to think about.

Boon Leong Ng from Singapore suggested a way to deal with both the above objections is for the outer structure to wrap around the curved surface of the cylinder before separating (see the bottom left diagram in Figure 10). The main difference between the incorrect solution in the top right diagram and this structure in the bottom left diagram is that the former contains only part of a cylinder while the latter contains the full cylinder. Nevertheless, the question remains: will there be dotted lines in the top and front views resulting from the parts where the outer structure separates from the cylinder? This is debatable, unlike the incorrect solution in the top right diagram.

To resolve the issue of whether there are dotted lines in the bottom left diagram, one may be tempted to make the 'supposedly dotted lines' coincide with the inner square by wrapping the outer structure around the curved surface of the cylinder a lot more (see the bottom right diagram in Figure 10). In this case, there will definitely be solid lines AB and CD where the outer structure separates from the cylinder because the surface of the outer structure is either horizontal or vertical at the plane of separation, but the solid lines will coincide with a side of the inner square. Unfortunately, the outer structure will also create a solid line in both the front and top views. So this structure is not feasible either.

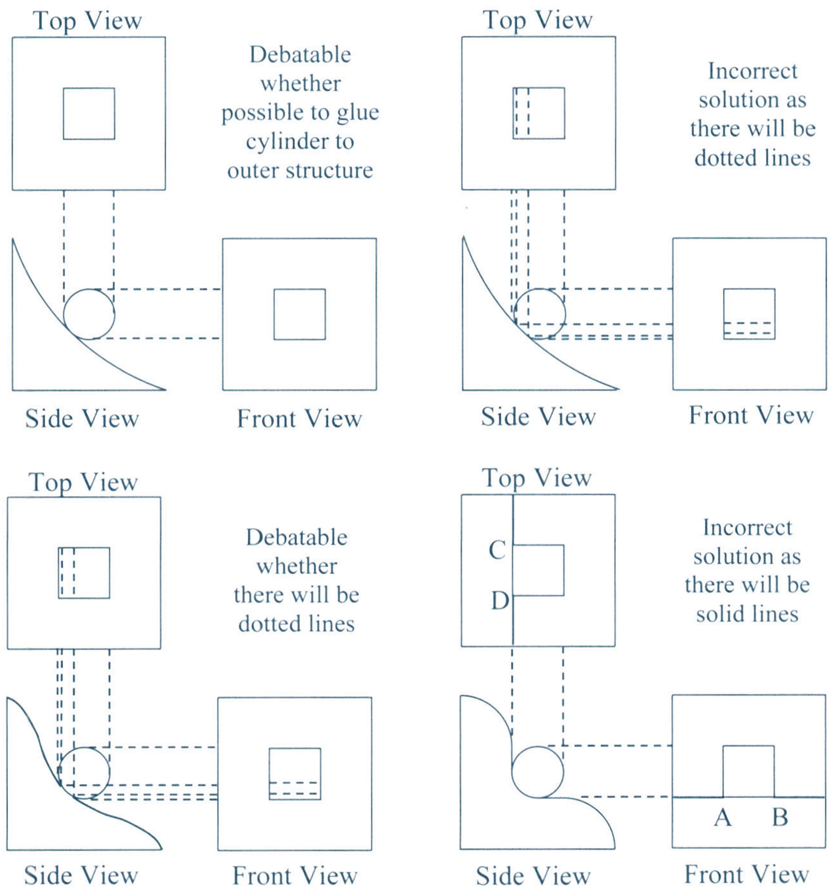

Figure 10. Some hypothetical and incorrect solutions if the inner square is a cylinder or part of a cylinder

What we have done so far is to investigate the case where the inner square is a cylinder or part of a cylinder. It seems that there are *essentially* only two solutions for the inner square as shown in Figure 8. The other structures (top left and bottom left diagrams in Figure 10) are hypothetical. Of course, other variations have to do with how the outer structure can curve. For example, Phineas Harper, Deputy Director of the Architecture Foundation from UK, contributed two rather aesthetically-pleasing solutions as shown in the top diagrams in Figure 11. Notice that the inner structures are essentially the same as those in Figure 8.

The first problem is that both structures in the top diagrams in Figure 11 cannot stand on their own. If they fall backwards, the top and front views will be affected (both the inner and outer squares will become non-square rectangles in the top view, and the front view will be totally different). To make the structures stand, the bottom part of the outer structures has to curve more. But the curve in the side view of the first structure is already a quadrant (or a quarter-circle), so the curvature cannot be increased without affecting the outer square in both the top and front views. One way to make both structures stand is shown in the bottom diagrams in Figure 11. Unfortunately, they may no longer be aesthetically pleasing.

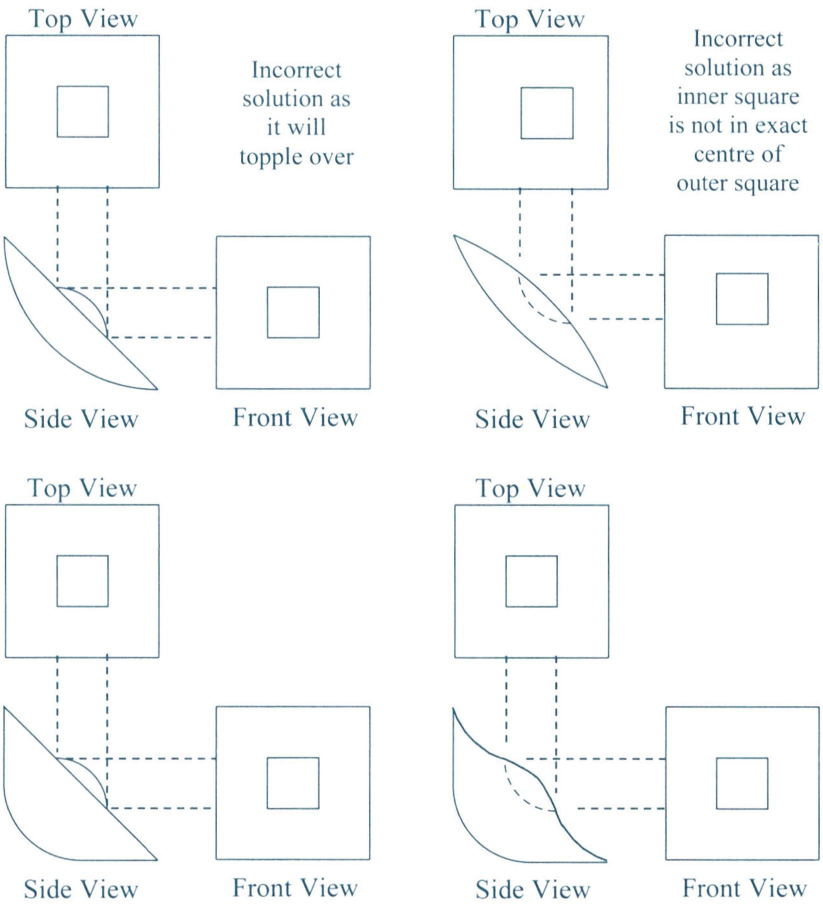

Figure 11. Some incorrect and correct solutions if the inner square is part of a cylinder

The second problem lies with the structure in the top right diagram in Figure 11: the inner square is not in the exact center of the outer square in both the top and front views. If the inner square in the top view is shifted to the left so that it will be in the exact center of the outer square, the dotted curve in the side view will shift up, resulting in a inner square in the front view shifting even further up. The readers can try to make the inner square in both the top and front views bigger, or change the curvature of the outer curves in the side view, but the same problem remains: we cannot make the inner square to be at the exact center of the outer square in both views at the same time. One way to make this work is shown in the bottom right diagram in Figure 11, where the top part of the outer structure has to be a more complex curved surface (notice that the shape and position of the inner structure are still the same as those of the inner structure in the right diagram in Figure 8). Alternatively, the top part of the outer structure can be a plane, just like the bottom left diagram in Figure 11. But no matter what we do to the outer structure, the shape and position for the inner structure – part of a cylinder – are essentially the same as those in the two diagrams in Figure 8.

If we do not require the structure to be made of wood, which the original puzzle actually did not specify, Figure 12 shows another possibility: a levitating cylindrical magnet inside an empty box with transparent surfaces. But the surfaces of the box must still have some thickness for it to stand, just like the bottom structure in Figure 2. The question is whether the thickness will show up as dotted lines in both the top and front views as the surfaces are transparent. Since this is arguable, we will leave this as a hypothetical solution for the readers to think about. Another simpler solution is just an opaque cylinder in the exact center of a glass cube (similar to the structures in Figure 7).

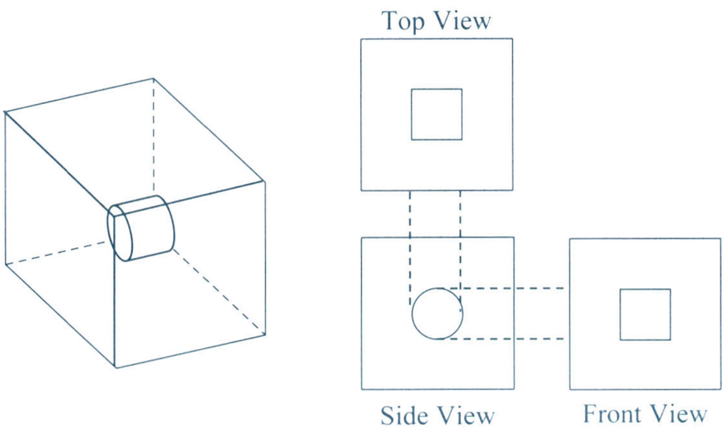

Figure 12. Hypothetical solution if the structure is not made of wood and the inner square is a cylinder

The general case.

If the inner square does not have to be a cylinder or part of a cylinder, there are far more solutions. Figure 13 shows some solutions contributed by the second author before the puzzle was submitted to the Guardian's puzzle column. In order to save space, we have combined some of the different features in the same structure. We can also swop, e.g. the inner structure of the bottom left diagram with the inner structure of the bottom right diagram. Moreover, there is some freedom in how the curved surface in each diagram can curve, meaning that there are infinitely many solutions although they may belong essentially to the same type.

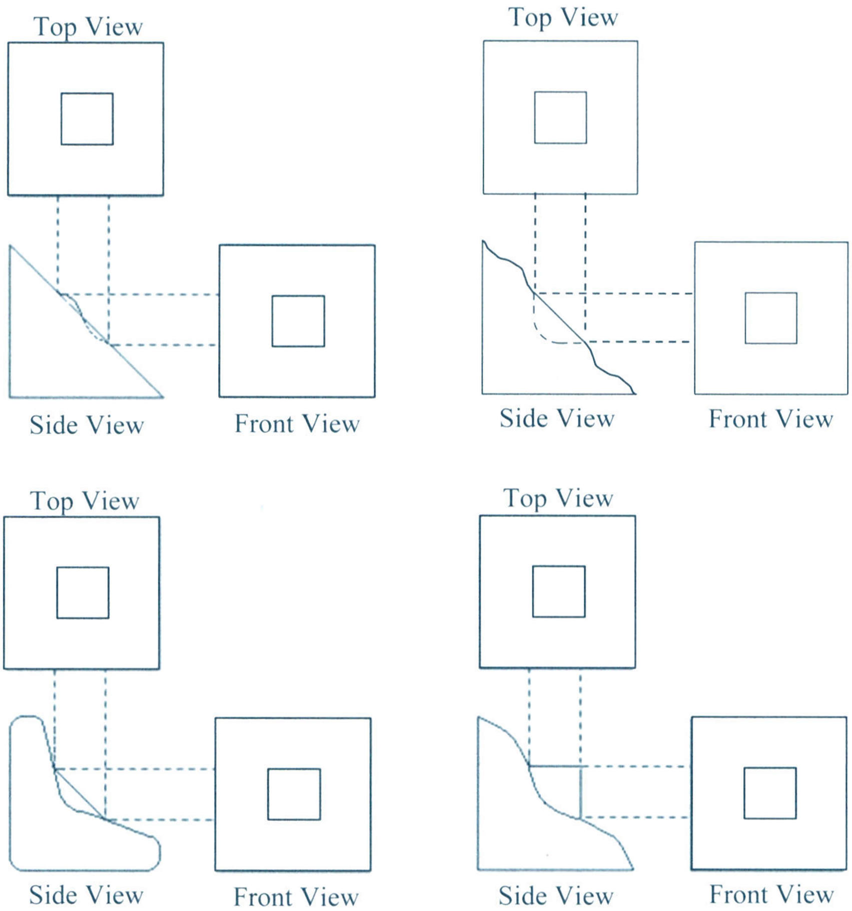

Figure 13. Some Solutions with Curved Surfaces

Conditions on the curved surfaces.

However, it does not mean that any curved surfaces will suffice. There are some important criteria as to how the surfaces should curve so that they will not show up as additional solid or dotted lines in the given top view and/or front view.

- If the curved surface is not smooth enough, it will result in a sharp line, creating an extra solid or dotted line in the top or front view. In particular, this happens when a curved surface is not tangential to a plane surface at the line of contact (e.g. see the top left diagram in Figure 14 and compare with the bottom right diagram in Figure 13).
- If the curved surface is vertical or horizontal, it may create an extra line in the top or front view respectively (e.g. see the top right diagram in Figure 14 and compare with the bottom right diagram in Figure 13 again; see also the bottom left diagram in Figure 14). On the other hand, the curved surface in the top left diagram in Figure 13 is horizontal at either end, but it will appear as a side of the inner square in the front view; however, the curved surface must not be vertical in the middle, or else it will create an extra solid line in the middle of the inner square in the top view.
- If the curved surface concaves upwards or downwards (compare the turning point of a 2D curve), it will create an extra dotted line or solid line in the top or front view, even if the curved surface is smooth (e.g. see the bottom right diagram in Figure 14).

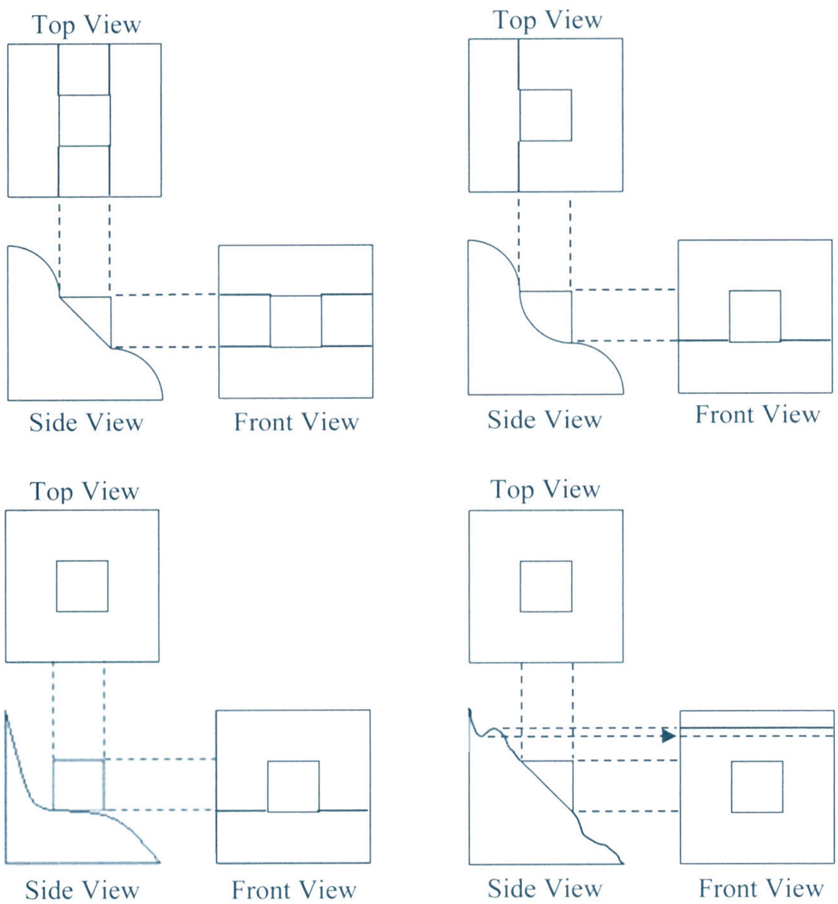

Figure 14. Some Incorrect Solutions with Curved Surfaces

Complicated solutions.
The second author even designed two complicated structures as shown in Figure 15. In the top diagram, starting from the right triangular side, the top surface is a plane, until it passes underneath the triangular prism in the center; then the top surface starts to curve up smoothly until the circular arc on the left.

The bottom diagram in Figure 15 is even more complicated to visualise. If the surface does not curve properly, e.g. if the surface curves from corner A to corner C, then this curve will appear as a curve in the front view. Therefore, it is very important to curve the surface in the following manner. From the line AB, the surface moves inwards and upwards, i.e. it will be more of a plane nearer to A than to B. From the line BC, the surface curves smoothly inwards, i.e. it has to be a curved surface nearer to B but curves inwards. From the line CD, the surface curves smoothly downwards. There must not be any sharp line on the curved surface; otherwise, it will appear in the top or front view. The issue with this structure is whether it is possible to curve in such a manner, and in particular, whether there will be any sharp line at the corners B, C and D, as indicated in the same figure.

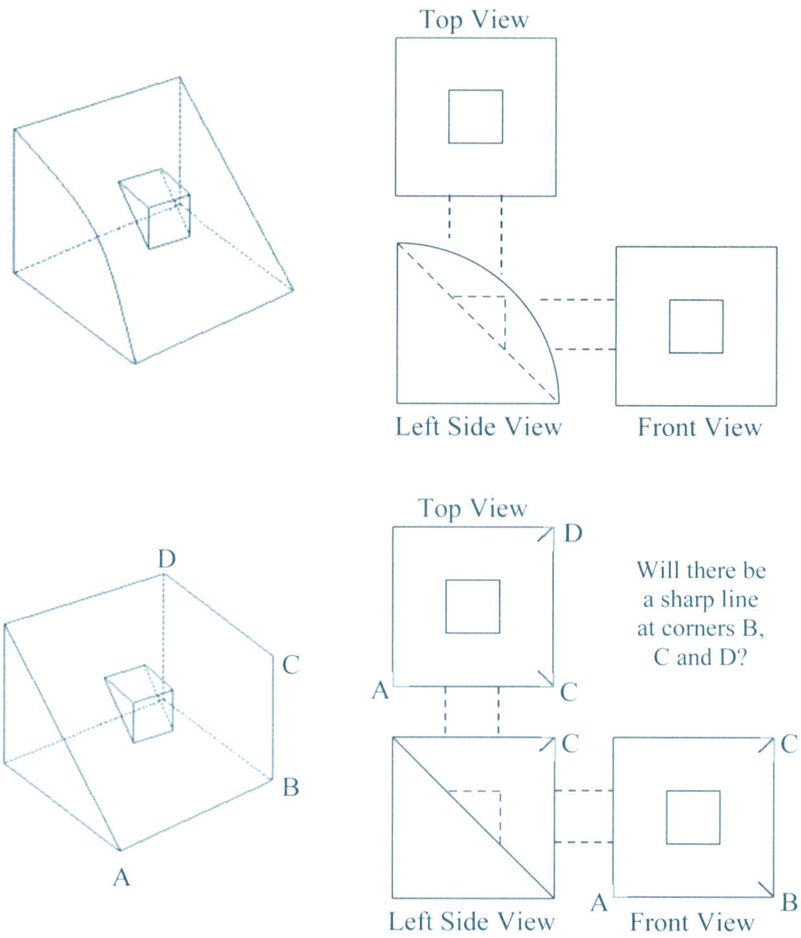

Figure 15. Hypothetical Complicated Solutions with Curved Surfaces

Boon Leong Ng from Singapore then raised another complicated possibility. If the bottom diagram in Figure 15 can be constructed, then it may also be possible to construct the structure as shown in Figure 16, where the side view is identical to the top and front views. From the lines FA and AB, the surface must curve inwards and upwards until it reaches the lower edges of the inner 'cube'. From the lines ED and DC, the surface must curve inwards and downwards until it reaches the upper edges of the inner 'cube'. Similarly for the lines EF and BC. The same issue is whether there will be any sharp line at each of the six corners from A to F. In addition, will there be any sharp line at each of the six corners, from U to Z, for the inner 'cube'?

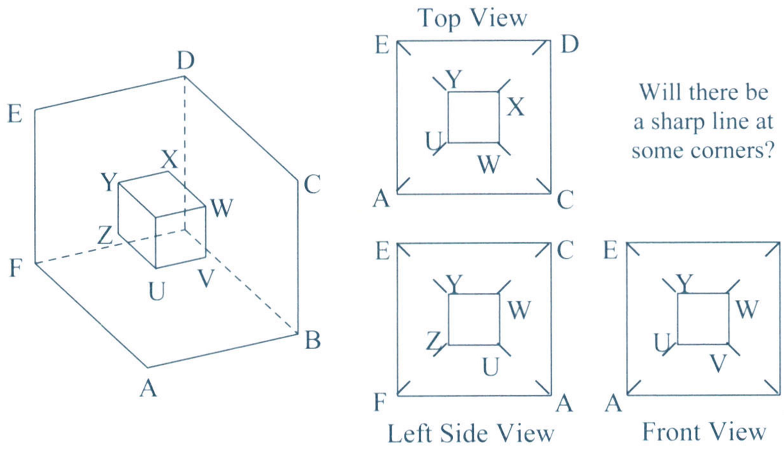

Figure 16. Another Hypothetical Complicated Solution with Curved Surfaces

Another way to look at the above issue is to consider the incorrect structure with flat surfaces as shown in Figure 17. How do we get rid of, say, the line FZ between the planes EFZY and AFZU? It is possible to get rid of the middle part of the line FZ by moving both planes near the line FZ a bit upwards so that it forms a smooth curved surface. The tricky part is what will happen near the endpoints, F and Z, of the line FZ when we do that? Will we also get a smooth curved surface, or will the line persist at F and at Z?

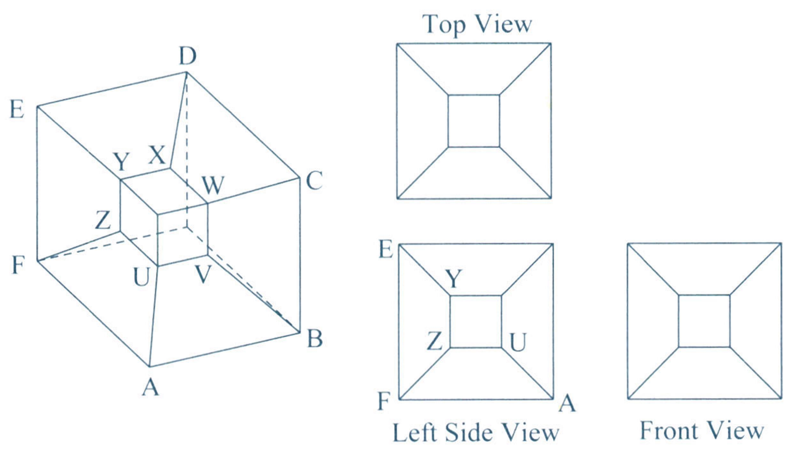

Figure 17. Incorrect Solution with Flat Surfaces

Figure 18 shows a concave Rubik's cube and a wooden bowl. Ignoring the gridlines, each face of the concave Rubik's cube is actually a smooth curved surface. However, all the edges of the cube curve inwards in order to produce a smooth concave surface. But what happens if the edges remain straight? Is it possible for the surface to curve smoothly inwards or will there be a sharp line near each corner of the surface? As for the wooden bowl, the corners are rounded in order to produce a smooth concave surface. What happens if the corners are sharp? Is it still possible for the surface to curve smoothly near the corners? Another way to look at this is to imagine securing a rubber canvas to a square frame and then depressing the canvas inwards to see if the canvas will still be smooth at the four corners. Resolving this issue will solve the problem of whether it is possible to construct the bottom structure in Figure 15 and the structure in Figure 16.

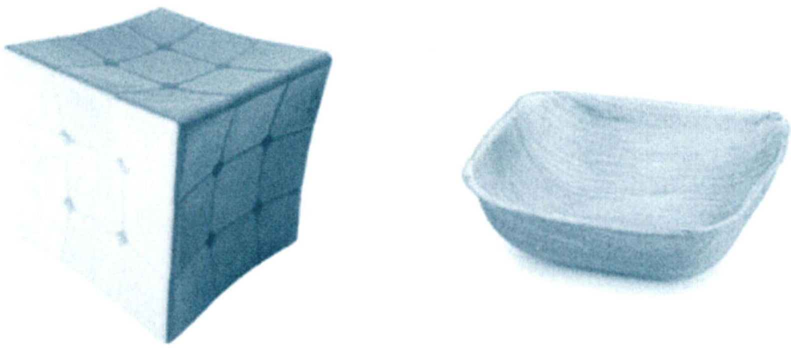

Figure 18. Concave Rubik's Cube and Wooden Bowl

(Please note that these pictures are taken from Internet and we do not know whether there are any copyright issues; alternatively, we can draw freehand if this article is accepted for publication.)

Another reader of the Guardian puzzle column has independently suggested another complicated structure. This is the ultimate solution because all the six views of the structure are identical: a small square in the exact center of a larger square. Imagine the concave Rubik's cube in Figure 18 but with straight edges. Then concave each face of the cube further inwards until we obtain the smaller cube as shown in Figure 19. For example, ABCG will concave inwards until UVWT, and AFEG will concave inwards until UZYT. Just like the bottom structure in Figure 15 and the structure in Figure 16, the structure in Figure 19 has the same problem of whether there is a sharp line at each corner.

Another interesting issue is that both the structures in Figure 16 and Figure 19 have the same top view, front view and left side view. According to Wikipedia (2016), "*usually* three views of a drawing give enough information to make a 3D object". The word 'usually' is italicized, meaning that there are exceptions. So it *seems* that the two structures in Figures 16 and 19 are examples of these exceptions if both of them can be constructed.

However, there is a bigger problem with the structure in Figure 19, e.g. how does the surface curve from the edge AG of the outer cube to the edge UT of the inner cube? Because all the faces of the outer cube concave inwards, by symmetry, the surface from AG to UT will just be an isosceles trapezium AGTU, with AU = GT. Other than creating the two lines AU and GT which will appear in any of the six views, the main problem is that AGTU is a plane with no thickness, i.e. it is impossible to construct the structure in Figure 19.

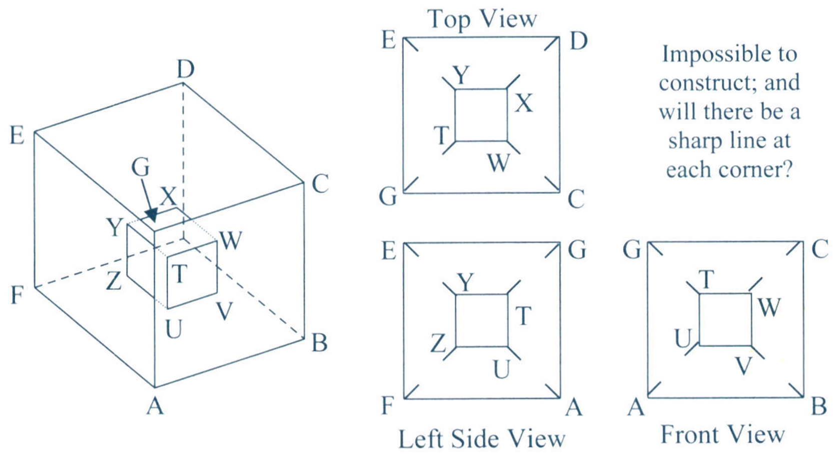

Figure 19. Ultimate Complicated Solution with Curved Surfaces (but impossible to construct)

To resolve the no-thickness problem, there is a need to give the trapezium AGTU a bit of thickness, either on one side or on both sides. To avoid the problem of creating the lines AU and GT, we have to create some thickness on both sides of the trapezium. In other words, AG will curve to the left of the trapezium and inwards to reach UT, and AG will also curve to the right of the trapezium and inwards to reach UT.

The first issue with this modification is what happens along, say, GT? Below GT, AG curves to the right and inwards to reach UT, but on the right side of GT, GC also curves downwards and inwards to reach TW. The question is how do these two surfaces meet along GT? Will it be a sharp line along GT? Or is it possible to smooth the intersection between the two surfaces? Or is it possible to even construct such complicated surfaces at all? Even if this is possible, there is the same old problem of whether there will be a sharp line at the corners G and T.

The second issue is the thickness of the 'trapezium' AGTU. In the top view, the 'trapezium' will be hidden from view by the surface that curves from GC to TW, and the surface that curves from EG to YT. But will it appear as dotted curves around GT as shown in the top view in Figure 20? (By symmetry, there will be four pairs of curves in each view.) This is another debatable issue for the readers to think about.

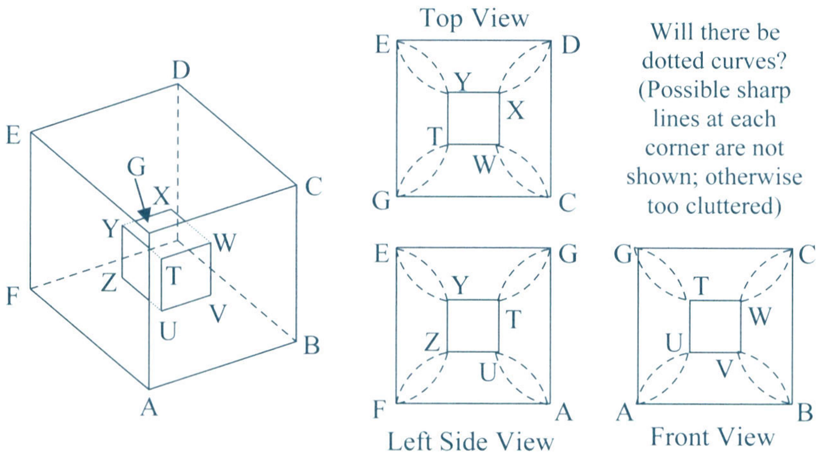

Figure 20. Ultimate Hypothetical Complicated Solution with Curved Surfaces (modified from Fig. 19)

We invite the readers to discuss whether it is possible to construct the bottom structure in Figure 15, the structure in Figure 16 and the structure in Figure 20, such that they match the given top and front views, without any additional sharp lines at the corners or any dotted curves. All these issues could be resolved if a clay model or an interactive 3D computer model of the structures could be constructed to show whether they are possible or impossible to construct, but the authors do not have such expertise, so we invite readers with the relevant skills to do so.

4. STRUCTURES WITH CURVED SURFACES ONLY?

All the structures that we have discussed so far contain some flat surfaces. The question is whether it is possible to construct a structure with curved surfaces only, and still satisfies the given top and front views. Take for example the left diagram in Figure 8. The part-cylinder in the center must have two flat surfaces on either end. Similarly for the outer structure, the left and right sides have to be flat for both the top and front views to have an outer square. Suppose we want to make the left side of the outer structure to be a curved surface. Even if we can maintain an outer square in the top view, this curvature will show up in the front view; and vice versa. Therefore, it is not possible to construct such a structure with curved surfaces only.

Since we cannot do anything about the flat surfaces on the left and right sides of the inner and outer structures, the next question is whether we can construct a structure such that its side view does not contain any straight line. The bottom right diagram in Figure 11 looks promising, but the top left portion and the bottom right portion in the side view are actually straight lines. As explained in that section, the curve in the side view of the top left structure is already a quadrant (or a quarter-circle), so the curvature cannot be increased without affecting the outer square in both the top and front views; and the structure has to stand. Therefore, the bottom part of the outer structure cannot be a simple curved surface.

One way to make it work is shown in Figure 21. In order for the structure to stand, the curved surface at the bottom part has to be more to the left in the side view because the structure is heavier on the left. We did not use the top surface of the outer structure in the bottom right diagram in Figure 11 because the curvature is not suitable for the right portion in the side view in Figure 21. Instead we use the top surface of the outer structure in the bottom right diagram in Figure 13 and then modify accordingly. Bearing in mind the conditions on the curved surfaces discussed in the previous section, the only vertical surface of the outer structure in the side view in Figure 21 must be on the extreme right so that it will coincide with the right side of the outer square in the top view; and the only horizontal surface of the outer structure in the side view must be right at the bottom so that it will coincide with the bottommost side of the outer square in the front view. Otherwise, it will result in additional lines in either the top or front view. There are other ways to curve the left portion in the side view but the curve will be more complicated.

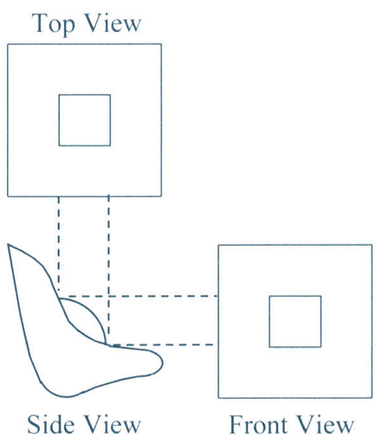

Figure 21. Solution without Any Straight Lines in Side View

5. CONCLUSION.

What is supposed to be a unique structure in the original puzzle in the 1980s, with both the top and front views consisting of a small square in the exact center of a larger square, turns out to be not so unique after all. Even if the structure is made of flat surfaces only, there are at least three known solutions. But if the structure can have curved surfaces, there are infinitely many solutions because there is some freedom in how the curved surface can curve, although there are certain restrictions on how the curved surface can curve in order to avoid creating dotted lines in the given top and front views. Some people may think that two orthographic projections are enough to construct a unique 3D model, but this puzzle serves to illustrate otherwise. In fact, if both the structures in Figures 16 and 20 can be constructed, then this will be an example that three orthographic projections are still not enough to construct a unique 3D model. The beauty of a connected world through the Internet, emails and social media is that people from all over the world are able to come together, without meeting physically, to contribute their ideas, thus generating so many structures with the given top and front views.

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