

Interview with IMO Contestants 2015

Introduction

David and Siah Yong exchanged perplexed glances as they watched the Interviewer deep in thought at the front of the classroom, wondering what he could possibly be musing over. The 2015 IMO Team, which included the two of them, had been gathered for an interview by – as you may have guessed – none other than the Interviewer himself.

Those who have been following previous editions of *Mathematical Medley* and remember the interview with the 2014 IMO Team (“*Ten Questions for the SIMO 2014 Team*”, Volume 40 No. 2) would recall that last year’s Interviewer was a rather mysterious character. As it turned out, this year’s Interviewer, who would like to be called by the very same name, was no different in this aspect. Though he had, by now, come to know the members of Singapore’s IMO Team rather well, he still stubbornly insisted on remaining anonymous in the public eye. In fact, whether the two Interviewers we speak of are actually the same person is left as a puzzle for the reader. (Ask any team member, however, and they will certainly be willing to divulge the answer. Observant readers may also be able to pick up clues along the way.)

After a long, uneasy silence, the Interviewer finally addressed the IMO Team. “I’ve made up my mind,” he started, “let’s not have an interview this year...”

“What do you mean?” Yijia interrupted. “Aren’t we assembled here for the interview today?”

“Calm down,” the Interviewer continued, “what I was about to explain was that we shan’t have a traditional interview, akin to that in last year’s edition. After all, five team members had already been part of the IMO Team last year. Wouldn’t it be repetitive to ask them similar questions again?”

“But I haven’t had a chance to describe myself yet,” Zhao Yu protested.

Zhao Yu Ma

“Anyway, here’s my suggestion: since there’s 6 IMO Problems and 6 team members, why don’t we have each member to talk about one question instead? I’m sure that the discussion will generate many meaningful insights since everyone has attempted the questions under a competition setting. So which question should each member describe?”

“Sounds like a plan. I’ll take Q4,” Siah Yong was first to volunteer.

“In that case, I’d like to talk about Q3. Geometry is my favourite subject,” David offered.

“If nobody objects, I’ll speak about Q5,” Sheldon proposed.

“Why don’t you choose a question first?” Yijia remarked to Zhao Yu.

“Perhaps I’d pick Q2 then,” the latter replied.

“Q1 for me... Wait, isn’t Dylan missing?” Just as Yijia inquired, the door flew open as Dylan rushed into the room. “I’m really sorry for being late, but I had some school activities to attend to,” he apologised.

“Aha!” exclaimed the Interviewer, “that leaves Q6 for you. Looks like we’ve sorted the questions out now. How about we present our takeaways from the respective problems in numerical order?”

“Hey, shouldn’t we begin with a fanciful introduction? I was particularly glad upon seeing all the juicy – or shall I say soupy? – bits inside last year’s introduction,” Siah Yong reminisced, while Zhao Yu echoed his sentiments.



Zhao Yu Ma

“Well, this was the introduction. Time for Yijia to start with his Q1,” smiled the Interviewer as he took a seat right at the back of the classroom.

Problem 1

Yijia strode to the front of the room and commenced the discussion. (Note: as in subsequent problem descriptions, some parts have been slightly edited to allow the text to be more perusable when written.)

“This question is a combinatorics question (with a bit of geometry). It is interesting as it has two parts. Part a) also seems to be a purely construction question which is quite rare. The difficulty in this question is that there are no obvious things to derive from the question immediately, and it does not tell us which n fails.

(a) The crucial part of this question is the $n = 3, 4$ case. $n = 3$ leads to the equilateral triangle immediately and $n = 4$ will be the $60 - 120$ rhombus.

There are multiple constructions, but a direct one will be to see that one can add equilateral triangles. Take another equilateral triangle (of similar side length), rotate it by some degree and combine one of the vertex with another vertex on the original construction to get from $n = k$ to $k + 2$ (ie. induction).

(b) The bad thing is that all our constructions are not centre-free (except for $n = 3$). The good thing is that it tells us a balanced centre-free set may not exist for certain n . Since the construction needed seems highly symmetric, it is natural to consider regular polygons, so that we get all odd n works (although you might have considered this as the construction for (a)).



Liu Yijia explaining his solution to Joseph Kuan and Mr Lu Shang-Yi

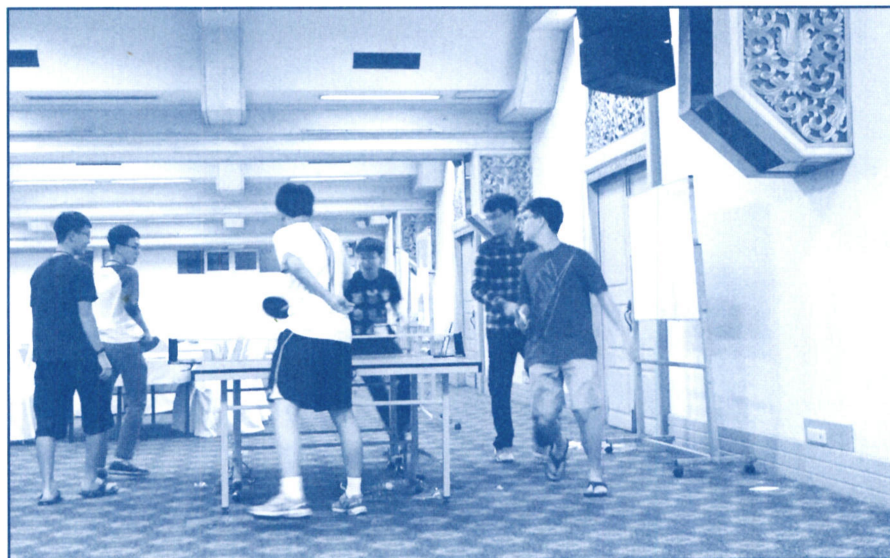
After trying to construct for even and having suspected that some n does not work, we set out to try and show all even n fails. It also points to some parity argument. Indeed, suppose such a construction exists. For even n let the points be A_1, A_2, \dots, A_n . Then consider choosing any two as a pair for A and B . There are $\binom{n}{2}$ such pairs, to be divided among n points (each pair (A, B) has a corresponding C that satisfy $AC = BC$, for which we say a pair (A, B) belongs to C). Since $\frac{n-1}{2}$ is not an integer, at least one point (WLOG A_1) has $\geq \frac{n}{2}$ pairs. Now we see that the centre-free condition implies that no two pairs belonging to A_1 have a common point, thus there are at least n distinct points (which excludes A_1) among the pairs belong to A_1 , contradiction.”

Problem 2

“My turn,” Zhao Yu announced before he recounted his experience with Q2.

“This question is a number theory question. Many people said that this was rather technical and had a very whacky solution, but personally I felt that it was a nice question, with a reasonable (not too much) amount of case-whacking needed.

I started off with making some observations, such as the fact that the condition is symmetric, so I can directly assume that $a \geq b \geq c$. Another observation would be that none of the numbers a, b, c can be 1. The next thing I did was to try the case of two numbers being equal, which quickly narrows down to two solutions, $(2, 2, 2)$ and $(2, 2, 3)$. With this, I was left with the case of all numbers being different from each other.



Zhao Yu's turn to strike - Some sort of "sports" to help relaxation before the actual IMO

At this point, I thought these were the only solutions to the question. As such, I tried to reject all other possible answers by considering cases based on the parity of a, b, c (e.g. all 3 even, all 3 odd, etc.). While trying the all even case, I found a key step to solving this problem, which was the fact that 2^{k_2} divides $2^{k_1} - 2^{k_2}$ if $k_1 \geq k_2$. Assuming $a < b < c$ and using this lemma on the two powers of 2, I got that $ca - b \mid b - a$, so $ca - b \leq b - a$. I repeated the same thing for another two powers of 2, and got a different inequality. With these two inequalities, it is easy to show a contradiction.

Unfortunately this method only seems to work for a, b, c all even. After trying the other cases, I was unable to arrive at a contradiction. Hence, I gave up on trying to solve this question by using parity. Looking back to my previous lemma, I realized that instead of just using $2^{k_1} - 2^{k_2}$, I could also turn to using the fact that 2^{k_2} also divides $2^{k_1} + 2^{k_2}$, which I didn't realize before. With an additional divisibility condition, I was able to use both to obtain the inequality of $3b \geq (c - 2)a$. Now, I realized that at high values of a, b and c , we would expect the inequality not to hold as the RHS will increase quadratically, while the LHS will increase linearly. Thus, I decided to start bounding the terms, and gradually, I managed to reduce the problem to the case $a = 2$ or 3 .

The rest is perhaps what others consider the "case whacky" part. In my opinion, the problem becomes a lot easier once it is reduced to two variables. The way I did it was by considering parity, and for each of the parity cases, there would be a power of 2 that was odd. And this was good as it means that that particular power of 2 would be equals to 1. From the case of $a = 3, b$ and c both odd, I got the solution $(3, 5, 7)$, while from the case of $a = 2, b$ even and c odd, we would get the final solution $(2, 6, 11)$



Dr Wong and Ang Yan Sheng coordinating Problem 2

In the process of checking cases, I actually rejected $(2, 6, 11)$ as a solution. Fortunately, this mistake was not penalized. I liked how the solution to this problem almost purely relies on logic, with no lemmas or theorems needed beforehand. I also found the key step to the problem (that 2^{k_2} divides $2^{k_1} - 2^{k_2}$ and $2^{k_1} + 2^{k_2}$ if $k_1 \geq k_2$) very nice, as well as the fact that when you subtract or add any two powers of 2, you will get an expression that is factorizable. On the other hand, as others have said, the solution is rather long and technical, and that was something I did not like about the question." Zhao Yu heaved a sigh of relief, his mind still flooded with the different cases he had to face during the IMO.

Problem 3

David rose from his seat and began.

“Q3 is a classic geometry problem, which is a popular choice at recent IMOs.

The question asks to demonstrate that two circles are tangent, which is especially difficult since the point of tangency K is a newly-defined feature of the question, with no prior properties known (worse still, the point of tangency could be completely undefined; see IMO 2011 Q6).

To do geometry questions, one needs both spatial imagination and some background knowledge of the tools of geometry. Most well-trained contestants will be able to recognize point Q ; it is also the intersection of HM with the circumcircle of triangle ABC . This comes readily due to the following (which we invite the reader to try to prove): reflecting H about M yields point H' , such that AH' is the diameter of the circumcircle.

From here, my own solution (shared with many other contestants) was to study a geometric transformation, specifically an *inversion* about H . What the transformation does is that it takes a point X , and maps it to another point along the ray XH , such that the new point X' satisfies $HX \cdot HX' = k$, for a constant k to be established. In this case, a smart choice would be $k = HA \cdot HA'$, A' being the foot of the altitude from A . This also implies that Q maps to M and back. Exploiting these coincidences combined with some knowledge about inversion (for example, circles and tangency is preserved under inversion), the problem cracks with little effort.



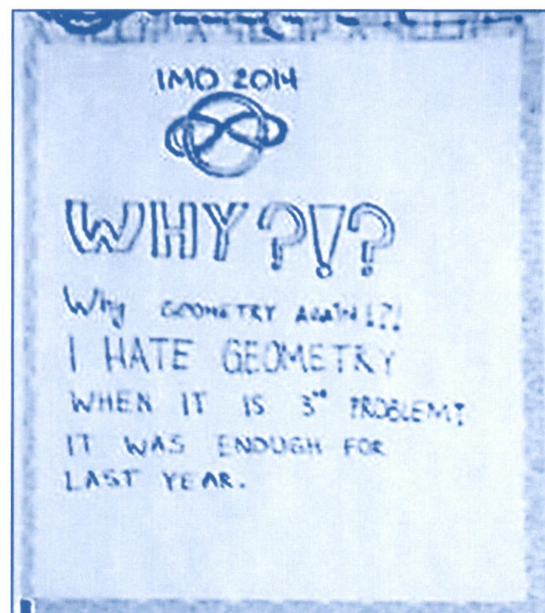
David selects his balls - Another "sports" to help relaxation

Problem 4

Siah Yong thoroughly loved story-telling. Not the typical recitation of children's bedtime stories that one may imagine, but the narration of almost any event, which he took great pride in doing. As such, it came as no surprise to his team members as he faced his audience and took a deep breath.

“Ladies and gentlemen,

Step right in and let me tell you a story. A story of what went down in Thailand this year. Now, maybe you've read the first three questions, but this, this is a brand new day. Day 2.

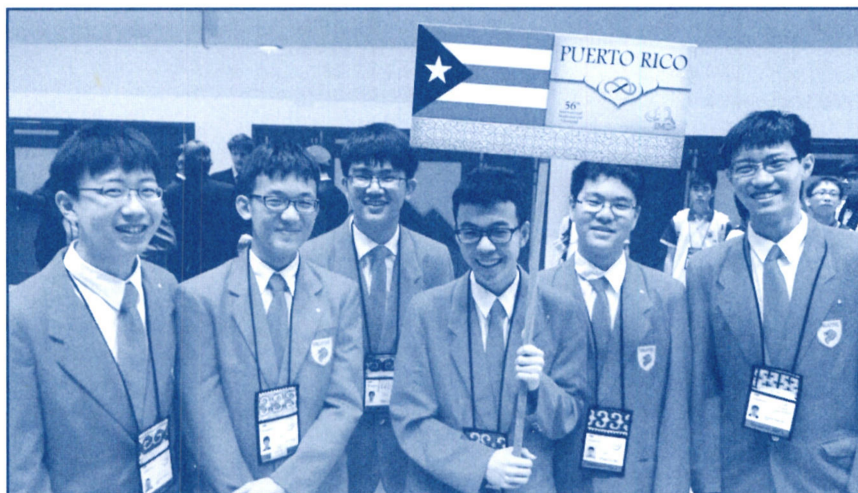


A disgruntled contestant, who certainly would've been unhappy in 2015 (again)

Without advanced knowledge (what's *inversion*? Never heard of it), it is still possible to tackle the questions, since in geometry “all roads (do) lead to Rome”. Another approach starts with taking the circumcircle of triangle HKA' (where A' is the extension of AF to the circumcircle) and showing that it is tangent to line MH .

All in all, Q3 is a wonderful problem, merging the familiar MHQ -line with the new point K . But perhaps it is time to finally see a Q3 of a different topic.” David chuckled, as he especially enjoyed most Q3 geometry problems after all.

There are rivalries. There are feuds. There are grudges, and blood debts, and age-old clan divisions. But nothing comes quite close to how I feel about geom. No. Do not want.



Siah Yong "steals" Puerto Rico's flag - The team usually succeeds in such quests

There are two places I would like to see geom. (Well, maybe 4) Q1/4 where I can feebly wriggle my arms in anger and slowly beat it into submission. Or Q3/6 where everyone else can feebly wriggle their arms in anger, competitive advantage lost. So you can imagine my face when Q4 for the last 3 years was geom. (No geom Q2/5 for last 3 years, living the dream)

Cheese is amazing, it ages so exquisitely, so elegantly, so magnificent in its final hours. Mathematicians aren't like that. Many of us stagnate, and then devolve into something resembling a head of cabbage with Alzhiemers. As an X man once said:

I'm too old for this shit. I'm like this banana, all soft and saggy.

So where does that lead us? In Sec 4, I took 2 hours to trigo the Q4 to death, I would have coorded it, but then I thought of the look of disappointment our trainer would have, so endure I did.

In JC1, I took over an hour to do the Q4, such amazing alacrity was no doubt due to our repeated song sessions of the parallelogram song, which aided in elucidation of a solution.

In JC2, when I saw the geometry Q4, I knew this was it. Twice I had defied it, now only one of us would survive.

In the early hours of the battle, we marched to battle. Well, I marched to battle, the others just steamrolled it. Many years of brutal, unending conflict with geometry had taught me one thing.

The Geometers Protocol:

If you can't solve it, it's probably Pascal or Ptolemy.

That didn't work. This was evidently the Voldemort of the IMO Geometry circuit, and two and a half hours in, I had made scant progress. All hope seemed lost, to have beaten geometry twice in the past two years only to be defeated in this final battle. I resorted to my final measure, I raised the card.

And went to the toilet.

With every leaden step, my mind cleansed itself, wiping every thought of the past and the future. I was there, in the moment. I ascended to a higher plane of existence. I experienced the briefest of true Nirvana that grace the daily lives of those truly deserving of being called Math Gods. For the briefest moment, I peeled the murky façade of daily life

back, and witnessed it all, the math that bound our world together. I realised that I was but a man trespassing on the realm of gods, that this was true wisdom that was never meant for the likes of mortals like myself.

It was enlightening, if I actually understood any of it. I felt ready, confident, in the zone. I could tackle any problem, no geometry question was too hard.

Needless to say, I was sorely misguided.

My delusions were shattered within ten minutes of returning to my seat. In desperation, and wholly out of options, I pounded the question in frustration, angle whacking it, the crudest of methods.

Upon which it promptly fell apart.

Lesson learnt? **Sometimes, if pushing doesn't work, don't pull, don't try thinking smart, or going out of the box, or being creative.**

Just push harder. Seriously.

Works for combi too.

By the way, **all important lessons are in bold.**" (Don't ask how he managed to produce words in bold while speaking. Mathematicians are sometimes Mathemagicians too.)

"Lastly, a shoutout to editors: thanks for leaving the bit about the thousand swans and exploding mushroom soup in!" Siah Yong said as he broke into a laugh...



Siah Yong "steals" Puerto Rico's flag - The team usually succeeds in such quests

Problem 5

Sheldon took his place at the front of the classroom confidently, remembering how dear Q5 had been to him at this IMO.

"Q5 is a Functional Equation problem, which falls under the category of Algebra.

[A common practice after the first day of the IMO is predicting the question types for Day 2. With Day 1 being CNG, we deduced that GAC had one of the highest chances of occurring. Furthermore, some of us ventured to conjecture that a Q5 Algebra question would probably be a Functional Equation due to the prevalence of this sub-category in recent years. Thus, it came as little surprise when we discovered that these guesses were indeed accurate the following day.]



Sheldon's idea of "sports"



Mr Thomas Teo, Dr Tay, and the Russian (whom we believe always coordinates Problem 5)



Dylan celebrates with his team mates

Naturally, it is often useful for one to guess the set of solutions for the function. My first instinct was to notice that $f(x) = x$ worked, which was frequently the only solution to such equations.

The difficulties in the question first arose when I began to try out substituting some constants, attempting to find some common values, such as $f(0)$ or $f(1)$. Only after scrutinising these equations did I suspect that there could be another linear solution. By trying $f(x) = ax + b$, I then realised that $f(x) = 2 - x$ actually worked too. While having another solution could mean more cases to try out, it could also provide a direction for the proof as one could surmise possible equations that had to be derived.

Soon, I found myself easily finishing the case for $f(x) = 2 - x$ within another half an hour. With the standard $f(x) = x$ solution left, I thought that most of the work had been completed. Unfortunately, I turned out to be horribly mistaken: $f(x) = 2 - x$ was by far the easier half of the question, which I gradually realised after getting stumped on the rest of the problem. After a while, I highlighted all my important workings and decided to take a toilet break.

When I returned, I suddenly derived another equation which seemed to solve the problem when combined with some of the earlier equations. Success! I excitedly reviewed all the relevant working. To my dismay, I found that one of the previous equations was incorrect as I had accidentally miswritten it. I now had to patch the incorrect equation instead. Sadly, this process of fixing an equation and revealing another wrong one recurred multiple times, taking nearly 2 hours in total. I finally managed to rectify all the steps in my working after another toilet break and reaching the essential condition that $f(x)$ was an odd function.

Some have criticised this question for being overly 'whacky' as it tends to involve many equations and substitutions, just like the cases in Q2. Looking at the number of steps I eventually took, I reluctantly have to agree. However, I still believe that the question statement itself, and how it encompasses 2 different solutions, is rather neat and well-set, invoking many common characteristics of functional equations in the process. Many shorter solutions also exist. Personally, I probably should have been more careful and methodical in writing my working so as to avoid the multitude of errors that misled me into thinking that I was already close to solving the problem. This would have saved me much more time as I was forced to review where the inaccuracies were thereafter instead. Overall, it was a relief that I found the errors and that the persistence paid off," Sheldon concluded, walking past the others back to his seat behind everyone else.

Problem 6

All eyes were on Dylan to present the final problem. He had originally bemoaned being excluded from the problem selection process between the team members, but now unhesitatingly began his presentation too. "This question

concerns the field of Combinatorics. During the actual duration of the Day 2 paper, I spent less than 15 minutes on this question as I had decided to play safe and concentrate all efforts on Q5; upon reflection, I regrettably realised I could have had greater success if I tried to tackle Q6 for longer.

My first intuition was to plot several graphs of points $(i, i + a_i)$ which fit the criteria; the generally increasing sequence should then be interpolated by a line of the form $y = x + b$ with low “error” (the expression bounded by 1007^2). I then attempted to construct the ‘worst case scenario’ where the interpolated line would give maximum “error” (1007^2). I also tried to set b as the “mean” of a_i ’s but failed to find a good enough interpretation of this average. I did not manage to achieve more significant results and understanding although the geometric interpretation above, complemented with the idea of “juggling” (jumping from i to $i + a_i$ and onwards) would have completed the proof. Another similar approach (or rather, interpretation) that I did not try was to plot points (i, a_i) and work along diagonal lines of the form $x + y = n$ (condition ii). The proof would be similar to the former method but perhaps more motivatable.

Combinatorics is my favourite and also my strongest field; the ideas and structure of questions such as in the above question fascinate me. I especially liked the ingenuity of the “juggling” idea, which placed the question in an almost entirely different new perspective; there are also little details to consider after this underlying concept is found, making the question very intuitive. However, I found that the question had little room for further exploration (apart from generalisation) due to the unique nature of the conditions.”



The team (with their medals), our team guide, and our mascot SIMON

Conclusion

“Alright, I suppose we’ve completed the interview,” Dylan stated, as the rest applauded.

“Yes indeed,” concurred the Interviewer. “But... how about one last question? It won’t take too long,” he added. “What was your favourite experience in Thailand?”

Zhao Yu: “Perhaps the most memorable experience for me was the activities we did in the games room. I heard from the rest that the previous IMOs didn’t have games rooms, so I felt pretty lucky that we had one this year. There were 6 table-tennis tables for participants to play on. For most of the duration of the IMO, all 6 tables were occupied. We got to play on the day we arrived at the venue, as well as during the table-tennis tournament organized by the IMO. Yijia and I played as a pair, while Siah Yong and Dylan played as another pair. Both of our teams lost our first match in the tournament and were then eliminated, but this tournament was still a very meaningful experience for me as I was able to interact with participants from other countries.”

David: “IMO in Thailand was a peculiar blend of the familiar (the Southeast Asian climate) and the foreign (almost everything was in Thai), which was a refreshing change after two IMOs held on the other side of the planet.”

Siah Yong: “The most exciting thing that happened in Thailand? Well, I would tell you, but I can’t. So I’ll tell you the err...fifth most exciting thing? Or you know, you could just hear it from someone who was around for X-men night.” (X-men night is an annual session held during the Singapore IMO Camp where past IMO participants share their experiences.)

Sheldon: “Beyond the competition, IMO is certainly about forming friendships with fellow mathematics lovers across the globe through games and math. Perhaps what makes these bonds even more meaningful is being able to meet up with one another again. During this year’s closing ceremony, I was pleasantly surprised to realise that I was sitting next to a Kazakh participant whom I had met last year because we had been assigned adjacent seats during the 2014 IMO closing ceremony too (a result of our scores being almost identical for both years). We immediately felt like close friends, and he even passed me the Kazakh cap he was wearing as a souvenir. We hope to meet again next year!”

Dylan: “The 2015 IMO in Chiang Mai, Thailand was especially memorable. Even though it was hosted in Southeast Asia and thus less exotic compared to other countries, the culture and tradition were extremely rich and shone through in the temples, street markets and people in general. In our few days of independence during coordination, we mixed around and played with the other countries, made close friends with the friendly Poland team, went out to the night market and waterfalls and all in all, had a very enjoyable and intoxicating experience. The most memorable experience was playing a Polish card game with the team; after returning to Singapore, we still play a translated version among ourselves!”

“And with that, the interview has ended. Thank you for your time!” The Interviewer declared, as the IMO team members left the room for a few rounds of their *Polish* card game.



More SIMONs