IMO 2015 Problems with Selected Solutions

Language : English
Day : 1

Friday, July 10, 2015

Problem 1. We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B and C in $\mathcal{S}$, there is no point P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geqslant 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geqslant 3$ for which there exists a balanced centre-free set consisting of n points.

Problem 2. Determine all triples (a, b, c) of positive integers such that each of the numbers

$$ab - c, \quad bc - a, \quad ca - b$$

is a power of 2.

(*A power of* 2 *is an integer of the form* 2^n, *where* n *is a non-negative integer.*)

Problem 3. Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocentre, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$, and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different, and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Handwritten solutions for Problem 1, 2 and 3 by SIMO Team members can be found in the following pages.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Language :	English
Day :	2

Saturday, July 11, 2015

Problem 4. Triangle ABC has circumcircle Ω and circumcentre O. A circle Γ with centre A intersects the segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(x + y)\big) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

Problem 6. The sequence $a_1, a_2, \ldots$ of integers satisfies the following conditions:

(i) $1 \leqslant a_j \leqslant 2015$ for all $j \geqslant 1$;

(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leqslant k < \ell$.

Prove that there exist two positive integers b and N such that

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leqslant 1007^2$$

for all integers m and n satisfying $n > m \geqslant N$.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Contestant	:	Tan Siah Yong
Problem	:	1
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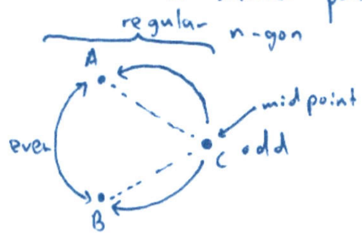
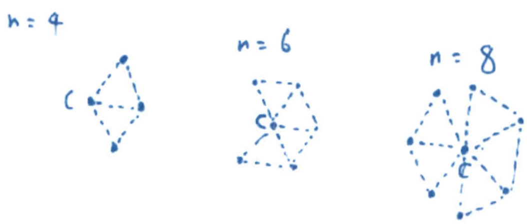
a) For odd n, consider a regular n-gon, as set S.

Any two points in S divides the n-gon into two (possibly empty) arcs. Since n odd, we have one arc with an even number of points in S (possibly 0), and the other with an odd number.

Consider the odd arc, its midpoint belongs in S, and is equidistant from our 2 selected points, so S is balanced.

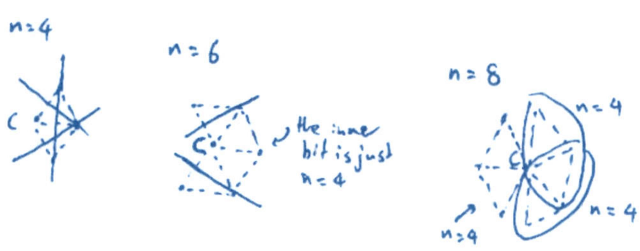


For even n, consider the following:



Dotted lines indicate same distance.

In each of these, all of the other points are equidistant from the one marked C, so any two of those can use C, picking C and any of the other points also gives an equidistant point.



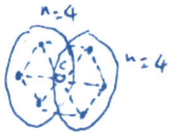
So $n = 4, 6, 8$ work.

See next page.

Consider a 6-ring.

Note all 6 in the ring are also equidistant from the point marked C.

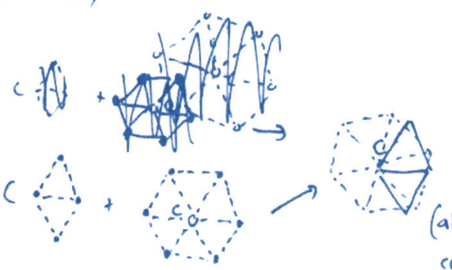
Furthermore,



Picking centre and ~~given~~ any point in the ring will work, and centre C is equidistant from any two points in the ring.

Overlapping the ring of 6 with an existing S (overlap on point marked C) with a suitable rotation applied to the ring so that no other points overlap adds 6 new points.

E.g. for 10,



All other points will be equidistant to C, and any point with C has an equidistant point.

(all lines ~~equidistant~~ represent same distance, but colour-coded for convenience)

Case 1: Point with C in base set S, as base set S works exists point in S equidistant to both selected points.

Case 2: Point with C on ring of 6. As ring of 6 works, exists point on ring equidistant to both selected points,

Also, our new set S′ has all other points equidistant to the point marked C. hence S′ is balanced

We can repeat this over and over, adding 6 each time, as S is finite, always exists a rotation so no point other than the one marked C overlaps.

Since we have constructed for n = 4, 6, 8, true for n ≡ 2, 4, 0 (mod 6), n ≥ 3, so true for all even n ≥ 3.

Hence, constructed balanced set for all n ≥ 3.

Contestant	:	Tan Siah Yong
Problem	:	1
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b) Note for odd n, a regular n-gon is balanced (as proven in part a)

For any 3 points A, B, C in S, P (centre $PA = PB = PC$) must lie on the perpendicular bisector of AB and AC, but for a regular n-gon, that point is the centre of the n-gon, which is not in S. Hence, the regular n-gon is balanced and centre-free for odd n.

For even n, consider ~~this~~ a balanced set S with $2k$ points. We make point-~~pa~~ segment pairs as follows, as S is balanced, for a segment AB, exists C such that $AC = BC$, so we make a point-segment pair ~~AAB~~. (C, AB). There are $\binom{2k}{2}$ segments and $2k$ points, so there are at least $\binom{2k}{2}$ point-segment pairs so by pigeonhole, there is a point ~~B~~ T such that T is involved as the point in at least $\left\lceil \frac{\binom{2k}{2}}{2k} \right\rceil = \left\lceil \frac{2k-1}{2} \right\rceil = k$ point segment pairs. Since there are exactly $2k-1$ points other than T, and each segment k involves 2 endpoints, exists at least 2 segments in T's k point-segment pairs that share the same endpoint, XY and XZ. (They cannot share both endpoints or they would be the same segment) Hence by definition of point-segment pair, $TY = TX = TZ$, for X, Y, Z in S, T in S with $TX = TY = TZ$, S not centre-free. Hence no balanced set with n even is centre free.

Hence, the integers $n \geq 3$ with balanced centre-free set is odd n.

Hence, done.

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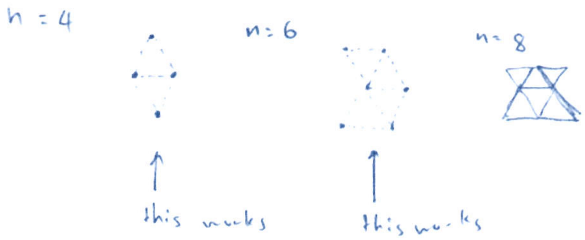
a) A regular n-gon for odd n.

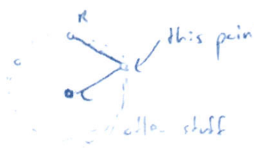
→ For any two points, one of the midpoints of the arcs between them exists.



pick a point as the centre, then add the 6 hexagon ring

So note that for n = 6k + 4, 6k you ~~have~~ have the base shape and the rings of 6 added. Any two in the outer hull have the centre. The centre and ~~any~~ anything in the outer hull have a point in the base shape / ring of 6

Centre and ring of 6



Centre and base shape

(since base shape works, we know some point in the base shape satisfies

e.g.

n = 8

Tadah

So all even n done too

All dotted lines same distance.

shape add 6-rings equidistant from centre, same as base shape

e.g.

overlap C, rotate rings a little every time.

Contestant	:	Tan Siah Yong
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b) Note for odd n, our n-gon is centre free, since $\perp$ bisector of AB, AC meet in centre of n-gon, always, and no point is there.

For even n... we select a point P. For each of the other $n-1$ points, their equidistant point must be ~~some other else~~ a distinct point, so $n-1$ points, each is equidistant from...

well, evidently I'm supposed to use $\binom{n}{2}$ is not divisible by n for n even to force a contradiction. (I'm picking pairs of points $\binom{n}{2}$, and even and odd grants us this great distinction.

For even $2k$, note that for $\binom{2k}{2}$ pairs of points, each pair has at least 1 point 'mapped' to it. Denote map as the point being equidistant to the pair.

$\left\lceil \frac{\frac{2k(2k-1)}{2}}{2k} \right\rceil = k$ so by pigeonhole, some point is mapped to at least k pairs. But aside from that point itself, there are only $2k-1$ other points, so two of the k pairs must share a ~~common~~ common point, making this point a centre.

So only for odd n, even n fails.

~~QED~~ Hence, done.

Contestant	:	Ma Zhao Yu
Problem	:	2
Page	:	1

Solutions are (2,2,2), (2,2,3), (3,5,7) and all permutations.

2. If any of the numbers $=1$, WLOG let it be a.

Then $b-c$, $bc-1$, $c-b$ all powers of 2

but one of these is negative/0 $\Rightarrow$ cannot be power.

$\Rightarrow a,b,c \geq 2$.

If any 2 numbers equal, WLOG let it be $a=b$.

$\Rightarrow a^2-c$, $ac-a$, $ca-a$. all powers of 2.

$\Downarrow$

but this is $a(c-1)$. $\Rightarrow$ a has to be a power of 2

Since $a>1$, a is ~~even~~

$\Rightarrow c-1$ has to be a power of 2.

If c is even, $c-1=1$. $\Rightarrow c=2$.

$\Rightarrow a^2-2$ is power of 2. Note $a=2^k$, $k>0$ $(a>1)$

$\Rightarrow 2^{2k}-2$ is a power of 2.

for $k=1$, $a=b=c=2$, and this works.

for $k>1$, note $2^{2k}-2>2^{2k-1}$, so it is not a power of 2 $\Rightarrow \Leftarrow$

If c is odd. $\Rightarrow a^2-c=1$. $\Rightarrow a^2=c+1=2^{2k}$, yet $c-1$ is a power of 2

$\Rightarrow 2^{2k}-2=$ pwr of 2. use above reasoning

$\Rightarrow k=1$

$\Rightarrow a=b=2$

$\Rightarrow c=3$

This works.

All numbers are different

If all are even,

WLOG $a<b<c$

Let $ab-c=2^{k_1}$, $bc-a=2^{k_2}$, $ca-b=2^{k_3}$

(uncancel) ~~$2^{k_2}-2^{k_1}=bc-a-ab+c$~~ $= b(c-a)+(c-a)=(b+1)(c-a)>0$

$\Rightarrow 2^{k_2}>2^{k_1}$

Contestant	:	Ma Zhao Yu
Problem	:	2
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~~$2^{k_2}-2^{k_1}=(b+1)(c-a)>0 \Rightarrow 2^{k_2}>2^{k_1}$~~

Yet note that $2^{k_1} \mid (b+1)(c-a)$, but $b+1$ is odd, so

$$2^{k_1} \mid c-a$$

$\Rightarrow ab-c \le c-a \Rightarrow (a+1)b \le 2c$ — ①

$2^{k_2}-2^{k_3} = bc-a-ca+b = (c+1)(b-a) > 0$.

Again, $2^{k_2} > 2^{k_3} \Rightarrow 2^{k_3} \mid (b-a)$

$\Rightarrow ca-b \le b-a \Rightarrow (c+1)a \le 2b$ — ②

$\Rightarrow 4b \ge 2a(c+1) > 2ac \ge a(a+1)b$.
②

But $a>1 \Rightarrow a(a+1) \ge 6 > 4 \Rightarrow \Leftarrow$.

$\Rightarrow$ no solution $\Rightarrow$ one of them must be odd.

~~Case 2: If a, b, c can be cycled such that $a>b>c$.~~

~~again let $ab-c=2^{k_1}$, $bc-a=2^{k_2}$, $ca-b=2^{k_3}$~~

~~$2^{k_1}-2^{k_2}=(b+1)(a-c)>0$.~~

~~$\Rightarrow 2^{k_1}>2^{k_2} \Rightarrow 2^{k_2} \mid a-c$.~~

~~$\Rightarrow bc-a \le a-c \Rightarrow (b+1)c \le 2a$.~~

~~$2^{k_1}-2^{k_3} = ab-c-ca+b = (a+1)(b-c) > 0$~~

~~$\Rightarrow 2^{k_1}>2^{k_3} \Rightarrow 2^{k_3} \mid b-c$~~

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2. Since this condition is symmetric, WLOG $a<b<c$. Since $a,b,c>1$, $c\geq 4$. $b\geq 3$

Let $ab-c=2^{k_1}$, $bc-a=2^{k_2}$, $ca-b=2^{k_3}$

$2^{k_3}-2^{k_1}=a(c-b)+c-b=(a+1)(c-b)\geq 0$.

$2^{k_2}-2^{k_3}=bc-a-ca+b=(c+1)(b-a)\geq 0$.

$\Rightarrow$ We know the order of these terms, $ab-c\leq ca-b\leq bc-a$.
$k_1\leq k_3\leq k_2$.

$2^{k_2}-2^{k_3}=(c+1)(b-a)$, $2^{k_2}+2^{k_3}=bc+ca-a-b=(c-1)(b+a)$

Since $\gcd(c+1,c-1)=2$, $4\nmid c+1$ or $c-1$

Case 1: If it is $c+1$, $2^{k_3}\mid(c+1)(b-a)$
↑ contributes at least $v_2(c+1)=1$.

$\Rightarrow 2^{k_3-1}\mid b-a \Rightarrow \frac{ca-b}{2}\leq b-a$.

Case 2: If it is $c-1$,

$2^{k_3-1}\mid b+a \Rightarrow \frac{ca-b}{2}\leq b+a$.

} both satisfy $ca-b\leq 2(b+a)$

$\Rightarrow ca-b\leq 2(b+a)$. $\Rightarrow 2(b+a)\geq ca-b \Rightarrow$ $\boxed{3b\geq(c-2)a}$.

$\frac{b}{b-1}=\frac{b}{b+1-2}\geq\frac{b}{c-2}\geq\frac{a}{3}$. If $a\geq 4$ If $b\geq 5$, $\frac{b}{b-1}\leq\frac{5}{4}\Rightarrow\frac{a}{3}\leq\frac{5}{4}\Rightarrow a\leq 3 \Rightarrow\Leftarrow$.

If $b=4$, $\frac{4}{c-2}\geq\frac{a}{3}\geq\frac{4}{3}\Rightarrow c-2\leq 3\Rightarrow c\leq 5\Rightarrow c=5$.
But $(2,4,5)$ and $(3,4,5)$ fail. $\Rightarrow\Leftarrow$

If $b=3$, $\frac{3}{c-2}\geq\frac{4}{3}\Rightarrow c-2\leq\frac{9}{4}\Rightarrow c\leq 4$. But $(2,3,4)$ fails $\Rightarrow\Leftarrow$.

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Page	:	4

$\Rightarrow a=2$ or 3.

If $a=2$,

$2b-c, bc-2, 2c-b$ are powers of 2.

not all of them are even.

If both are odd, bc is odd $\Rightarrow bc-2=1$ ($\Rightarrow bc=3$ ↯) $\Rightarrow b=1 \Rightarrow \Leftarrow$.

$\Rightarrow$ one even one odd.

If b is odd, $2c-b=1$. $\Rightarrow 2c=b+1$ $\Rightarrow \Leftarrow$ since $(b<c)$

If c is odd, $2b-c=1$ $\Rightarrow 2b=c+1$ $\Rightarrow c=2b-1$.

$\Rightarrow 3b \geq (c-2)a = (2b-3)a = 4b-6$.

$\Rightarrow b \leq 6$.

If $b=3, c=5$ (fails)

$b=4, c=7$ (fails)

$b=5, c=9$ (fails)

$b=6, c=11$ (fails) $\Rightarrow \Leftarrow$.

If $a=3$ $3b-c, bc-3, 3c-b$ powers of 2.

If b or c is even, bc even $\Rightarrow bc-3=1$ $\Rightarrow bc=4$ but $b,c > a=3$ $\Rightarrow \Leftarrow$

$\Rightarrow b, c$ both odd.

$3b \geq (c-2)a =$

$\Rightarrow b \geq c-2$. $\Rightarrow b=c-2$ (same parity, $b<c$).

$\Rightarrow 3(c-2)-c, 3c-(c-2)$ } powers of 2.

$= 2c-6, 2c+2$.

$= 2(c-3), 2(c+1)$ $\Rightarrow c-3, c+1$ powers of 2 $\Rightarrow c=7$.

$\Rightarrow \boxed{(a,b,c)=(3,5,7)}$

Contestant	:	Ma Zhao Yu
Problem	:	2
Page	:	Scrap 1

ab−c power of 2.

~~check a=1~~. note order!

cycle.

22−2 ✓.

$a=b=c=2 \Rightarrow ab-c=bc-a=ca-b=4-2=2=2^1$ (works)

a=1:
b−c,
bc−1,
c−b fail.

$ab-c=bc-a$.

$b(a-c)=c-a$.

cannot unless c=a. b=1. not same power if

a=b

$a=b=c$: $a^2-a \Rightarrow a(a-1)$ a=2 ✓.

$a=b$: a^2-c, $a(c-1)$.

even even power of 2 power of 2, odd?

a>1. $\Rightarrow c-1=1$.

c=2

a^2-2, $a(1)$

4

a power of 2 $\Rightarrow a=b=c=2$.

$a\neq b\neq c$. all different powers.

a biggest.

$(bc-a)-(ab-c) = b(c-a)+(c-a)$

$=(b+1)(c-a)$.

odd (not odd!)

Contestant : Ma Zhao Yu
Problem : 2
Page : Scrap 2

$c-a \geq bc-a$ or $ab-c$. X

$\Rightarrow c-a \geq ab-c$

$\Rightarrow 2c \geq a(b+1)$

similarly, $2c \geq b(a+1)$.

$(bc-a)-(ca-b) = c(b-a)+(b-a) = (c+1)(b-a)$.

$b-a \geq bc-a$. X

$b-a \geq ca-b$

$2b \geq c(a+1)$, $2c \geq (b)(a+1)$.

⇓

$4b \geq a(a+1)(b)$. fail. lol.

Contestant	:	Ma Zhao Yu
Problem	:	2
Page	:	Scrap 3

$ab-c$, $bc-a$, $ca-b$ gg.

$bc=a+1$. $a=bc-1$

$(bc-1)b-c$.

b^2c-b-c.

bc^2-b-c.

$ab-c = 2\times(bc-a)$

3, 5, 7 works ? !

$3b-c$, $bc-3$, $3c-b$!

$(b+1)(c-a)$

$ab+bc+ca - a - b - c$.

Contestant	:	Ma Zhao Yu
Problem	:	2
Page	:	Scrap 4

$2(b+a) \geq ca - b$

$3b + 2a \geq ca$

$3b \geq (c-2)a$

$a > 4$

2, 3, 4.

$\frac{b}{c-2} \geq \frac{a}{3}$

$\frac{3}{2} \geq \frac{4}{3}$

$a = 3$

$3b \geq (2b-3) 2$

$3b \geq 4b - 6$

Contestant	:	Ma Zhao Yu
Problem	:	2
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c: even

$3c-2,\ (2c-1)c-2\ ,\ 1$

$2^k+2=3c$

powers of 2.

bound

$1\to$

$2\to$

$3\to$

$(2c-1)2k-2$

$2((2c-1)k-1)$

$3b-c,\ bc-3,\ 3c-b$

0,0

$c-b\le 2$

$c=b+2$

$3b-b-2$

$2b-2$

$2(b-1)$

$,\ 3b+6-b$

$2b+6$

$2(b+3)$

Contestant	:	Ma Zhao Yu
Problem	:	2
Page	:	Scrap 6

~~$bc-a\le 2b+2$~~

$a<b<c$ odd

$ab-c\le 2b-1$

$ab-c\le 2a+2$

$(b+1)(a+c)$

$(b+1)($

$a<b<c$.

$a(b+c)(a-1)$

$(c-b)(a+1)$

$\frac{ab-c}{c}\le 2a+2$ or $2a-2$.

$3\times 5-7\le 8$

Syndan

$ab-c\le 2a+2$

$cb-a$ or $ca-b\le 2c+2$ ✓?

(2)

$7\times 3-5\le 2(c+2)$.

(4) die

$a=2$

$2b-c,\ bc-2,\ 2c-b$ pairs of ?

o e e o.

$2c-b=1$.

$b=2c-1$.

$4c-2-c=3c-2$

Contestant	:	Ma Zhao Yu
Problem	:	2
Page	:	Scrap 7

5, 7, 19 (16) 3+16

$(ab-c)(bc-a)(ca-b)$

$(ab^2c - bc^2 - a^2b + ac)(ca-b)$

3, 5, 7 $a^2b^2c^2 - abc^3$

mod 4

$a<b<c$

$a \geq 4$

1 3 1

3, 5, 7

$ab-c$, $bc-a$

$c : (4b+1)$ (3, 1, 3)

$c<ab$

$ab-c=1$

$c=ab-1$

$ab-c$

$ab-4b+1$

$a(4b+1)-b$

$4ab+a-b$

$(b+1)(c-a)$

$(c+1)(b-a)$

$(a+1)(b-c)$

7, 11, 13

$ab-c+bc-a$

$(b-1)(a+c)$

$(b+1)(a-c)$

2c

$\frac{ab-c}{2} \leq b+1$

$ab-c \leq 2b-1$

Contestant	:	David Lin
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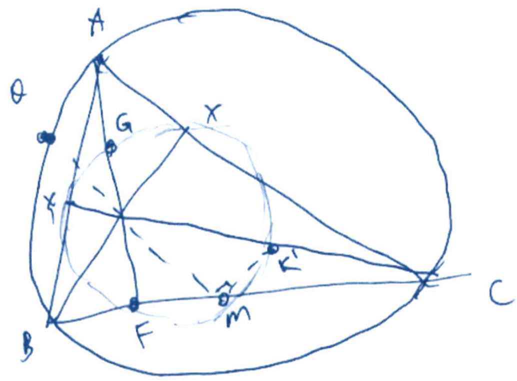
Where is K?

K' is on ⊙FXY and ∠HQ'K' = 90°
or ∠HMK' = 90°



So ⊙KQH ⟷ line MK'
⊙FKM ⟷ ⊙AK'Q

It suffices to show that line MK' is tangent to ⊙AKQ.

Let G = AF ∩ ⊙FXY (G ≠ F). It is clear that G is the center of ⊙AXY, since GX = GY ([redacted] FA bisects ∠XFY)
and G lies on the diameter AH of ⊙XYA.

So [redacted] GQ = GA. But [redacted] GK' // MQ ⊥ QA.

(since MX = MY (∠BYC = ∠BXC = 90°), so [redacted] M, G are diametrically opposite and [redacted] GK' ⊥ K'M ⊥ MQ).

∴ K'M tangent to ⊙QK'A.

Contestant	: David Lin
Problem	: 3
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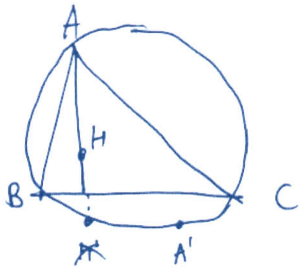
3. Lemma: AA' is the diameter of Γ

Then H, M, A' collinear.



~~Proof: Reflect H about BC to get H'.~~

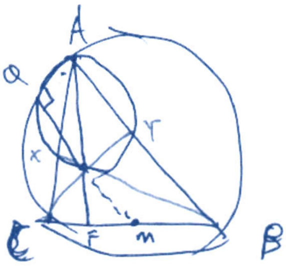
~~Since $\angle BH'C = 180° - \angle A$, H' lies on Γ~~

Proof: $BH \parallel A'C$, $CH \parallel A'B$, so $A'H$ bisects BC (since $A'BHC$ is a parallelogram)

Since $\angle HQA = 90°$ and $\angle AQA' = 90°$,

H, Q, A' collinear.

$\Rightarrow HMQ$ collinear.



Then $\angle AFM = \angle MQA = 90°$.

$\Rightarrow A, F, M, Q$ concyclic $\Rightarrow HM \cdot HQ = HF \cdot HA$.

Consider the feet of the altitudes from B, C (in $\triangle ABC$), and let them be X, Y respectively.

Note $AHXYQ$ concyclic since $\angle AQH = \angle AXH = \angle AYH$ concyclic.

Also, it is obvious that $HA \cdot HF = HB \cdot HX = HC \cdot HY = HM \cdot HQ$.

Invert about H with power $HA \cdot HF$, and denote the image of object X as X'.

Note that under this inversion: $A \leftrightarrow F$ (represents $A' = F$, $F' = A$)

$B \leftrightarrow X$

$C \leftrightarrow Y$

$M \leftrightarrow Q$.

Line $BC \leftrightarrow$ circle $\odot HXY$

$\odot ABC \leftrightarrow \odot FXY$.

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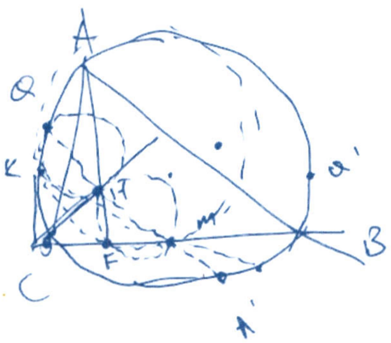

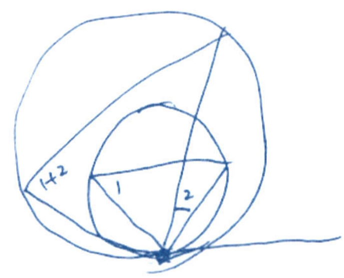

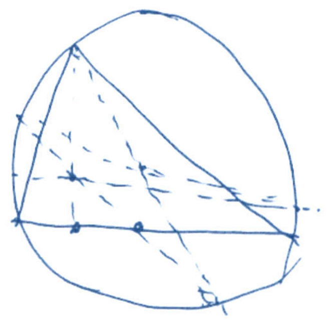

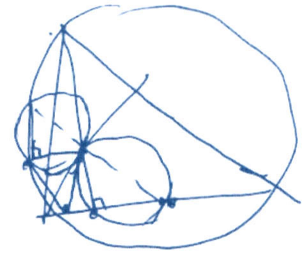

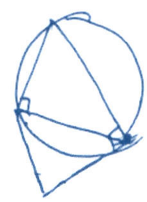

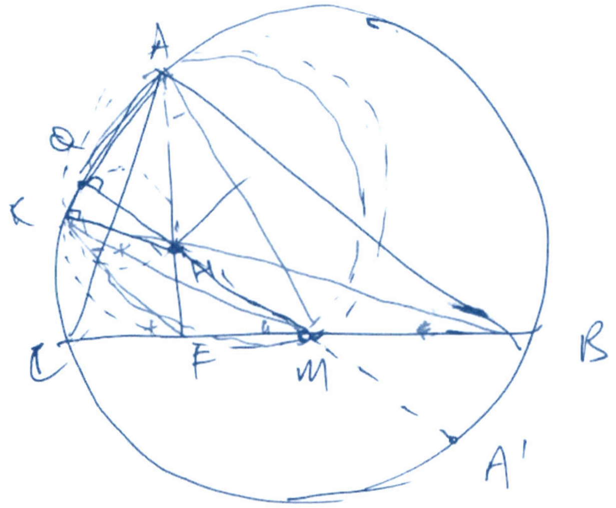

MF

∠KQH − ∠K[illegible]

= ∠KQA' − ∠KBM.

− ∠MKB

= $\overset{\frown}{KA'}$ − $\overset{\frown}{KC}$ − ∠MKB

= ∠CKA' − ∠MKB

= ∠FKH?

~~$\overset{\frown}{KA'}$ − $\overset{\frown}{KC}$ − ∠AM~~

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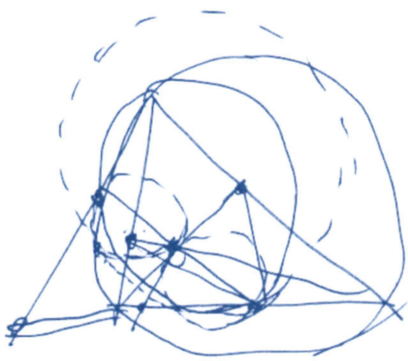

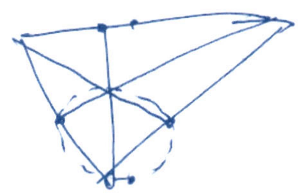

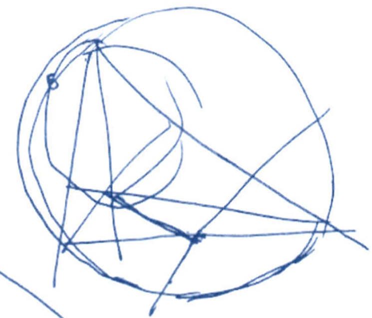

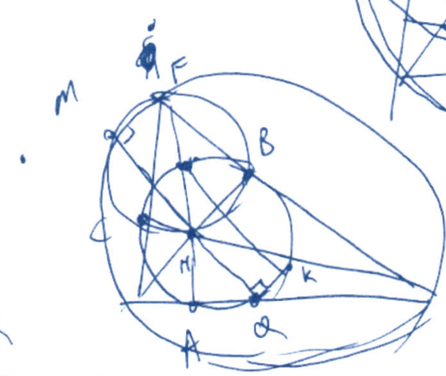

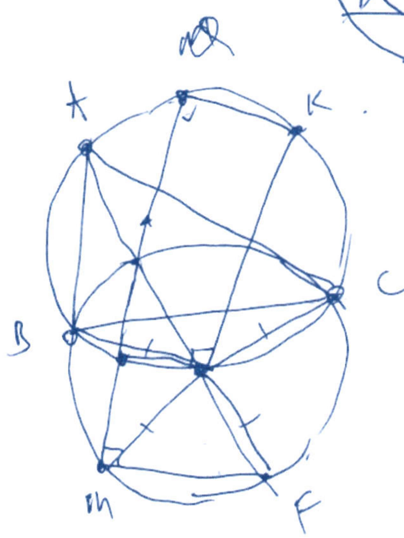

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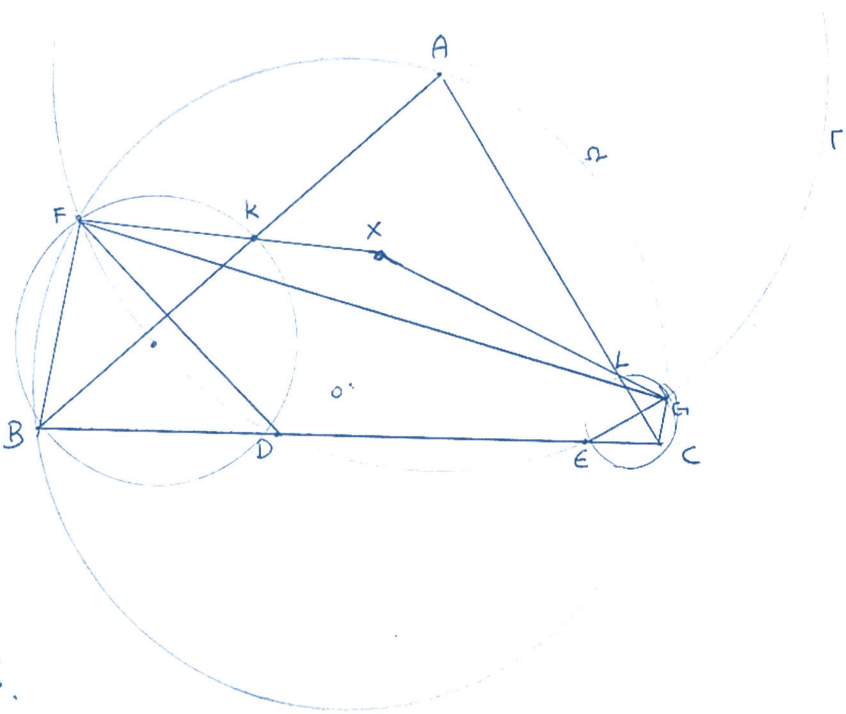

Directed angles mod $180°$.

We have $\angle BFD = \angle BFG - \angle DFG = 180° - \angle GCB - \angle CEG = \angle EGC$.

(BFGC concyclic; FDEG concyclic)

$\therefore \angle DFC = \angle BFC - \angle BFD = \angle BGC - \angle EGC = \angle BGE$.

Also, $\angle FAK = \angle FAB = \angle FCB = \angle FCD$ and $\angle FKA = 180° - \angle BKF = 180° - \angle BDF = \angle FDC$

$\therefore \triangle FAK \sim \triangle FCD$

$\therefore \angle KFA = \angle DFC$.

Similarly $\triangle GAL \sim \triangle GBE$ $\therefore \angle AGL = \angle BGE$.

$\therefore \angle KFA = \angle AGL$.

Note $FA = GA$, $FO = GO$ $\therefore AO$ is perpendicular bisector of FG.

$\therefore \angle FAO = \angle OAG$.

Let FK and GL intersect AO at X' and X'' respectively.

No more directed angles.

$\angle FAX' = \angle GAX''$, $\angle X'FA = \angle X''GA$, $FA = GA$

$\therefore \triangle FAX' \cong \triangle GAX''$ $\therefore AX' = AX''$.

$X = X' = X''$.

$\therefore X$ lies on AO.

#.

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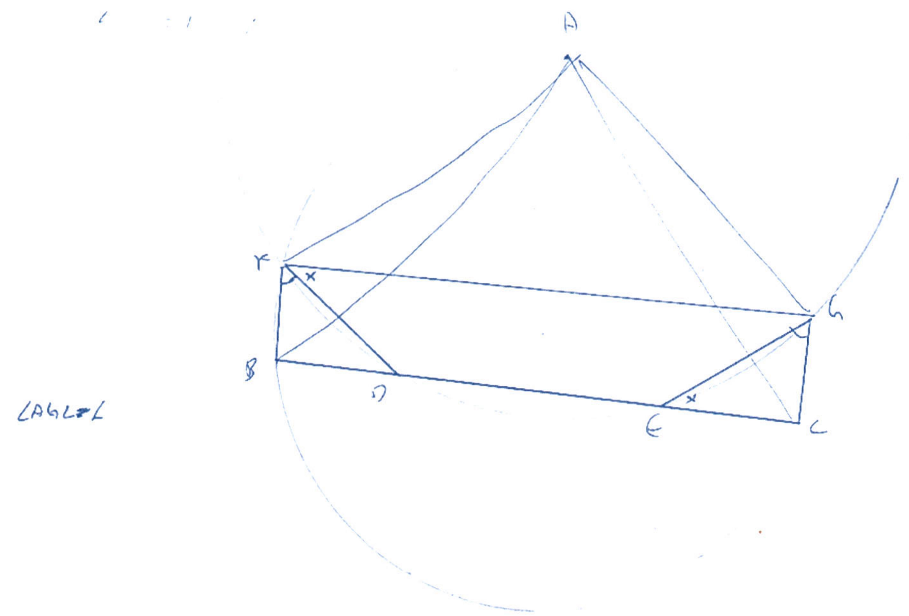

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Problem 5

Let $P(x,y)$ be the proposition $f(x+f(x+y))+f(xy)=x+f(x+y)+yf(x)$.

$P(x,1)$: $f(x+f(x+1))+f(x)=x+f(x+1)+f(x) \Rightarrow f(x+f(x+1))=x+f(x+1)$.

Suppose $x+f(x+1)=c\ \forall x \Rightarrow f(x+1)=c-x \Rightarrow f(x)=c-x+1$. Now $P(x,y)$ yields $y+c+1-xy=~~x~~+c+1-~~x~~-y+y(c+1-x) \Rightarrow cy=y \Rightarrow c=1$.

So $f(x)=2-x$ is a solution. Now suppose ~~∄~~ $\nexists c$ such that $x+f(x+1)=c\ \forall x$, i.e. $x+f(x+1)$ takes on at least 2 values over all x. Call x a stationary point if $f(x)=x$. So ~~f(x)+~~ $x+f(x+1)$ is a stationary point $\Rightarrow$ we have ≥ 2 stationary points in $f(x)$.

$P(0,y)$: $f(f(y))+f(0)=f(y)+yf(0) \Rightarrow f(f(y))-f(y)=(y-1)f(0)$. Let 2 stationary points in f be c_1, c_2. Substituting $y=c_1, c_2$ gives $(c_1-1)f(0)=(c_2-1)f(0)=0 \Rightarrow f(0)=0$ (since $c_1 \neq c_2$ so both cannot be 1 at the same time). So $f(f(y))-f(y)=0 \Rightarrow f(f(x))=f(x)\ \forall x$, i.e. $f(x)$ is a stationary point too.

$P(x,-x)$: $f(x)+f(-x^2)=x+xf(x)$. Substituting $x=-1$, $f(-1)+f(-1)=-1+f(-1) \Rightarrow f(-1)=-1$. Substituting $x=1$, $f(1)+f(-1)=1+f(1) \Rightarrow f(1)=1$.

~~$P(1,f(y+1)-1)$: $f(f(y+1)+1)+f(f(y+1)-1)=1+f(y+1)+f(y+1)-1=2f(y+1)$.~~

$P(1,f(y)-1)$: $f(f(y)+1)+f(f(y)-1)=f(y)+1+f(y)-1=2f(y)$ — ①

$P(1,y-1)$: $f(f(y)+1)+f(y-1)=f(y)+1+y-1=f(y)+y$ — ②

① − ②: $f(f(y)-1)-f(y-1)=f(y)-y$ — ③

~~$P(-1,y)$: $f(f($~~

$P(-1,y+1)$: $f(f(y)-1)+f(-y-1)=f(y)-1-y-1$ — ④

④ − ③: $f(y-1)+f(-y-1)=-2 \Rightarrow f(y)=-2-f(-y-2)$. — ⑤

~~$P(-1,2-y)$: $f(f(1-y)-1)+f(y$~~

$P(-1,1-y)$: $f(f(-y)-1)+f(y-1)=f(-y)-1+y-1$. Note that from ⑤, $f(f(-y)-1)=f(-2-f(y-2)-1)=-2-f(f(y-2)+1)$, and $f(-y)=-2-f(y-2)$.

So $P(-1,1-y)$ becomes $-2-f(f(y-2)+1)+f(y-1)=-2-f(y-2)+y-2 \Rightarrow f(f(y-2)+1)=f(y-1)+f(y-2)-y+2$. Substituting y with $y+2$, we have $f(f(y)+1)=f(y)+f(y+1)-y$ — ⑥

② − ⑥: $f(y-1)+f(y+1)=2y$ — ⑦

~~$x+f(x)$~~ $x+f(x+1)$ is a stationary point $\Rightarrow x-1+f(x)$ is a stationary point. Substituting $y=x+f(x)$ into ⑦, we obtain

$f(x+f(x)-1)+f(x+f(x)+1)=2x+2f(x) \Rightarrow f(x+f(x)+1)=x+f(x)+1$.

$P(x+1,-1)$: $f(x+1+f(x))+f(-x-1)=x+1+f(x)-f(x+1) \Rightarrow f(x+1)+f(-x-1)=0 \Rightarrow f$ is odd ($f(x)=-f(-x)$).*

~~$P(x,f(x)-x)$: $f(x+f(x))+f(x(f(x)-x))=x+f(x)+f(x)(f(x)-x) \Rightarrow f(x(f(x)-x))=f(x)(f(x)-x) \Rightarrow f(x(x-f(x)))=f(x)(x-f(x))$.~~

~~$P(x-f(x),f(x))$: $f(x+f(x))+$~~ $f(x)+f((x-f(x)$

$P(-x,-y)$: $f(-x+f(-x-y))+f(xy)=-x+f(-x-y)-yf(-x) \Rightarrow -f(x+f(x+y))+f(xy)=-x-f(x+y)+yf(x)$.

$P(x,y)-P(-x,-y)$: $2f(x+f(x+y))=2x+f(x+y))$ ~~$\Rightarrow f(x+f(x+y))=x$~~ $f(x+f(x+y))=x+f(x+y)$. Now letting $y=-x$, we have $f(x)=x\ \forall x$.*

$\therefore$ The sol^{n}s are: $f(x)=2-x\ \forall x$, or $f(x)=x\ \forall x$. Checking, both work.

$f(x)=2-x$: LHS=RHS=$2+y-xy$; $f(x)=x$: LHS=RHS=$2x+y+xy$.

~~*Alternatively, ⑦−⑤:~~

*Alternatively, ⑦−⑤: $f(y+1)-f(-y-1)=2y+2 \Rightarrow 2f(y+1)=2y+2 \Rightarrow f(y+1)=y+1 \Rightarrow f(x)=x\ \forall x$.

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$f(x+f(x+y))+f(xy)=x+f(x+y)+yf(x)$.

$x=0$: $f(f(y))+f(0)=f(y)+yf(0) \Rightarrow f(f(y))=f(y)+(y-1)f(0) \Rightarrow f(f(1))=f(1)$.

$x=y=0$: $f(f(0))+f(0)=f(0) \Rightarrow f(f(0))=0$.

$y=0$: $f(x+f(x))+f(0)=x+f(x)$.

$x=1$: $f(f(y+1)+1)+f(y)=1+f(y+1)+yf(1)$.

$y=1$: $f(x+f(x+1))+f(x)=x+f(x+1)+f(x) \Rightarrow f(x+f(x+1))=x+f(x+1)$.

f clearly unbounded. (vary y) $\quad f(2f(1)-1)=2f(1)-1$.

$f(f(2))=f(2)+f(0)$.

nts either $f(x)=x$ or $f(f(x))=x$.

$x=x-f(x)$, $y=f(x)$: $f(x)+f((x-f(x))f(x))=x+f(x)+f(x)f(x-f(x))$. ($-f(x)$)

$\Downarrow$

$f((x-f(x))f(x))=x+f(x)f(x-f(x))$. ($-f(x)$)

$y=-1$: $f(x+f(x-1))+f(-x)=x+f(x-1)-f(x)$ ~~$x \to x+1$~~ $\quad f(x+1+f(x))+f(-x-1)=x+1+f(x)-f(x+1)$.

$y=-x$: $f(x+f(0))+f(-x^2)=x+f(0)-xf(x)$. $x=-1$: $f(-1+f(0))=-1+f(0)$. $\quad$ nts $f(x+f(x)+1)=x+f(x)+1$.

$x+f(x+1)$ fixed: done. Assume otherwise. $\to f(x)+f(-x^2)=x-xf(x)$.

$\hookrightarrow f(x)=x+2$. $\quad f(-1)+f(-1)=-1+f(-1) \Rightarrow f(-1)=-1$. $\quad f(1)+f(-1)=1-f(1) \Rightarrow f(1)=1$.

$xy=x+y \Rightarrow (x-1)(y-1)=1 \Rightarrow y=\frac{1}{x-1}+1=\frac{x}{x-1}$; $f\left(x+f\left(\frac{x^2}{x-1}\right)\right)+f\left(\frac{x^2}{x-1}\right)=x+f\left(\frac{x^2}{x-1}\right)+\frac{x}{x-1}f(x)$.

c s.t. $f(c)=c$. $f(f(c))=f(c)+(c-1)f(0) \Rightarrow f(0)=0$. (at least 2 values of c)

$\Rightarrow f(f(x))=f(x)$, $f(x+f(x))=x+f(x)$, $f(x+f(x+1))=x+f(x+1)$ $\quad f(x+f(x)-1)=x+f(x)-1$.

$f(2f(x))=2f(x)$. $\quad x \to f(x)-1$: $f(2f(x)-1)=2f(x)-1$.

$y=f(x)-x$: $f(x+f(f(x)))+f(x(f(x)-x))=x+f(x)+f(x)(f(x)-x) \Rightarrow f(x(f(x)-x))=f(x)(f(x)-x)$.

$y=f(x+1)-x$: $f(x+f(x+1))+f(x(f(x+1)-x))=x+f(x+1)+(f(x+1)-x)f(x) \Rightarrow f(x(f(x+1)-x))=(f(x+1)-x)f(x)$.

$y \to f(x+y)-x$: $f(x+f(x+y))+f(x(f(x+y)-x))=x+f(x+y)+(f(x+y)-x)f(x) \Rightarrow f(x(f(x+y)-x))-f(xy)=(f(x+y)-x-y)f(x)$.

$x=-1$: $f(f(y-1)-1)+f(-y)=f(y-1)-1+yf(-1)$. $\quad y \to y-x$ $\quad x=1$: $f(f(y+1)-1)-f(y)=(f(y+1)-y-1)f(1)$.

$f(x+f(y))+f(x(y-x))=x+f(y)+(y-x)f(x)$.

$f(y+f(x+y))+f(xy)=xy+f(x+y)+xf(y)$.

$f(f(x)+f(f(x)+f(y)))+f(f(x)f(y))=f(x)+f(f(x)+f(y))+f(x)f(y)$. swap: $f(f(x)+f(f(x)+f(y)))-f(x)=f(f(y)+f(f(x)+f(y)))-f(y)$.

$x=y$: $f(x+f(2x))+f(x^2)=x+f(2x)+xf(x) \Rightarrow f(3f(x))+f(f(x)^2)=3f(x)+f(x)^2$.

Sol^{n}s

$f(x)=x$

$f(x)=2-x$

$ax+b$:

$ax+a^2x+a^2y+ab+b+axy+b$

$=x+ax+ay+b+axy+by$.

x: $a+a^2=1+a \Rightarrow a=\pm 1$

y: $a^2=a+b \Rightarrow b=a^2-a$.

c: $ab+b=0 \Rightarrow b=0$ or $a=-1$

$a=-1 \Rightarrow b=2$.

$f(2-y)=y$ $\quad$ yep.

$y+2-xy=x+2-x-y+2x-xy$

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$f(x)+f(-x^2)=x-xf(x) \Rightarrow (x+1)f(x)-x=-f(-x^2)$.
$(1-x)f(-x)+x=-f(-x^2)$. $\Big\}$ $(x+1)f(x)+(x-1)f(-x)=2x$.

$f(0)=0, f(1)=1, f(-1)=-1$.

$x=1: f(f(y+1)+1)+f(y)=f(y+1)+y+1$

$f(f(y)+1)+f(y-1)=f(y)+y$.

$f(f(y)+1)+f(f(y)-1)=2f(y)$.

$f(f(y+1)-1)-f(y)=f(y+1)-y-1$.

$f(f(y)-1)-f(y-1)=f(y)-y$.

$x=-1: f(f(y-1)-1)+f(-y)=-1+f(y-1)-y$.

$f(f(y)-1)+f(-y-1)=f(y)-y-2$.

$f(y)-y+f(y-1)+f(-y-1)=f(y)-y-2$.

$f(y-1)+f(-y-1)=-2$.

$f(y)+f(-y-2)=-2$.

$-2-f(f(y-1)+1)$
$\quad ||$
$f(-3-f(y-1))$

$x=-1, y=1-y: f(-1+f(-y))+f(y-1)=-1+f(-y)+y-1 \Rightarrow f(-1+f(-y-1))+f(y)=f(-y-1)+y-1$.

$y=f(y)+1: f(f(y)-1)+f(f(y)+1)=-1+f(y)-f(y)-1$.

$-f(f(y-1)+1)+f(y)=-f(y-1)+y-1$.

$f(f(y-1)+1)=f(y-1)+f(y)+1-y$.

$f(f(y)+1)=f(y)+f(y+1)-y$.

$f(y+1)+f(y-1)=2y$

$f(x+f(x-y))-f(xy)=x+f(x-y)-yf(x)$.

$f(-x+f(y-x))-f(xy)=-x+f(y-x)-yf(x)$.

$f(-x-f(x+y))+f(xy)=-x-f(x+y)+yf(x)$
$\quad ||$
$-f(x+f(x+y))$

$f(x+f(x+y))=x+f(x+y)$.

Problem 6. The sequence $a_1, a_2, \ldots$ of integers satisfies the following conditions:

(i) $1 \leqslant a_j \leqslant 2015$ for all $j \geqslant 1$;

(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leqslant k < \ell$.

Prove that there exist two positive integers b and N such that

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leqslant 1007^2$$

for all integers m and n satisfying $n > m \geqslant N$.

Solution 1. We visualize the set of positive integers as a sequence of points. For each n we draw an arrow emerging from n that points to $n + a_n$; so the *length* of this arrow is a_n. Due to the condition that $m + a_m \neq n + a_n$ for $m \neq n$, each positive integer receives at most one arrow. There are some positive integers, such as 1, that receive no arrows; these will be referred to as *starting points* in the sequel. When one starts at any of the starting points and keeps following the arrows, one is led to an infinite path, called its *ray*, that visits a strictly increasing sequence of positive integers. Since the length of any arrow is at most 2015, such a ray, say with starting point s, meets every interval of the form $[n, n + 2014]$ with $n \geqslant s$ at least once.

Suppose for the sake of contradiction that there would be at least 2016 starting points. Then we could take an integer n that is larger than the first 2016 starting points. But now the interval $[n, n + 2014]$ must be met by at least 2016 rays in distinct points, which is absurd. We have thereby shown that the number b of starting points satisfies $1 \leqslant b \leqslant 2015$. Let N denote any integer that is larger than all starting points. We contend that b and N are as required.

To see this, let any two integers m and n with $n > m \geqslant N$ be given. The sum $\sum_{i=m+1}^{n} a_i$ gives the total length of the arrows emerging from $m + 1, \ldots, n$. Taken together, these arrows form b subpaths of our rays, some of which may be empty. Now on each ray we look at the first number that is larger than m; let $x_1, \ldots, x_b$ denote these numbers, and let $y_1, \ldots, y_b$ enumerate in corresponding order the numbers defined similarly with respect to n. Then the list of differences $y_1 - x_1, \ldots, y_b - x_b$ consists of the lengths of these paths and possibly some zeros corresponding to empty paths. Consequently, we obtain

$$\sum_{i=m+1}^{n} a_i = \sum_{j=1}^{b} (y_j - x_j),$$

whence

$$\sum_{i=m+1}^{n} (a_i - b) = \sum_{j=1}^{b} (y_j - n) - \sum_{j=1}^{b} (x_j - m).$$

Now each of the b rays meets the interval $[m + 1, m + 2015]$ at some point and thus $x_1 - m, \ldots, x_b - m$ are b distinct members of the set $\{1, 2, \ldots, 2015\}$. Moreover, since $m + 1$ is not a starting point, it must belong to some ray; so 1 has to appear among these numbers, wherefore

$$1 + \sum_{j=1}^{b-1} (j + 1) \leqslant \sum_{j=1}^{b} (x_j - m) \leqslant 1 + \sum_{j=1}^{b-1} (2016 - b + j).$$

The same argument applied to n and $y_1, \ldots, y_b$ yields

$$1 + \sum_{j=1}^{b-1} (j + 1) \leqslant \sum_{j=1}^{b} (y_j - n) \leqslant 1 + \sum_{j=1}^{b-1} (2016 - b + j).$$

So altogether we get

$$\left|\sum_{i=m+1}^{n}(a_i-b)\right| \leqslant \sum_{j=1}^{b-1}\big((2016-b+j)-(j+1)\big)=(b-1)(2015-b)$$
$$\leqslant \left(\frac{(b-1)+(2015-b)}{2}\right)^2 = 1007^2,$$

as desired.

Solution 2. Set $s_n = n + a_n$ for all positive integers n. By our assumptions, we have

$$n+1 \leqslant s_n \leqslant n+2015$$

for all $n \in \mathbb{Z}_{>0}$. The members of the sequence $s_1, s_2, \ldots$ are distinct. We shall investigate the set

$$M = \mathbb{Z}_{>0} \setminus \{s_1, s_2, \ldots\}.$$

Claim. At most 2015 numbers belong to M.

Proof. Otherwise let $m_1 < m_2 < \cdots < m_{2016}$ be any 2016 distinct elements from M. For $n = m_{2016}$ we have

$$\{s_1, \ldots, s_n\} \cup \{m_1, \ldots, m_{2016}\} \subseteq \{1, 2, \ldots, n+2015\},$$

where on the left-hand side we have a disjoint union containing altogether $n + 2016$ elements. But the set on the right-hand side has only $n + 2015$ elements. This contradiction proves our claim. ☐

Now we work towards proving that the positive integers $b = |M|$ and $N = \max(M)$ are as required. Recall that we have just shown $b \leqslant 2015$.

Let us consider any integer $r \geqslant N$. As in the proof of the above claim, we see that

$$B_r = M \cup \{s_1, \ldots, s_r\} \tag{1}$$

is a subset of $[1, r+2015] \cap \mathbb{Z}$ with precisely $b + r$ elements. Due to the definitions of M and N, we also know $[1, r+1] \cap \mathbb{Z} \subseteq B_r$. It follows that there is a set $C_r \subseteq \{1, 2, \ldots, 2014\}$ with $|C_r| = b - 1$ and

$$B_r = \big([1, r+1] \cap \mathbb{Z}\big) \cup \big\{r + 1 + x \,\big|\, x \in C_r\big\}. \tag{2}$$

For any finite set of integers J we denote the sum of its elements by $\sum J$. Now the equations (1) and (2) give rise to two ways of computing $\sum B_r$ and the comparison of both methods leads to

$$\sum M + \sum_{i=1}^{r} s_i = \sum_{i=1}^{r} i + b(r+1) + \sum C_r,$$

or in other words to

$$\sum M + \sum_{i=1}^{r}(a_i - b) = b + \sum C_r. \tag{3}$$

After this preparation, we consider any two integers m and n with $n > m \geqslant N$. Plugging $r = n$ and $r = m$ into (3) and subtracting the estimates that result, we deduce

$$\sum_{i=m+1}^{n}(a_i - b) = \sum C_n - \sum C_m.$$

Since C_n and C_m are subsets of $\{1, 2, \ldots, 2014\}$ with $|C_n| = |C_m| = b - 1$, it is clear that the absolute value of the right-hand side of the above inequality attains its largest possible value if either $C_m = \{1, 2, \ldots, b-1\}$ and $C_n = \{2016 - b, \ldots, 2014\}$, or the other way around. In these two cases we have

$$\left|\sum C_n - \sum C_m\right| = (b-1)(2015-b),$$

so in the general case we find

$$\left|\sum_{i=m+1}^{n} (a_i - b)\right| \leqslant (b-1)(2015-b) \leqslant \left(\frac{(b-1)+(2015-b)}{2}\right)^2 = 1007^2,$$

as desired.

Comment 1. The sets C_n may be visualized by means of the following process: Start with an empty blackboard. For $n \geqslant 1$, the following happens during the n^{th} step. The number a_n gets written on the blackboard, then all numbers currently on the blackboard are decreased by 1, and finally all zeros that have arisen get swept away.

It is not hard to see that the numbers present on the blackboard after n steps are distinct and form the set C_n. Moreover, it is possible to complete a solution based on this idea.