Singapore International Mathematical Olympiad 2015
National Team Selection Test
Day 0

1. Let O and G denote the circumcentre and the centroid of the triangle ABC respectively. Let the perpendicular bisectors of AG, BG, CG intersect mutually at D, E, F respectively. Show that O is the centroid of the triangle DEF.

2. Let n be a positive integer. Suppose there are n coins with values \$1, \$2, \$3, …, \$$n$. Person A arranges the coins in a straight line. Thereafter, Person A and Person B alternatively take one coin from one of the ends of the straight line with Person B starting first until all the coins are exhausted. Suppose both Person A and Person B are equally intelligent and both want to maximize the amount of money they can take. Determine who will take more money and by how much.

3. Let n be an odd positive integer greater than 3. Let k denote the least positive integer such that $kn+1$ is a perfect square and m the least positive integer such that mn is a perfect square. Prove that n is a prime number if and only if $k > \frac{n}{4}$ and $m > \frac{n}{4}$.

4. The sequence $(a_n)_{n\geq 1}$ of natural numbers satisfies the following relation:

$$a_{n+2} = \left\lfloor \frac{2a_{n+1}}{a_n} \right\rfloor + \left\lfloor \frac{2a_n}{a_{n+1}} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x. Prove that there exists a natural number m such that $a_m = 4$ and $a_{m+1} = \{3, 4\}$.

Time allowed: 4 hours

Singapore International Mathematical Olympiad 2015
National Team Selection Test
Day 1

5. Let $n \geq 2$ be an integer, and let A_n be the set
$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leq k < n\}.$$
For each n, determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

6. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$.

7. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as
$$\max_{1 \leq i \leq n} |x_1 + \cdots + x_i|.$$
Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus in the i^{th} step he chooses x_i among the remaining numbers so as to minimize the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

   Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence George might obtain, the resulting values satisfy the inequality $G \leq cD$.

Time allowed: 4.5 hours

Singapore International Mathematical Olympiad 2015
National Team Selection Test
Day 2

8. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

9. Define the function $f : (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

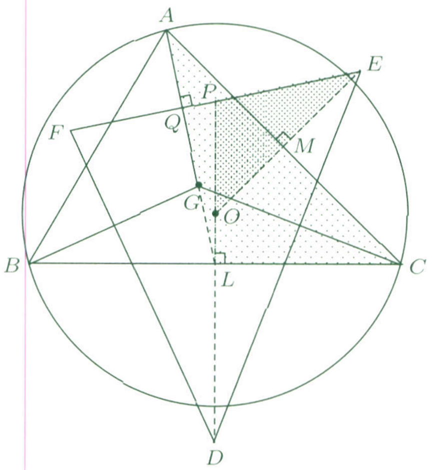
10. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

Time allowed: 4.5 hours

Solutions to National Team Selection Test 2015

1. Let O and G denote the circumcentre and the centroid of the triangle ABC respectively. Let the perpendicular bisectors of AG, BG, CG intersect mutually at D, E, F respectively. Show that O is the centroid of the triangle DEF.

   Solution. Refer to the figure below. It is easy to show that D, E, F are the circumcentres of the triangles BCG, ACG, ABG respectively. Let N, M, L be the midpoints of AB, AC, BC respectively. It is easy to see that the lines DL, EM, FN intersect at O. Let DL extended intersects EF at P. It suffices to show that P is the midpoint of EF.



   Let AG intersect EF at Q. Then A, E, M, Q are concyclic and hence $\angle CAL = \angle OEP$. Similarly, C, L, O, M are concyclic and $\angle ACL = \angle EOP$. It follows that the triangles ACL and EOP are similar and hence, $EP/AL = OP/CL$. Similarly, the triangles ABL and FOP are similar and we have $FP/AL = OP/BL$. Since $BL = CL$, these two equations imply that $EP = FP$. This completes the proof.

2. Let n be a positive integer. Suppose there are n coins with values \$1, \$2, \$3, $\ldots$, \$$n$. Person A arranges the coins in a straight line. Thereafter, Person A and Person B alternatively take one coin from one of the ends of the straight line with Person B starting first until all the coins are exhausted. Suppose both Person A and Person B are equally intelligent and both want to maximize the amount of money they can take. Determine who will take more money and by how much.

   Solution. Label the positions of the coins as $1, 2, 3, \ldots, n$.

   Case 1: $n = 2m + 1$ is odd.

When $n = 1$, obviously Person B gains \$1 over Person A. Let $n = 2m+1 > 1$. Person A can always force Person B to take coins from the odd positions by taking coins from even positions at every step. So he will arrange coins with values $\$1, \$2, \cdots, \$(m+1)$ in the odd positions and coins with values $\$(m+2), \$(m+3), \ldots, \$(2m+1)$ at the even positions.

Hence wealth of Person A less wealth of Person $B = (2m+1)(m+1) - (m+1)(m+2) = m^2 - 1$.

That is, Person A will have the winning strategy for all odd values of $n > 3$ with a gain of $m^2 - 1$ over Person B. When $n = 3$, the coins will be arranged as $\$1, \$3, \$2$, and both of them will collect the same amount of cash.

Case 2: $n = 2m$ is even.

Person B can always force Person A to take all the coins from the odd positions by taking coins from even positions at every step, or Person B can force Person A to take all coins from the even positions by taking coins from odd positions at every step.

Let Person A arranges the coins in the order: $\$a_1, \$b_1, \ldots, \$a_m, \$b_m$.

If $\sum_{i=1}^{m} a_i \geq \sum_{i=1}^{m} b_i$, Person B will force Person A to take all the coins with values b_i's in the even positions. If $\sum_{i=1}^{m} b_i \geq \sum_{i=1}^{m} a_i$, Person B will force Person A to take all the coins with values a_i's in the odd positions.

Since $\sum_{i=1}^{m} a_i + \sum_{i=1}^{m} b_i = \frac{n(n+1)}{2}$, Person A can take at most half of the sum.

When $n \equiv 0 \pmod 4$, Person A will arrange the coins in the following order:

$$\$1, \$3, \$5, \$7, \ldots, \$n-3, \$n-1, \$n, \$n-2, \ldots, \$8, \$6, \$4, \$2$$

Then A and B take equal amount of the sum of the coins.

When $n \equiv 2 \pmod 4$, Person A will arrange the coins in the following order:

$$\$1, \$3, \$5, \$7, \ldots, \$n-5, \$n-3, \$n-1, \$n, \$n-2, \$n-4, \ldots, \$8, \$6, \$4, \$2$$

In this case, Person B gains \$1 over Person A.

3. Let n be an odd positive integer greater than 3. Let k denote the least positive integer such that $kn+1$ is a perfect square and m the least positive integer such that mn is a perfect square. Prove that n is a prime number if and only if $k > \frac{n}{4}$ and $m > \frac{n}{4}$.

Solution. If $n = p$ is a prime number, then $m = p$ so it is clear that $m > \frac{n}{4}$. For k we have $kp = (y-1)(y+1)$, so

$$y - 1 = k_1, y + 1 = k_2 p \quad \text{or} \quad y - 1 = k_1 p, y + 1 = k_2.$$

In either case, we have $2y > p$. If $4k \leq p$, then we would have

$$y^2 = kp + 1 < \frac{p^2}{4} + 1 \Rightarrow p^2 < 4y^2 < p^2 + 4,$$

which is impossible since there are no perfect squares between p^2 and $p^2 + 4$.

We will now show there does not exist an odd composite number $n > 3$ such that $4m > n$ and $4k > n$. Suppose such an n exists. Then we can write

$$n = x^2 p_1 \cdots p_s,$$

with $p_1, \cdots, p_s$ different prime numbers. Clearly, $m = p_1 \cdots p_s$ and we have

$$4p_1 \cdots p_s = 4m > n = x^2 p_1 \cdots p_s.$$

This implies $x = 1$ and thus

$$n = p_1 \cdots p_s.$$

Let y denote the smallest number greater than 1 such that $y^2 - 1$ is divisible by n. Clearly, $kn + 1 = y^2$, and it is easy to see that $k > \frac{n}{4} \Leftrightarrow 2y > n$.

Now we can write n in the form $n = pr$, where $p = p_i$ for some i, and r is the product of the remaining p_j, for $j \neq i$. Since n is composite $r > 1$. By the Chinese remainder theorem, there exists a unique T, $0 \leq T < n$ such that $T \equiv 1 \pmod{r}$ and $T \equiv -1 \pmod{p}$. Now consider the number $S = n - T$. We have $S \equiv -1 \pmod{r}$ and $S \equiv 1 \pmod{p}$ so $T^2 \equiv S^2 \equiv 1 \pmod{n}$. Thus both T and S are candidates for y. Clearly, one of them is less than $\frac{n}{2}$, so $k < \frac{n}{4}$, a contradiction.

4. The sequence $(a_n)_{n \geq 1}$ of natural numbers satisfies the following relation:

$$a_{n+2} = \left\lfloor \frac{2a_{n+1}}{a_n} \right\rfloor + \left\lfloor \frac{2a_n}{a_{n+1}} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x. Prove that there exists a natural number m such that $a_m = 4$ and $a_{m+1} = \{3, 4\}$.

Solution. Let $n \geq 3$. Then $a_n = \left\lfloor \frac{2a_{n-1}}{a_{n-2}} \right\rfloor + \left\lfloor \frac{2a_{n-2}}{a_{n-1}} \right\rfloor > \frac{2a_{n-1}}{a_{n-2}} + \frac{2a_{n-2}}{a_{n-1}} - 2 \geq 4 - 2 = 2$. Thus $a_n \geq 3$ for all $n \geq 3$.

Next we claim that for all $n \geq 3$,

$$a_{n+1} = a_n \quad \text{or} \quad a_{n+2} < \max\{a_n, a_{n+1}\}.$$

Suppose that $a_{n+1} \neq a_n$ for some $n \in \mathbb{N}$. If $a_n = \max\{a_n, a_{n+1}\}$, then $\frac{2a_{n+1}}{a_n} < 2$. On the other hand,

$$\begin{aligned} a_{n+1}, a_n \geq 3 &\Rightarrow \tfrac{2a_n}{a_{n+1}} \leq \tfrac{2a_n}{3} \\ &\Rightarrow a_{n+2} = \left\lfloor \tfrac{2a_{n+1}}{a_n} \right\rfloor + \left\lfloor \tfrac{2a_n}{a_{n+1}} \right\rfloor \leq 1 + \tfrac{2a_n}{3} \leq \tfrac{a_n}{3} + \tfrac{2a_n}{3} = a_n. \end{aligned}$$

Now if $a_{n+2} \geq \max\{a_n, a_{n+1}\} = a_n$, the above inequalities will be equalities and thus $a_n = a_{n+1} = 3$ which contradicts our assumption that $a_n \neq a_{n+1}$. The case where $a_{n+1} = \max\{a_n, a_{n+1}\}$ is dealt with similarly.

Lastly, we claim that there is some natural number k such that $a_k = a_{k+1}$. Suppose not. Then from the above, we have

$$a_{n+2} < \max\{a_n, a_{n+1}\},$$

and

$$a_{n+3} < \max\{a_{n+1}, a_{n+2}\} < \max\{a_n, a_{n+1}\}.$$

Therefore,

$$\max\{a_{n+2}, a_{n+3}\} < \max\{a_n, a_{n+1}\}.$$

Thus $\max\{a_{2k}, a_{2k+1}\}$ is a strictly decreasing sequence of natural numbers, which is impossible.

Therefore there exists some natural number $k \geq 1$ such that $a_k = a_{k+1}$. This implies $a_{k+2} = 4$. Now if $a_k = a_{k+1} = \{3, 4\}$, then $a_{k+3} \in \{3, 4\}$ and we are done. If not, then $a_k = a_{k+1} > 4$, and there exists some natural number m such that $a_{k+m} = a_{k+m+1}$. Using a similar argument to above, we have if m is odd,

$$\begin{aligned} a_{k+1} &= \max\{a_{k+1}, a_{k+2} = 4\} \\ &> \max\{a_{k+3}, a_{k+4}\} > \cdots > \max\{a_{k+m}, a_{k+m+1}\} \geq a_{k+m}. \end{aligned}$$

If m is even,

$$\begin{aligned} a_{k+1} &= \max\{a_{k+1}, a_{k+2} = 4\} \\ &> \max\{a_{k+3}, a_{k+4}\} > \cdots > \max\{a_{k+m-1}, a_{k+m}\} \geq a_{k+m}. \end{aligned}$$

Therefore, if two equal consecutive terms in the sequence are greater than 4, then there is another pair of equal consecutive terms in the sequence with a smaller value. We can continue this until we find two equal consecutive terms which are less than or equal to 4. This completes the proof as the next terms after the pair are 4, 3 or 4, 4.

5. Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leq k < n\}.$$

For each n, determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

Solution. Ans: $(n-2)2^n + 1$.

Part 1. First we show that every integer greater $(n-2)2^n+1$ can be represented as such a sum. This is achieved by induction on n. For $n=2$, the set A_n consists of the two elements 2 and 3. Every positive integer m except for 1 can be represented as the sum of elements of A_n in this case: as $m = 2+2+\cdots+2$ if m is even, and as $m = 3+2+2+\cdots+2$ if m is odd.

Now consider some $n>2$, and take an integer $m>(n-2)2^n+1$. If m is even, then consider

$$\frac{m}{2} \geq \frac{(n-2)2^n+2}{2} = (n-2)2^{n-1}+1 > (n-3)2^{n-1}+1.$$

By induction hypothesis, there is a representation of the form

$$\frac{m}{2} = (2^{n-1}-2^{k_1}) + (2^{n-1}-2^{k_2}) + \cdots + (2^{n-1}-2^{k_r})$$

for some k_i with $0 \leq k_i < n-1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}),$$

giving us the desired representation as a sum of elements of A_n.

If m is odd, consider

$$\frac{m-(2^n-1)}{2} > \frac{(n-2)2^n+1-(2^n-1)}{2} = (n-3)2^{n-1}+1.$$

By induction hypothesis, there is a representation of the form

$$\frac{m-(2^n-1)}{2} = (2^{n-1}-2^{k_1}) + (2^{n-1}-2^{k_2}) + \cdots + (2^{n-1}-2^{k_r})$$

for some k_i with $0 \leq k_i < n-1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}) + (2^n-1),$$

giving us the desired representation of m once again.

Part 2. It remains to show that there is no representation for $(n-2)2^n+1$. Note that $(n-2)2^n+1 \equiv 1 \pmod{2^n}$. Let N be the smallest positive integer that satisfies $N \equiv 1 \pmod{2^n}$, and which can be represented as a sum of elements of A_n. Consider a representation of N, i.e.,

$$N = (2^n-2^{k_1}) + (2^n-2^{k_2}) + \cdots + (2^n-2^{k_r}), \tag{5.1}$$

where $0 \leq k_1, k_2, \ldots, k_r < n$. Suppose first that two of the terms in the sum are the same, i.e., $k_i = k_j$ for some $i \neq j$. If $k_i = k_j = n-1$, then we can simply remove these two terms to get a representation for

$$N - 2(2^n - 2^{n-1}) = N - 2^n$$

as a sum of elements of A_n, which contradicts our choice of N. If $k_i = k_j = k < n-1$, replace the two terms by $2^n - 2^{k+1}$, which is also an element of A_n, to get a representation for

$$N - 2(2^n - 2^k) + 2^n - 2^{k+1} = N - 2^n.$$

This is a contradiction once again to the minimality of N. Therefore, all k_i have to be distinct, which means that

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \leq 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

One the other hand, taking (5.1) modulo 2^n, we find that

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \equiv -N \equiv -1 \pmod{2^n}.$$

Thus we must have $2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} = 2^n - 1$, which is only possible if each element of $\{0, 1, \ldots, n-1\}$ occurs as one of the k_i. This gives us

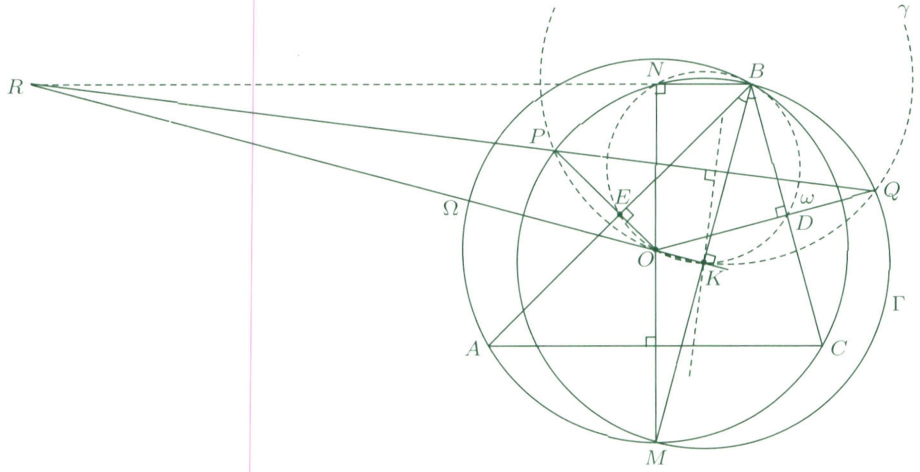
$$N = n2^n - (2^0 + 2^1 + \cdots + 2^{n-1}) = (n-1)2^n + 1.$$

In particular, this means that $(n-2)2^n + 1$ cannot be represented as a sum of elements of A_n.

6. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$.

   Solution. Let K be the midpoint of BM, i.e., the centre of Γ. Notice that $AB \neq BC$ implies that $K \neq O$. Clearly, the lines OM and OK are perpendicular bisectors of AC and MB, respectively. Therefore R is the intersection of the lines PQ and OK.

   Let N be the second point of intersection of Γ with the line OM. Since BM is a diameter of Γ, the lines BN and AC are both perpendicular to OM. Hence $BN \parallel AC$, and it suffices to prove that BN passes through R. Our plan for doing this is to interpret the lines BN, OK and PQ as the radical axes of three appropriate circles.



Let ω be the circle with diameter BO. Since $\angle BNO = \angle BKO = 90°$, the points N and K lie on K.

Next we show that the points O, K, P and Q are concyclic. To this end, let D and E be the midpoints of BC and AB, respectively. Clearly, D and E lie on the rays OQ and OP, respectively. By our assumptions about the triangles ABC, the points B, E, O, K and D lie in this order on ω. It follows that $\angle EOR = \angle EBK = \angle KBD = \angle KOD$, so the line KO externally bisects $\angle POQ$. Since the point K is the centre of Γ, it also lies on the perpendicular bisector of PQ. So K coincides with the midpoint of the arc POQ of the circumcircle γ of the triangle POQ.

Thus the lines OK, BN and PQ are pairwise radical axes of the circles ω, γ and Γ. Hence they are concurrent at R, as required.

7. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \le i \le n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus in the i^{th} step he chooses x_i among the remaining numbers so as to minimize the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence George might obtain, the resulting values satisfy the inequality $G \le cD$.

Solution. Ans: $c = 2$.

If the initial numbers are $1, -1, 2$ and -2, then Dave may arrange them as $1, -2, 2, -1$, while George may get the sequence $1, -1, 2, -2$, resulting in $D = 1$ and $G = 2$. So we obtain $c \geq 2$.

Therefore, it remains to prove that $G \leq 2D$. Let $x_1, x_2, \ldots, x_n$ be the numbers Dave and George have at their disposal. Assume that Dave and George arrange them into sequences $d_1, d_2, \ldots, d_n$ and $g_1, g_2, \ldots, g_n$, respectively. Put

$$M = \max_{1 \leq i \leq n} |x_i|, \quad S = |x_1 + \cdots + x_n|, \quad \text{and} \quad N = \max\{M, S\}.$$

We claim that

$$S \leq D, \tag{7.1}$$

$$M \leq 2D, \tag{7.2}$$

$$G \leq N = \max\{M, S\}. \tag{7.3}$$

These inequalities yield the desired estimate, as

$$G \leq \max\{M, S\} \leq \max\{M, 2S\} \leq 2D.$$

The inequality (7.1) is a direct consequence of the definition of the price.

To prove (7.2), consider an index i with $|d_i| = M$. Then we have

$$\begin{aligned} M = |d_i| &= |(d_1 + \cdots + d_i) - (d_1 + \cdots + d_{i-1})| \\ &\leq |d_1 + \cdots + d_i| + |d_1 + \cdots + d_{i-1}| \leq 2D, \end{aligned}$$

as required.

It remains to establish (7.3). Put $h_i = g_1 + g_2 + \cdots + g_i$. We will prove by induction on i that $|h_i| \leq N$. The base case $i = 1$ holds, since $|h_1| = |g_1| \leq M \leq N$. Notice also that $|h_n| = S \leq N$.

For the induction step, assume that $|h_{i-1}| \leq N$. We distinguish two cases.

Case 1. Assume that no two of the numbers $g_i, g_{i+1}, \ldots, g_n$ have opposite signs.

Without loss of generality, we may assume that they are all nonnegative. Then one has $h_{i-1} \leq h_i \leq \cdots \leq h_n$, thus

$$|h_i| \leq \max\{|h_{i-1}|, |h_n|\} \leq N.$$

Case 2. Among the numbers $g_i, g_{i+1}, \ldots, g_n$ there are positive and negative ones.

Then there exists some index $j \geq i$ such that $h_{i-1}g_j \leq 0$. By the definition of George's sequence we have

$$|h_i| = |h_{i-1} + g_i| \leq |h_{i-1} + g_j| \leq \max\{|h_{i-1}|, |g_j|\} \leq N.$$

Thus, the induction step is established.

8. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

Solution 1. Let k be the number of rectangles in the dissection. The set of all points that are corners of exactly one of the rectangles can be divided into three disjoint subsets:

- A, which consists of the four corners of the original rectangle R, each of which is the corner of exactly one of the smaller rectangles,
- B, which contains points where exactly two of the rectangles have a common corner (T-junctions as in Figure 8.1),
- C, which contains points where four of the rectangles have a common corner (crossings as in Figure 8.1).

Figure 8.1: A T-junction and a crossing

We denote the number of points in B by b and the numbers of points in C by c. Since each of the k rectangles has exactly four corners, we get

$$4k = 4 + 2b + 4c.$$

It follows that $2b \leq 4k - 4$, so $b \leq 2k - 2$.

Each of the n given points has to lie on a side of one of the smaller rectangles (but not of the original rectangle R). If we extend this side as far as possible along borders between rectangles, we obtain a line segment whose ends are T-junctions. Note that every point in B can only be an endpoint of at most one such segment containing one of the given points, since it is stated that no two of them lie on a common line parallel to the sides of R. This means that

$$b \geq 2n.$$

Combining our two inequalities for b, we get

$$2k - 2 \geq b \geq 2n,$$

thus $k \geq n + 1$, which is what we wanted to prove.

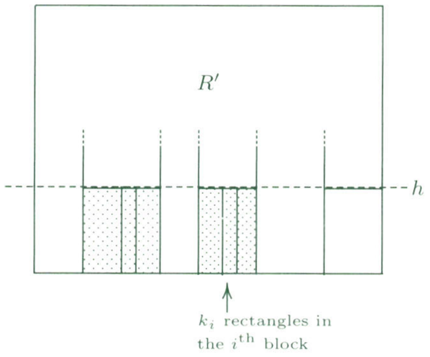
Solution 2. Let k denotes the number of rectangles. In the following, we refer to the directions of the sides of R as 'horizontal' and 'vertical' respectively. Our goal is to prove the inequality $k \geq n + 1$ for fixed n. Equivalently, we can prove the inequality $n \leq k - 1$ for each k, which will be done by induction on k. For $k = 1$, the statement is trivial.

Now assume that $k > 1$. If none of the line segments that form the borders between the rectangles is horizontal, then we have $k - 1$ vertical segments dividing R into k rectangles. On each of them, there can only be one of the n points, so $n \leq k - 1$, which is exactly what we want to prove.

Otherwise, consider the lowest horizontal line h that contains one or more of the these line segments. Let R' be the rectangle that results when everything that lies below h is removed from R. See the figure below.

The rectangles that lie entirely below h form blocks of rectangles separated by vertical line segments. Suppose there are r blocks and k_i rectangles in the i^{th} block. The left and right borders of each block has to extend further upwards beyond h. Thus we can move any points that lie on these borders upwards, so that they now lie inside R'. This can be done without violating the conditions, one only needs to make sure that they do not get to lie on a common horizontal line with one of the other given points.

All other borders between rectangles in the i^{th} block have to lie entirely below h. There are $k_i - 1$ such line segments, each of which can contain at most one of the given points. Finally, there can be one point that lies on h. All other points have to lie in R' (after moving some of them as explained in the previous paragraph).



We see that R' is divided into $k-\sum_{i=1}^{r} k_i$ rectangles. Applying the induction hypothesis to R', we find that there are at most

$$\left(k-\sum_{i=1}^{r} k_i\right)-1+\sum_{i=1}^{r}(k_i-1)+1=k-r$$

points in R. Since $r \geq 1$, this means that $n \leq k-1$, which completes our induction.

9. Define the function $f:(0,1) \to (0,1)$ by

$$f(x)=\begin{cases} x+\frac{1}{2} & \text{if } x<\frac{1}{2}, \\ x^2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0<a<b<1$. We define the sequences a_n and b_n by $a_0=a, b_0=b$, and $a_n=f(a_{n-1})$, $b_n=f(b_{n-1})$ for $n>0$. Show that there exists a positive integer n such that

$$(a_n-a_{n-1})(b_n-b_{n-1})<0.$$

Solution. Note that

$$f(x)-x=\frac{1}{2}>0$$

if $x<\frac{1}{2}$ and

$$f(x)-x=x^2-x<0$$

if $x \geq \frac{1}{2}$. So if we consider $(0,1)$ as being divided into two subintervals $I_1=(0,\frac{1}{2})$ and $I_2=[\frac{1}{2},1)$, the inequality

$$(a_n-a_{n-1})(b_n-b_{n-1})=(f(a_{n-1})-a_{n-1})(f(b_{n-1})-b_{n-1})<0$$

holds if and only if a_{n-1} and b_{n-1} lie in distinct subintervals.

Let us now assume, to the contrary, that a_k and b_k always lie in the same subinterval. Consider the distance $d_k=|a_k-b_k|$. If both a_k and b_k lie in I_1, then

$$d_{k+1}=|a_{k+1}-b_{k+1}|=|a_k+\frac{1}{2}-b_k-\frac{1}{2}|=d_k.$$

If, on the other hand, a_k and b_k both lie in I_2, then $\min(a_k,b_k) \geq \frac{1}{2}$ and $\max(a_k,b_k)=\min(a_k,b_k)+d_k \geq \frac{1}{2}+d_k$, which implies

$$\begin{aligned} d_{k+1} &= |a_{k+1}-b_{k+1}|=|a_k^2-b_k^2|=|(a_k-b_k)(a_k+b_k)| \\ &\geq |a_k-b_k|(\tfrac{1}{2}+\tfrac{1}{2}+d_k)=d_k(1+d_k) \geq d_k. \end{aligned}$$

This means the difference d_k is non-decreasing, and in particular $d_k \geq d_0 > 0$ for all k.

We can even say more. If a_k and b_k lie in I_2, then

$$d_{k+2} \geq d_{k+1} \geq d_k(1+d_k) \geq d_k(1+d_0).$$

If a_k and b_k lie in I_1, then a_{k+1} and b_{k+1} both lie in I_2, and so we have

$$d_{k+2} \geq d_{k+1}(1+d_{k+1}) \geq d_{k+1}(1+d_0) = d_k(1+d_0).$$

In either case, $d_{k+2} \geq d_k(1+d_0)$, and inductively we get

$$d_{2m} \geq d_0(1+d_0)^m.$$

For sufficiently large m, the right-hand side is greater than 1, but a_{2m} and b_{2m} both lie in $(0,1)$, we must have $d_{2m} < 1$, a contradiction.

Thus there must be a positive integer n such that a_{n-1} and b_{n-1} do not lie in the same subinterval, which proves the desired statement.

10. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

Solution. Ans: $(p, x, y) \in \{(3,2,5), (3,5,2)\} \cup \{(2, n, 2^k - n) \mid 0 < n < 2^k\}$.

For $p = 2$, clearly all pairs of two positive integers x and y whose sum is a power of 2 satisfy the condition. Thus we assume in the following that $p > 2$, and we let a and b be positive integers such that $x^{p-1} + y = p^a$ and $x + y^{p-1} = p^b$. Assume further, without loss of generality, that $x \leq y$, so that $p^a = x^{p-1} + y \leq x + y^{p-1} = p^b$, which means $a \leq b$ and thus $p^a \mid p^b$.

Now we have

$$p^b = y^{p-1} + x = (p^a - x^{p-1})^{p-1} + x.$$

We take this equation modulo p^a and take into account that $p-1$ is even, which gives us

$$0 \equiv x^{(p-1)^2} + x \pmod{p^a}.$$

If $p \mid x$, then $p^a \mid x$, since $x^{(p-1)^2-1} + 1$ is not divisible by p in this case. However, this is impossible, since $x \leq x^{p-1} < p^a$. Thus we know that $p \nmid x$, which means that

$$p^a \mid x^{(p-1)^2-1} + 1 = x^{p(p-2)} + 1.$$

By Fermat's little theorem, $x^{(p-1)^2} \equiv 1 \pmod{p}$, thus $p \mid (x+1)$. Let p^r be the highest power of p that divides $x+1$. Thus $r \geq 1$. By the binomial theorem, we have

$$x^{p(p-2)} = \sum_{k=0}^{p(p-2)} \binom{p(p-2)}{k} (-1)^{p(p-2)-k} (x+1)^k.$$

Except for the terms corresponding to $k = 0, k = 1$ and $k = 2$, all terms in the sum are clearly divisible by p^{3r} and thus by p^{r+2}. The remaining terms are:

$$-\frac{p(p-2)(p^2-2p-1)}{2}(x+1)^2,$$

which is divisible by p^{2r+1} and thus also by p^{r+2};

$$p(p-2)(x+1),$$

which is divisible by p^{r+1}, but not p^{r+2} by our choice of r; and the final term -1 corresponding to $k = 0$. It follows that the highest power of p that divides $x^{p(p-2)} + 1$ is p^{r+1}.

On the other hand, we already know that p^a divides $x^{p(p-2)}+1$, which means that $a \leq r + 1$. Moreover,

$$p^r \leq x + 1 \leq x^{p-1} + y = p^a.$$

Hence we either have $a = r$ or $a = r + 1$.

If $a = r$, then $x = y = 1$ needs to hold in the inequality above, which is impossible for $p > 2$ as $p \mid x + 1$. Thus $a = r + 1$. Now since $p^r \leq x + 1$, we get

$$x = \frac{x^2 + x}{x + 1} \leq \frac{x^{p-1} + y}{x + 1} = \frac{p^a}{x + 1} \leq \frac{p^a}{p^r} = p,$$

so we must have $x = p - 1$ for p to divide $x + 1$.

It follows that $r = 1$ and $a = 2$. If $p \geq 5$, we obtain using $x = p - 1$ that

$$p^a = x^{p-1} + y > (p-1)^4 = (p^2 - 2p + 1)^2 > (3p)^2 > p^2 = p^a,$$

a contradiction. So the only case that remains is $p = 3$, and indeed $x = 2$ and $y = p^a - x^{p-1} = 3^2 - 2^2 = 5$ satisfy the conditions. For $x \geq y$, we get the other solution $(3, 5, 2)$.