ENUMERATING (*k*; *l*)-CRITICAL AND SUPERCRITICAL PERMUTATIONS

YEO WAN JIN

NUS High School of Mathematics and Science, Singapore, Singapore 129957

YEO YAO RUI

Department of Mathematics, University of Minnesota, Minneapolis, MN 55455

ABSTRACT. In this note we study enumerative aspects of permutations that have longest increasing or decreasing subsequence of a certain length, but not any of its subpermutations. This is motivated by a two-player game in which players pick numbers from a certain range with the goal of achieving a permutation of a particular longest increasing or decreasing subsequence. We provide bounds for when the permutations exist, and compute a formula based on the Robinson-Schensted correspondence and hook length formula. Along the way we also study another natural class of permutations related to the one above via a combinatorial identity.

1. INTRODUCTION

The Young tableau is a useful tool to understand longest subsequences of permutations. For example, it gives a quick proof of the classical theorem that any permutation of $\{1, \ldots, nm+1\}$ contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$. Our goal in this note is to enumerate permutations for which deleting the last element contributes to a longest increasing or decreasing sequence. This study is motivated from the following game stated in [3], for which the solution to a winning strategy is an open problem:

> Let n, m be positive integers. Consider the following game for players A and B: both take turns picking elements $x_1, x_2, \ldots$ from $\{1, \ldots, nm+1\}$ without replacement, starting with player A, until the sequence $x_1 x_2 \cdots$ contains an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$.

Our enumeration will tell us the possibilites of permutations that will result from this game without assuming we have optimal play.

E-mail addresses: h1110126@nus.edu.sg, yeoxx038@umn.edu.

2000 *Mathematics Subject Classification.* 05A05.

Key words and phrases. Young tableau, permutations, longest subsequences, Robinson-Schensted correspondence, hook length formula.

Our computations in this note is a variant of known results in the literature which only studies longest increasing subsequences. The one closest to our computation is a result of Stanley [4, corollary 3], which gives a formula for the number of permutations of S_{nm} with length of longest increasing and decreasing subsequences n and m respectively (certainly this formula is more simple than ours). In particular, the technique used by Stanley is essentially the same we will use, i.e. exploiting the Robinson-Schensted correspondence and hook length formula. Another related computation is a theorem of Rains [2], which tells us the number of permutations of S_n with no increasing subsequence of length greater than k by finding the expected value of $|\mathrm{Tr}(U)^n|^2$ over U in the unitary group $U(k)$ with respect to the Haar measure.

It should also be mentioned that probabilistic methods have been also used to study the expected value of longest increasing subsequences. A result of [6] tells us the expected value for the length of longest increasing subsequence of any permutation in S_n (without any constraints) is asymptotic to $2\sqrt{n}$.

We should point out that understanding longest subsequences has many applications in computer science. In addition, the game above generalizes the *patience sorting algorithm* [1] in which players only seek to achieve a longest increasing subsequence of some length.

We now explain the organization of the note. We will recall certain aspects of the calculus of Young tableau such as the Robinson-Schensted correspondence and hook length formula in section 2. We then present our main enumerative result of (k, l)-critical permutations in section 3, followed by studying a related class of permutations in section 4, before ending with our final remarks in section 5.

2. Permutations and the Young tableau

In this section we will recall the aspects of the calculus of the Young tableau that we will need to use. This discussion follows [3, chapter 3].

Let us write S_n to be the set of all permutations on $n \geq 1$ letters, which we usually use the numbers $1, \ldots, n$. We will write a permutation of S_n as $x_1 \cdots x_n$ to mean that the i^{th} spot is occupied by letter x_i. For every permutation $\sigma = x_1 \cdots x_n \in S_n$, we also define its k^{th} *subpermutation* to be $\sigma_k := x_1 \cdots x_k$. The basic object we will be looking at in this note are subsequences of a permutation. Given $\sigma = x_1 \cdots x_n \in S_n$, a *subsequence of length* k is an ordered subset $(x_{i_1}, \ldots, x_{i_k})$ of $\{x_1, \ldots, x_n\}$, where $i_1 < \cdots < i_k$. This subsequence is *increasing* (respectively, *decreasing*) if $x_{i_1} < \cdots < x_{i_k}$ (respectively, if $x_{i_1} > \cdots > x_{i_k}$).

Our goal now is to develop ways on how to compute the length of longest increasing or decreasing subsequences more easily. Given a permutation $\sigma = x_1 \cdots x_n \in S_n$, we define its *Young tableau* Y_σ to be a left-justified diagram with n boxes filled with the numbers $\{1, \ldots, n\}$, and defined inductively as follow. Start with a box and fill it with x_1. Inductively, after writing down the Young tableau Y_{σ_k} for its k^{th} subpermutation σ_k, we get $Y_{\sigma_{k+1}}$ using the following algorithm:

- Set R to be the first row of Y_{σ_k}.
- **While** x_{k+1} is less than some element of row R, **do**:
  - Let y be the smallest element of R greater than x_{k+1} and replace y by x_{k+1}.
  - Set $x_{k+1} := y$ and set R to be the next row down.
- Now x_{k+1} is greater than every element of R. Place x_{k+1} at the end of row R and **stop**.

For example, the Young tableau of $4375612 \in S_7$ is constructed via the following steps.

$$\begin{array}{|c|}\hline 4\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 3&7\\\hline 4\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 3&5\\\hline 4&7\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 3&5&6\\\hline 4&7\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&5&6\\\hline 3&7\\\cline{1-2} 4\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4&7\\\cline{1-2}\end{array}$$

A main result surrounding Young tableau that we will need is the following.

Theorem 2.1. *Let Y_σ be the Young tableau of a permutation $\sigma \in S_n$. Then the number of boxes in the first row (respectively, first column) of Y_σ is the length of a longest increasing subsequence (respectively, longest decreasing subsequence) of S_n.* □

Another result we need is the Robinson-Schensted correspondence. To state this we need to know the general definition of a Young tableau. A *(standard) Young tableau of shape* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *and size* n is a left-justified diagram with n boxes and k rows such that $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$, row i has λ_i boxes for $i = 1, \ldots, k$, and the fillings are strictly increasing across rows and columns. λ is also call a *partition* of n.

Theorem 2.2 (Robinson-Schensted correspondence). *The set S_n is in bijection with the set of pairs of Young tableaux $\mathcal{T}_n := \{(P, Q) : P, Q$ has size n and same shape$\}$.*

Proof sketch. This theorem is proved by an important construction, so we outline the construction of the bijection without proving it indeed is a bijective (see subsection 3.1 of [3] for details). Given $\sigma \in S_n$, its image $(P_\sigma, Q_\sigma) \in \mathcal{T}_n$ is defined as follow. Define each Y_{σ_k} as in the construction before theorem 2.1, and define its corresponding Z_{σ_k} by adding a box filled with k below the column where x_k is added into Y_{σ_k}. Then let $(P_\sigma, Q_\sigma) = (Y_{\sigma_n}, Z_{\sigma_n})$. □

For example, the image of $4375612 \in S_7$ has P_σ the Young tableau gotten from the construction above theorem 2.1, and Q_σ gotten from the construction below.

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6&7\\\cline{1-2}\end{array}$$

In particular, note that at each step Y_{σ_k} and Z_{σ_k} have the same shape and size.

We will also need an enumerative aspect of the Young tableau. Let us say a *Young diagram* Y is a Young tableau with no fillings. For every box c of a Young diagram Y, define its *hook length* h_c to be the number of boxes below and to the right of c, including c itself.

Theorem 2.3 (Hook length formula). *Let Y be a Young diagram of shape λ and size n. The number of Young tableau of shape λ and size n is*

$$\frac{n!}{\prod_{c \in Y} h_c}.$$

□

Example 2.4. The Young diagrams

each has total possible number of Young tableau with this shape

$$\frac{7!}{5 \cdot 4 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 21$$

by the hook length formula, and the Robinson-Schensted correspondence together with theorem 2.1 tells us there are $2 \cdot 21^2 = 882$ permutations of S_7 with length of longest increasing subsequence 3 and length of longest decreasing subsequence 3. □

3. The (k, l)-critical permutation

In this section we will study permutations arising from the game written in the introduction without optimal play. Say a permutation $\sigma \in S_\alpha$ is (k, l)*-critical* if $\sigma_\alpha = \sigma$ has a longest increasing subsequence of length k or a longest decreasing subsequence of length l, but not $\sigma_{\alpha-1}$. For example, $3216547 \in S_7$ is a $(3, 4)$-critical permutation. We will assume $k \geq 2$ and $l \geq 2$ in this section, because clearly there are no $(1, l)$- or $(k, 1)$-critical permutations.

It is evident that the game stated in the introduction is all about trying to force or avoid a (k, l)-critical permutation, and in this section we will give a formula for enumerating these permutations in terms of the Young tableau. Let us define $X_{k,l}^{\alpha}$ to be the number of (k, l)-critical permutations of S_α. We start by giving an easy but subtle bound on α.

Proposition 3.1. *$X_{k,l}^{\alpha}$ is nonzero at* $\min\{k, l\} \leq \alpha \leq (l - 1)(k - 1) + 1$, *and zero elsewhere.*

Proof. For a (k, l)-critical permutation $\sigma \in S_\alpha$, consider its image into $\mathcal{T}_\alpha$ under the Robinson-Schensted correspondence. Theorem 2.1 says P_σ and Q_σ has first row and first column having k and l boxes respectively. This gives the desired bound for the places where it is possibly nonzero as the number of boxes of P_σ is α. To be more precise, the minimum number of boxes is if there is only one row of length k or one column of length l in P_σ, totalling $\min\{k, l\}$. The maximum number of boxes is if Q_σ is the rectangle with $l - 1$ rows and $k - 1$ columns, plus one more box in the top-right or bottom-left corner (a more rigorous explanation of this is given in the proof of proposition 3.2). Also note that it is indeed nonzero at these places. If $k \leq l$, take any tableau with k boxes in the first row, less than k boxes in the second row, and some number of columns less than l. If $k \geq l$, take any tableau with l boxes in the first column, less than k boxes in the first row, and have the last column only one box.

Finally, note that the bound is indeed valid as the upper bound is always at least the lower bound. This is because

$$(l - 1)(k - 1) + 1 \geq \min\{k - 1, l - 1\} + 1 = \min\{k, l\},$$

where the inequality is valid since $k, l \geq 2$. We are done. □

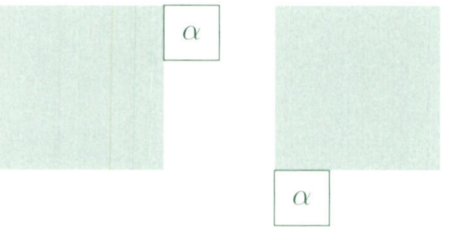
The following is the main result of this note. It tells us how to count $X_{k,l}^{\alpha}$. Although the equation is highly symmetrical, it is also quite massive. The proof will tell us how to interpret the equation.

Proposition 3.2. *Let $T_{k,l}^{\tau,\alpha}$ be the number of Young tableau with shape $\tau = (\tau_1, \tau_2, \ldots, \tau_i)$ and size α, first row having k boxes, and first column having l boxes. Then*

$$X_{k,l}^{\alpha} = \sum_{\substack{\tau=(\tau_1,\tau_2,\ldots,\tau_i) \\ \tau_1+\cdots+\tau_i=\alpha \\ i<l \text{ and } \tau_2<\tau_1 \text{ and } \tau_1=k}} T_{k,i}^{\tau,\alpha} T_{k-1,i}^{(\tau_1-1,\ldots,\tau_i),\alpha-1} + \sum_{\substack{\tau=(\tau_1,\tau_2,\ldots,\tau_l) \\ \tau_1+\cdots+\tau_l=\alpha \\ \tau_1<k \text{ and } \tau_l=1}} T_{\tau_1,l}^{\tau,\alpha} T_{\tau_1,l-1}^{(\tau_1,\ldots,\tau_{l-1}),\alpha-1}.$$

Proof. It suffices to assume we are the bound given in proposition 3.1. By the (proof of the) Robinson-Schensted correspondence, a critical permutation $\sigma \in S_\alpha$ corresponds to a pair

$(P_\sigma, Q_\sigma) \in \mathcal{T}_\alpha$ such that the box filled with α in Q_σ must be the top-right or bottom-left box. The cases corresponds to the generic left and right diagrams below respectively.



Hence we have two cases, and their sum gives the claimed equality.

Case 1: The top-right box of Q_σ is filled in with α. If the Young tableau corresponding to the Young diagram of σ has shape $\tau = (\tau_1, \dots, \tau_i)$, then this case happens only when $\tau_1 = k$ and $\tau_2 < \tau_1$ and $i < l$ because this will allow us to remove the top-right box and still get something in $\mathcal{T}_n$. The Robinson-Schensted correspondence tells us to enumerate the critical permutations $\sigma \in S_\alpha$ it suffices to enumerate pairs (P_σ, Q_σ) in $\mathcal{T}_\sigma$ such that Q_σ has top-right box filled with α. There are $T^{\tau,\alpha}_{k,i}$ possibilities for P_σ and $T^{(\tau_1-1,\dots,\tau_i),\alpha-1}_{k-1,i}$ possibilities for Q_σ, so the number of such σ is the sum of $T^{\tau,\alpha}_{k,i}T^{(\tau_1-1,\dots,\tau_i),\alpha-1}_{k-1,i}$ over all such τ with $\tau_1 + \dots + \tau_i = \alpha$.

Case 2: The bottom-left box of Q_σ is filled in with α. If the Young tableau corresponding to the Young diagram of σ has shape $\tau = (\tau_1, \dots, \tau_i)$, then this case happens only when $i = l$ and $\tau_1 < k$ and $\tau_i = 1$ because this will allow us to remove the top-right box and still get something in $\mathcal{T}_n$. This is exactly the same as the above case, except Q_σ has bottom-left box filled with α. There are $T^{\tau,\alpha}_{k,l}$ possibilities for P_σ and $T^{(\tau_1,\dots,\tau_{l-1}),\alpha-1}_{k,l-1}$ possibilities for Q_σ, so the number of such σ is the sum of $T^{\tau,\alpha}_{k,l}T^{(\tau_1,\dots,\tau_{l-1}),\alpha-1}_{k,l-1}$ over all such τ with $\tau_1 + \dots + \tau_i = \alpha$. □

Theorem 3.3. *$X^\alpha_{k,l}$ can be computed by*

$$X^\alpha_{k,l} = \sum_{\substack{\tau=(\tau_1,\tau_2,\dots,\tau_i)\\ \tau_1+\dots+\tau_i=\alpha\\ i<l \text{ and } \tau_2<\tau_1 \text{ and } \tau_1=k}} \frac{\alpha!(\alpha-1)!}{\prod_{c\in Y^{\tau,\alpha}_{k,i}} h_c \prod_{d\in Y^{(\tau_1-1,\dots,\tau_i),\alpha-1}_{k-1,i}} h_d} + \sum_{\substack{\tau=(\tau_1,\tau_2,\dots,\tau_l)\\ \tau_1+\dots+\tau_l=\alpha\\ \tau_1<k \text{ and } \tau_l=1}} \frac{\alpha!(\alpha-1)!}{\prod_{c\in Y^{\tau,\alpha}_{\tau_1,l}} h_c \prod_{d\in Y^{(\tau_1,\dots,\tau_{l-1}),\alpha-1}_{\tau_1,l-1}} h_d}.$$

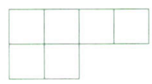
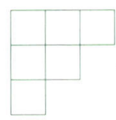
Proof. Substitute the hook length formula into proposition 3.2. □

Notice this main theorem tells us that we can now input the conditions into a computer program to churn out the possible Young diagrams. By pairing up Young diagrams with the same shape (and forcing either the top-right or bottom-left box of the second one to be filled with α) we can then implement the reverse of the Robinson-Schensted correspondence bijection to obtain (k, l)-critical permutations.

Example 3.4. Let us now give an example of what we have discussed by trying to evaluate the number of $(4,3)$-critical permutations in S_6. Theorem 3.3 tells us the number is

$$\begin{aligned} X^6_{4,3} &= \frac{6!5!}{\prod_{c\in Y^{(4,2),6}_{4,2}} h_c \prod_{c\in Y^{(3,2),5}_{3,2}} h_d} + \frac{6!5!}{\prod_{c\in Y^{(3,2,1),6}_{3,3}} h_c \prod_{c\in Y^{(3,2),5}_{3,2}} h_d} \\ &= \frac{6!5!}{(5\cdot 4\cdot 2\cdot 1\cdot 2\cdot 1)(4\cdot 3\cdot 1\cdot 2\cdot 1)} + \frac{6!5!}{(5\cdot 3\cdot 1\cdot 3\cdot 1\cdot 1)(4\cdot 3\cdot 1\cdot 2\cdot 1)} \\ &= 125, \end{aligned}$$

corresponding to the enumeration over the following two Young diagrams

 and  .

□

Example 3.5. It is clear that $X^l_{2,l} = X^l_{l,2} = l$ as this corresponds to enumeration over the two Young diagrams of shape $(2, 1^{l-1})$ and (1^l) respectively. □

A table of values of $X^\alpha_{k,l}$ of small order is given in appendix A. We now derive an easy consequence of our discussion. For convenience, let us define the following two expressions:

$$A^\alpha_{k,l} := \sum_{\substack{\tau=(\tau_1,\tau_2,\dots,\tau_i)\\ \tau_1+\dots+\tau_i=\alpha\\ i<l \text{ and } \tau_2<\tau_1 \text{ and } \tau_1=k}} T^{\tau,\alpha}_{k,i} T^{(\tau_1-1,\dots,\tau_i),\alpha-1}_{k-1,i},$$

$$B^\alpha_{k,l} := \sum_{\substack{\tau=(\tau_1,\tau_2,\dots,\tau_l)\\ \tau_1+\dots+\tau_l=\alpha\\ \tau_1<k \text{ and } \tau_l=1}} T^{\tau,\alpha}_{\tau_1,l} T^{(\tau_1,\dots,\tau_{l-1}),\alpha-1}_{\tau_1,l-1}.$$

That is, $X^\alpha_{k,l} = A^\alpha_{k,l} + B^\alpha_{k,l}$. Also define $I^\alpha_{k,l}$ (resp. $D^\alpha_{k,l}$) to be the expected length of longest increasing (resp. decreasing) subsequence of a (k,l)-critical permutation in S_α.

Proposition 3.6. $k/2 \le I^\alpha_{k,k}, D^\alpha_{k,k} \le k$.

This proposition will follow once we prove a relationship between $A^\alpha_{k,l}$ and $B^\alpha_{k,l}$.

Lemma 3.7. *Define $A^\alpha_{k,l}$ and $B^\alpha_{k,l}$ as above. Then $A^\alpha_{k,l} = B^\alpha_{l,k}$.*

Proof. Note that by two applications of the hook length formula,

$$\begin{aligned} A^\alpha_{k,l} &= \sum_{\substack{\tau=(\tau_1,\tau_2,\dots,\tau_i)\\ \tau_1+\dots+\tau_i=\alpha\\ i<l \text{ and } \tau_1=k}} T^{\tau,\alpha}_{k,i} T^{(\tau_1-1,\dots,\tau_i),\alpha-1}_{k-1,i} \\ &= \sum_{\substack{\tau=(\tau_1,\tau_2,\dots,\tau_k)\\ \tau_1+\dots+\tau_k=\alpha\\ \tau_1<l \text{ and } \tau_k=1}} T^{\tau,\alpha}_{\tau_1,k} T^{(\tau_1,\dots,\tau_{k-1}),\alpha-1}_{\tau_1,k-1} \\ &= B^\alpha_{l,k}. \end{aligned}$$

This gives us the equality we seek. □

Proof of proposition 3.6. Using lemma 3.7,

$$\begin{aligned} I^{\alpha}_{k,k} &= \frac{1}{X^{\alpha}_{k,k}} \left(kA^{\alpha}_{k,k} + \sum_{\substack{\tau=(\tau_1,\tau_2,\dots,\tau_k) \\ \tau_1+\dots+\tau_k=\alpha \\ \tau_1<k \text{ and } \tau_k=1}} \frac{\alpha!(\alpha-1)!\tau_1}{\prod_{c\in Y^{\tau,\alpha}_{\tau_1,k}} h_c \prod_{c\in Y^{(\tau_1,\dots,\tau_{k-1}),\alpha-1}_{\tau_1,k-1}} h_d} \right) \\ &\geq \frac{kA^{\alpha}_{k,k}}{A^{\alpha}_{k,k}+B^{\alpha}_{k,k}} \\ &= \frac{kA^{\alpha}_{k,k}}{2A^{\alpha}_{k,k}} \\ &= \frac{k}{2}, \end{aligned}$$

and

$$\begin{aligned} D^{\alpha}_{k,k} &= \frac{1}{X^{\alpha}_{k,k}} \left(\sum_{\substack{\tau=(\tau_1,\tau_2,\dots,\tau_i) \\ \tau_1+\dots+\tau_i=\alpha \\ i<k \text{ and } \tau_2<\tau_1 \text{ and } \tau_1=k}} \frac{\alpha!(\alpha-1)!i}{\prod_{c\in Y^{\tau,\alpha}_{k,i}} h_c \prod_{d\in Y^{(\tau_1-1,\dots,\tau_i),\alpha-1}_{k-1,i}} h_d} + kB^{\alpha}_{k,k} \right) \\ &\geq \frac{kB^{\alpha}_{k,k}}{A^{\alpha}_{k,k}+B^{\alpha}_{k,k}} \\ &= \frac{kB^{\alpha}_{k,k}}{2B^{\alpha}_{k,k}} \\ &= \frac{k}{2}. \end{aligned}$$

The upper bounds $I^{\alpha}_{k,k}, D^{\alpha}_{k,k} \leq k$ are trivial by definition of a (k,l)-critical permutation or simple manipulation of equations in proposition 3.2. □

In fact, lemma 3.7 implies that $X^{\alpha}_{k,l}$ is symmetrical about k and l, as we will expect from the definitions. It agrees with our intuition that that there should be equal number of permutations of longest increasing and decreasing subsequence of length k and l respectively, or l and k respectively.

Corollary 3.8. *$X^{\alpha}_{k,l} = X^{\alpha}_{l,k}$. In addition, $X_{k,l} = X_{k',l}$ if $k, k' \geq \alpha$ and $X_{k,l} = X_{k,l'}$ if $l, l' \geq \alpha$.*

Proof. Use lemma 3.7 to get $X^{\alpha}_{k,l} - X^{\alpha}_{l,k} = (A^{\alpha}_{k,l} - B^{\alpha}_{k,l}) + (A^{\alpha}_{l,k} - B^{\alpha}_{l,k}) = 0$. The second remark is clear. □

4. Another related enumeration

An obvious other variant we can consider is the case where a permutation $\sigma \in S_\alpha$ has $\sigma_\alpha = \sigma$ a longest increasing subsequence of length k and a longest decreasing subsequence of length l, but not $\sigma_{\alpha-1}$. Let us call this kind of permutations (k,l)-*supercritical*. Note that we can allow $k = 1$ and $l = 1$ for supercritical permutations since they make sense here.

We should warn that a $(k, 1)$-permutation does not enumerate the number of permutations with longest increasing subsequence of length k because we are also take into account supercriticalbility. Therefore our enumeration will be lesser in general compared to this case.

As in the previous section, let us start by giving a bound for $\mathcal{X}_{k,l}^{\alpha}$, the number of (k, l)-supercritical permutations in S_{α}.

Proposition 4.1. *$\mathcal{X}_{k,l}^{\alpha}$ is nonzero at $k + l - 1 \le \alpha \le \max\{k(l-1)+1, l(k-1)+1\}$, and zero elsewhere.*

Proof. Again, consider the image of a (k, l)-supercritical permutation under the Robinson-Schensted correspondence. This time, the minimum number of boxes in the Young diagram is if there is only one row and column of k and l boxes respectively, a sum of $k + l - 1$. The maximum number of boxes is if the Young diagram is the rectangle with $k(l-1)$ boxes plus one more box in the bottom left corner, or the rectangle with $l(k-1)$ plus one more box in the top right corner. It is indeed nonzero in the bound, since we can just take any tableau of $\alpha - 1$ boxes with $k - 1$ boxes in the first row, $l - 1$ boxes in the second row, and then add another box in the first row.

Finally, note that the bound is indeed valid as the upper bound is always at the least the lower bound. To confirm this, we compute

$$k(l-1)+1-(k+l-1) = kl-2k-l+2 = (k-1)(l-2) \ge 0,$$
$$l(k-1)+1-(k+l-1) = kl-2l-k+2 = (k-2)(l-1) \ge 0,$$

as desired. □

We can enumerate (k, l)-supercritical permutations in a similar fashion as in section 3. The arguments uses exactly the same idea as in the previous section, so we will not reprove them. We should note that the analogous bound we gave in proposition 3.6 is better and works for all k, l in this case.

Theorem 4.2. *Define $T_{k,l}^{\tau,\alpha}$ as in proposition 3.2, and assume the bound of $\mathcal{X}_{k,l}^{\alpha}$ given in proposition 4.1. Then $\mathcal{X}_{k,l}^{\alpha} = \mathcal{X}_{l,k}^{\alpha}$, and $\mathcal{X}_{k,l} = \mathcal{X}_{k',l}$ if $k, k' \ge \alpha$ and $\mathcal{X}_{k,l} = \mathcal{X}_{k,l'}$ if $l, l' \ge \alpha$, and*

$$\mathcal{X}_{k,l}^{\alpha} = \sum_{\substack{\tau=(\tau_1,\tau_2,\ldots,\tau_l)\\ \tau_1+\cdots+\tau_l=\alpha\\ \tau_2<\tau_1 \text{ and } \tau_1=k}} T_{k,l}^{\tau,\alpha} T_{k-1,l}^{(\tau_1-1,\ldots,\tau_l),\alpha-1} + \sum_{\substack{\tau=(\tau_1,\tau_2,\ldots,\tau_l)\\ \tau_1+\cdots+\tau_l=\alpha\\ \tau_1=k \text{ and } \tau_l=1}} T_{k,l}^{\tau,\alpha} T_{k,l-1}^{(\tau_1,\ldots,\tau_{l-1}),\alpha-1}.$$

If we define the first and second terms of the above expression as $\mathcal{A}_{k,l}^{\alpha}$ and $\mathcal{B}_{k,l}^{\alpha}$ respectively, then $\mathcal{A}_{k,l}^{\alpha} = \mathcal{B}_{l,k}^{\alpha}$. The expected length $\mathcal{I}_{k,l}^{\alpha}$ and $\mathcal{D}_{k,l}^{\alpha}$ of longest increasing or subsequence respectively of a (k, l)-supercritical permutation in S_{α} is

$$\mathcal{I}_{k,l}^{\alpha} = \frac{k\mathcal{A}_{k,l}^{\alpha} + l\mathcal{B}_{k,l}^{\alpha}}{\mathcal{X}_{k,l}^{\alpha}} = \mathcal{D}_{k,l}^{\alpha}.$$

In particular, $\min\{k, l\} \le \mathcal{I}_{k,l}^{\alpha}, \mathcal{D}_{k,l}^{\alpha} \le \max\{k, l\}$. □

A table of values of $\mathcal{X}_{k,l}^{\alpha}$ of small order is given in appendix B. We now explain the relationship between (k, l)-critical and (k, l)-supercritical permutations, other than the obvious *or* to *and* word change in the definitions. Firstly we have the observation that

$$A_{k,l}^{\alpha} = \mathcal{A}_{k,l-1}^{\alpha} + A_{k,l-1}^{\alpha} \text{ and } B_{k,l}^{\alpha} = \mathcal{B}_{k-1,l}^{\alpha} + B_{k,l}^{\alpha}.$$

This follows by definition. In fact this observation tells us the following:

Proposition 4.3. $X^\alpha_{k,k} = \mathcal{X}^\alpha_{k-1,k} + X^\alpha_{k-1,k}$.

Proof. Use lemma 3.7 and theorem 4.2 to see that $A^\alpha_{k,k-1} = B^\alpha_{k-1,1}$ and $\mathcal{A}^\alpha_{k,k-1} = \mathcal{B}^\alpha_{k-1,1}$ □

However the equality $X^\alpha_{k,l} = \mathcal{X}^\alpha_{k-1,l} + X^\alpha_{k-1,l}$ does not hold in general, as one can check by using the tables in the appendices. Nevertheless we do have an equality below relating $X^\alpha_{k,l}$ to $\mathcal{X}^\alpha_{k,l}$, which tells us that we can calculate a sum of $X^\alpha_{k,l}$ by calculating a sum of $\mathcal{X}^\alpha_{k,l}$. Denote $\delta_{p,q}$ to be the Kronecker delta function.

Proposition 4.4. *Assume $l \geq 2$. Then*

$$\sum_{i=0}^{l-2} X^\alpha_{2+i,l-i} = 2\delta_{\alpha,l} + \sum_{i=0}^{l-3} \left(X^\alpha_{2+i,l-1-i} + \mathcal{X}^\alpha_{2+i,l-i-1}\right)$$

Proof. Expanding the left hand side gives

$$\begin{aligned}\sum_{i=0}^{l-2} X^\alpha_{2+i,l-i} &= \sum_{i=0}^{l-2} \left(A^\alpha_{2+i,l-i} + B^\alpha_{2+i,l-i}\right)\\ &= A^\alpha_{2,l} + B^\alpha_{2,l} + A^\alpha_{l,2} + B^\alpha_{l,2} + \sum_{i=1}^{l-3}\left(\mathcal{A}^\alpha_{2+i,l-i-1} + A^\alpha_{2+i,l-i-1} + \mathcal{B}^\alpha_{1+i,l-i} + B^\alpha_{1+i,l-i}\right)\\ &= B^\alpha_{2,l} + A^\alpha_{l,2} + \sum_{i=0}^{l-3}\left(X^\alpha_{2+i,l-1-i} + \mathcal{X}^\alpha_{2+i,l-i-1}\right),\end{aligned}$$

where the second equality is using the definitions of $\mathcal{A}^\alpha_{k,l}, A^\alpha_{k,l}, \mathcal{B}^\alpha_{k,l}, B^\alpha_{k,l}$, and the last equality is by a telescoping-sum-like summation. We are done by observing $A^\alpha_{l,2}$ and $B^\alpha_{2,l}$ are nonzero only if they are enumerated over the Young tableau of shape (l) and $(1,\ldots,1)$ respectively, where $(1,\ldots,1)$ has l ones. This Young tableau is unique by definition or by using the hook length formula and observing the product of hook lengths is $l!$. As a sanity check, note that if we take $l = 2$ then $X^\alpha_{2,2} = 2\delta_{\alpha,2}$. □

Proposition 4.4 can be generalized to the case where we do not have to start enumeration at $i = 0$. By adapting the proof of proposition 4.4 and iterating the above proposition over l gives the following result almost immediately.

Theorem 4.5. *Assume $k, l \geq 2$, without loss of generality with $k \leq l$. Then*

$$\sum_{i=k-2}^{l-2} X^\alpha_{2+i,l-i} = B^\alpha_{k,l} + A^\alpha_{l,k} + \sum_{i=k-2}^{l-3}\left(X^\alpha_{2+i,l-1-i} + \mathcal{X}^\alpha_{2+i,l-i-1}\right)$$

In particular, taking $k = 2$ implies

$$\sum_{i=0}^{l-2} X^\alpha_{2+i,l-i} = \begin{cases} 2(l-1) + \sum_{j=3}^{l+1}\sum_{i=0}^{j-3}\mathcal{X}^\alpha_{2+i,j-i-1} & \text{if } \alpha \leq l,\\ \sum_{j=3}^{l+1}\sum_{i=0}^{j-3}\mathcal{X}^\alpha_{2+i,j-i-1} & \text{otherwise.}\end{cases}$$

Proof. As mentioned the first equality is a direct adaption of the proof of proposition 4.4, so we only comment on the second equality. Note that $A^\alpha_{2,l} = B^\alpha_{l,2}$ by lemma 3.7, is clearly one when $\alpha = l$, and zero otherwise. By subtracting the $X^\alpha_{2+i,1-l-i}$ sum on the left hand side, and summing over l, we get the desired equality. □

5. Conclusion and final remarks

We have established some enumeration techniques for critical permutations via the calculus of the Young tableau. A similar study for supercritical permutations and its relation to critical permutations is conducted as well. Although we have not pursued this in the present note, it may be worthwhile to understand a generalization of the enumerations given above, which is the case where repetition of numbers are allowed. This will most probably use the idea of semistandard tableau, since the Robinson-Schensted- Knuth correspondence gives a bijection between pairs of semistandard Young tableau and square matrices with entries 0 and 1, which can be thought of as a generalized permutation (see [5, section 7.11] for an exposition of this correspondence). It will also be a good further work to understand more about how the $A^{\alpha}_{k,l}$ and $B^{\alpha}_{k,l}$ relate to each other.

Problem 5.1. *Extend section 3 to generalized permutations.*

Problem 5.2. *Extend proposition 3.5 to all of k and l.*

A promising step for problem 5.2 is to use supercritical permutations to analyze. We have the following proposition as a possible first step towards this study.

Proposition 5.3. *Fix α and k. Then $\{A^{\alpha}_{k,l}\}_{l\geq 2} = \{B^{\alpha}_{l,k}\}_{l\geq 2}$ is an increasing sequence. In fact,*

- *the sequence is constant at $l < \frac{\alpha-1}{k-1}$, and is identically $A^{\alpha}_{k,2}$ if defined at this interval,*
- *the sequence is constant at $l > \alpha + 1 - k$ and is identically $A^{\alpha}_{k,\alpha+2-k}$,*
- *the sequence is strictly increasing at $\frac{\alpha-1}{k-1} \leq l \leq \alpha + 1 - k$.*

Proof. A similar idea as in the previous section tells us that

$$A^{\alpha}_{k,l+1} - A^{\alpha}_{k,l} = \left(A^{\alpha}_{k,2} + \sum_{j\leq l} \mathcal{A}^{\alpha}_{k,j}\right) - \left(A^{\alpha}_{k,2} + \sum_{j\leq l-1} \mathcal{A}^{\alpha}_{k,j}\right) = \mathcal{A}^{\alpha}_{k,l}.$$

We are done by noting that $\mathcal{A}^{\alpha}_{k,l}$ is nonzero only at $k + l - 1 \leq \alpha \leq l(k-1) + 1$ by a same proof as in proposition 4.1. □

6. Acknowledgements

The first author would like to thank Chia Vui Leong for giving her the opportunity to work on this project through the high school mathematics program.

References

[1] David Aldous and Persi Diaconis, Longest increasing subsequences, from patience sorting to the Baik-Deift-Johansson theorem, *Bull. Amer. Math. Soc.* **36** (1999), 413–432.

[2] Eric Rains, Increasing subsequences and the classical groups, *Electron. J. Combin.* **5** (1998), #R12.

[3] Bruce Sagan, *The Symmetric Group*, Springer, 1991.

[4] Richard Stanley, Increasing and decreasing subsequences and their variants, *Proceedings of the International Congress of Mathematicians* (2006), 545–579.

[5] Richard Stanley, *Enumerative Combinatorics (Volume 2)*, Cambridge University Press, 1999.

[6] Anatoly Vershik and Sergei Kerov, Asymptotics of the Plancherel measure of the symmetric group and the limiting shape of Young tableaux, *Dokl. Akad. Nauk. U.S.S.R.* **233** (1977), 1024–1027.

Appendix A. Table of values for $X^{\alpha}_{k,l}$

In appendix A we give some tables of values for $X^{\alpha}_{k,l}$ for α up to 8. We will make use of the bound in proposition 3.1 and only list values at those places. We will also make use of corollary 3.8 to insist that pairs (k,l) have $2 \leq k \leq l \leq \alpha$.

α	(k,l)	$X^{\alpha}_{k,l}$	α	(k,l)	$X^{\alpha}_{k,l}$	α	(k,l)	$X^{\alpha}_{k,l}$
2	(2, 2)	2		(4, 6)	106		(3, 6)	2156
3	(2, 3)	3		(5, 5)	40		(3, 7)	2044
	(3, 3)	2		(5, 6)	21		(3, 8)	2227
4	(2, 4)	4		(6, 6)	2		(4, 4)	9114
	(3, 3)	12	7	(2, 7)	7		(4, 5)	11368
	(3, 4)	7		(3, 4)	175		(4, 6)	6916
	(4, 4)	2		(3, 5)	546		(4, 7)	5775
5	(2, 5)	5		(3, 6)	501		(4, 8)	5734
	(3, 3)	20		(3, 7)	496		(5, 5)	6664
	(3, 4)	32		(4, 4)	1260		(5, 6)	3927
	(3, 5)	29		(4, 5)	1316		(5, 7)	3766
	(4, 4)	24		(4, 6)	860		(5, 8)	3333
	(4, 5)	13		(4, 7)	871		(6, 6)	1190
	(5, 5)	2		(5, 5)	552		(6, 7)	637
6	(2, 6)	6		(5, 6)	348		(6, 8)	596
	(3, 4)	125		(5, 7)	319		(7, 7)	84
	(3, 5)	125		(6, 7)	31		(7, 8)	43
	(3, 6)	121		(7, 7)	2		(8, 8)	2
	(4, 4)	250	8	(2, 8)	8			
	(4, 5)	125		(3, 5)	1372			

Appendix B. Table of values for $\mathcal{X}^{\alpha}_{k,l}$

In appendix B we give some tables of values for $\mathcal{X}^{\alpha}_{k,l}$ for α up to 8. We will make use of the bound in proposition 4.1 and only list values at those places. We will also make use of theorem 4.2 to insist that pairs (k,l) have $1 \leq k \leq l \leq \alpha$.

α	(k,l)	$\mathcal{X}^{\alpha}_{k,l}$	α	(k,l)	$\mathcal{X}^{\alpha}_{k,l}$	α	(k,l)	$\mathcal{X}^{\alpha}_{k,l}$
2	(1, 2)	1		(2, 5)	25		(4, 4)	400
3	(1, 3)	1		(3, 3)	160	8	(1, 8)	1
	(2, 2)	4		(3, 4)	100		(2, 5)	392
4	(1, 4)	1	7	(1, 7)	1		(2, 6)	280
	(2, 3)	9		(2, 4)	70		(2, 7)	49
5	(1, 5)	1		(2, 5)	126		(3, 4)	3626
	(2, 3)	10		(2, 6)	36		(3, 5)	3136
	(2, 4)	16		(3, 3)	525		(3, 6)	441
6	(1, 6)	1		(3, 4)	875		(4, 4)	6300
	(2, 4)	45		(3, 5)	225		(4, 5)	1225