Exploration Of The Isoperimetric Inequality

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1 Introduction

Whether it is economics, mathematics, physics, architecture or art, people have always played with concepts of area and perimeter to improve efficiency or simply create aesthetic beauty in the organic structures that arise.

The origin of the Isoperimetric Inequality[1] and other area/volume-optimisation problems dates back centuries. Bubbles on the surface of water, hexagonal beehives and rounded eggs are some of the examples of nature that make use of these principles of maximising volume for surface area; these phenomena fascinate the human mind and probes curiosity.

In the following report, we will work through a type of generalisation to the Isoperimetric Inequality[1], namely the Shortest Perimeter Problem. The report will uncover subtle relationships between area and perimeter that will help to shed light on the problem.

2 Exploration Of The Isoperimetric Inequality

2.1 Shortest Total Perimeter Problem

The Shortest Total Perimeter Problem is defined as follows:

(i) A farmer has n pigs. He needs to use fence to mark out n pig pens, each of area 1, in the pasture. Assume the fences are infinitely thin and flexible. What is the minimum length of fence required, and the structure of fence that obtains this minimum length?

Another definition of the problem is:

(ii) Let C be a set of continuous curves on the plane that contains n bounded regions, each region having an area of 1. What is the minimum total perimeter of C, and the structure of C that achieves this minimum?

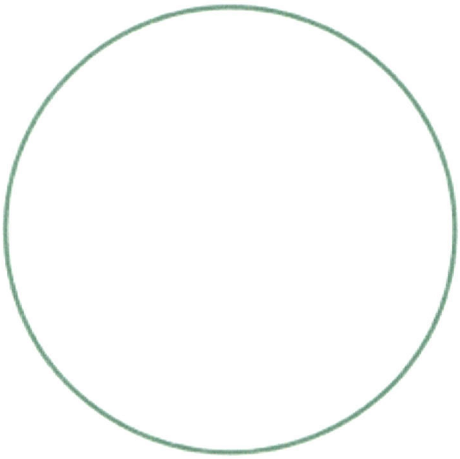
Note that definitions (i) and (ii) are equivalent.

2.2 Solved Cases

This problem, though original, has two already solved cases.

2.2.1 n=1

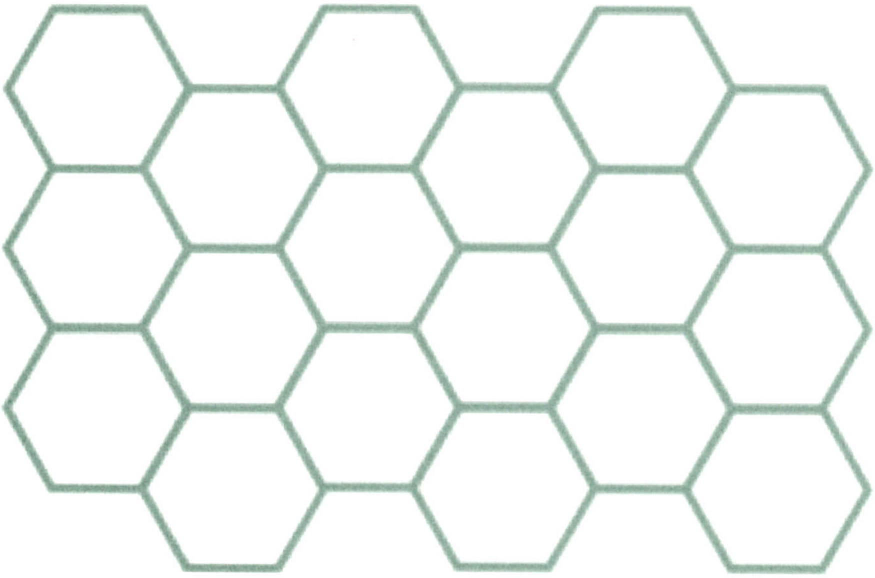
From the Isoperimetric Inequality[1], we find that a **circle** achieves the minimum perimeter of curve needed to mark out a single unit area on the plane.



Let A and r be the area and radius of the circle respectively. Since $A = \pi r^2 = 1$ thus the circle has radius $\frac{1}{\sqrt{\pi}}$. Consequently, the minimum perimeter achieved is $2\pi r = 2\sqrt{\pi} \approx 3.545$.

2.2.2 n$\rightarrow \infty$

From the Honeycomb Conjecture[3], we find that a **hexagonal lattice** achieves the minimum perimeter needed to mark out numerous unit areas on the plane.



Let A and r be the area and edge length of the circle respectively. Since $A = \frac{3\sqrt{3}}{2}r^2 = 1$ thus the hexagon has edge length $\sqrt{\frac{2}{3\sqrt{3}}}$. Since each hexagon has 6 edges but each edge is shared between 2 hexagons, total perimeter $\approx 3n\sqrt{\frac{2}{3\sqrt{3}}} = n\sqrt[4]{12}$. Thus minimum perimeter achieved tends to $n\sqrt[4]{12}$ as n approaches infinity.

Do note however that it is yet to be proven that the hexagonal tiling is the unique structure that achieves minimal perimeter for infinitely large areas.

2.3 n=2

2.3.1 Definitions and Lemma

Definition 2.1 *Let* P_{C_0} *be the length of continuous finite curve* C_0.

Definition 2.2 *Let* A_{C_0,X_0Y_0} *be the area bounded by continuous curve* C_0 *from* X_0 *to* Y_0 *with the line segment* X_0Y_0. *(Note: if curve* C_0 *intersects line segment* X_0Y_0, *take the area bounded to be the absolute value of the difference between sum of areas bounded on the two sides of the line segment)*

Lemma 2.3 *Given a fixed line segment* MN *on the plane of length* d, *construct continuous curve* C *from* M *to* N *such that* $A_{C,MN} = A$ *for some fixed area* A. *Then the minimum perimeter of* C *is achieved when* C *is a circular arc.*

Proof: Note that curve C_{min} that achieves minimum perimeter exists[2]. Let C_{min} have perimeter L_{min}.

Now C_{min} cannot intersect MN. Suppose on the contrary that C_{min} intersects MN. Then let sum of areas bounded on the left and right of MN be A_L and A_R respectively, and WLOG $A_L > A_R > 0$. Now C_{min} has perimeter L_{min} and bounds an area of $A_L - A_R = A$ with MN. Now reflect the curves to the left of MN across MN. Note that the resulting curve C' remains continuous and has perimeter L_{min} but bounds an area of $A_L + A_R > A$. Set up the coordinate system with origin $O = M$, let N lie on positive x-axis and C' lie on the top half (positive y coordinates) of the plane. Parametrise C' as $(x(t), y(t)), 0 \leq t \leq 1$ where $x(0) = 0$, $x(1) = d$, $y(0) = y(1) = 0$. Now note that we can find curve $C'' = (x(t), ky(t)), 0 \leq t \leq 1$ where $k < 1$ from M to N (because $x(0) = 0$, $x(1) = d$, $ky(0) = ky(1) = 0$), that bounds an area A with MN and has perimeter $< L_{min}$. This is a contradiction.

Thus C_{min} cannot intersect MN.

Now C_{min} is convex. Suppose on the contrary that C_{min} is concave. Then there exists points P and Q on C_{min} such that PQ lies entirely outside C_{min}. Now reflect the curve between PQ across line segment PQ. Note that the resulting curve C' remains continuous and has perimeter L_{min} but bounds an area greater that A. Set up the coordinate system with origin $O = M$, let N lie on positive x-axis and C' lie on the top half (positive y coordinates) of the axes. Parametrise C' as $(x(t), y(t)), 0 \leq t \leq 1$ where $x(0) = 0$, $x(1) = d$, $y(0) = y(1) = 0$. Now note that we can find curve $C'' = (x(t), ky(t)), 0 \leq t \leq 1$ where $k < 1$ from M to N (because $x(0) = 0$, $x(1) = d$, $ky(0) = ky(1) = 0$), that bounds an area A with MN and has perimeter $< L_{min}$. This is a contradiction.

Thus C_{min} is convex.

Case I: $A = \frac{1}{8}\pi d^2$.

C_{min} is a semicircle centred at the midpoint of MN and radius $r = \frac{d}{2}$. Then $L_{min} = \pi r = \frac{\pi d}{2}$. Suppose on the contrary that C_{min} is not a semicircle. Then let l be the line along MN. C_{min} bounds an area of A with l and by the Isoperimetric Inequality[1], $L_{min} > \sqrt{2\pi A} = \sqrt{2\pi(\frac{1}{8}\pi d^2)} = \sqrt{\frac{\pi^2 d^2}{4}} = \frac{\pi d}{2}$. But the perimeter achieved by a semicircle is $\frac{\pi d}{2} < L_{min}$. This is a contradiction.

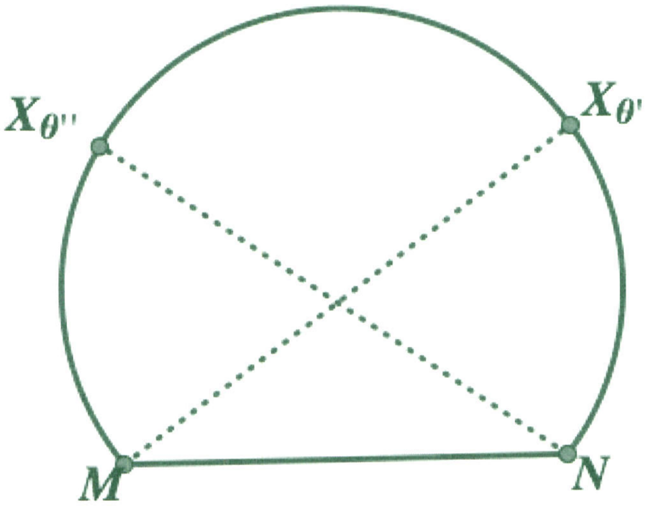
Thus C_{min} is a semicircle, $L_{min} = \frac{\pi d}{2}$. It follows that C_{min} is a circular arc.

Case II: $A > \frac{1}{8}\pi d^2$.

Set up a coordinate system with origin $O = M$, let N lie on the positive x-axis and curve C_{min} lie on the top half (positive y coordinates) of the plane.

Now define l_θ to be a line passing through M of angle θ anticlockwise to the x-axis. Let $f(\theta)$ be the area bounded by C_{min} and l_θ Let l_θ intersect curve C_min at point X_θ. Let $g(\theta) = \frac{f(\theta)}{|MX_\theta|^2}$. There exists θ_0 such that $f(\theta_0) = 0$ and $f(\theta) > 0$ for all$0 \leq \theta < \theta_0$. Since $g(\theta_0) = 0$ and $g(0) > 1/8$ there exists $0 \leq \theta' < \theta_0$ such that $g(\theta') = \frac{1}{8}$ and thus $f(\theta') = \frac{1}{8}\pi(MX_{\theta'})^2$. Now consider C_{min} from M to $X_{\theta'}$. Note that that segment of curve must have minimum perimeter among all curves bounding an area of $f(\theta')$ with $MX_{\theta'}$. Since $f(\theta') = \frac{1}{8}\pi(MX_{\theta'})^2$ thus using the result from Case I, that portion of curve C_{min} from M to $X_{\theta'}$ is a semicircle. Let this portion of curve be C'.

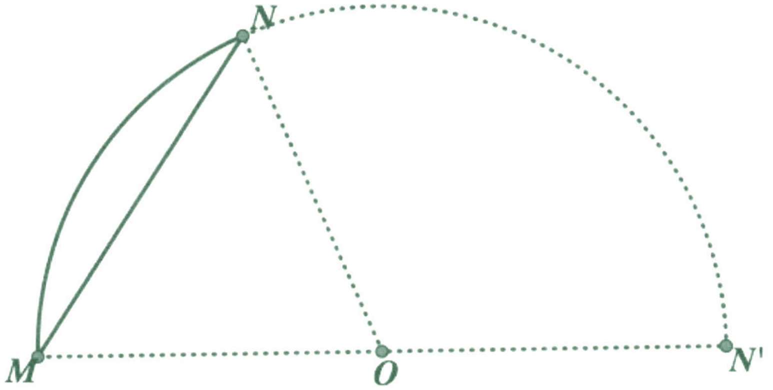
Similarly, there exists a portion of curve C_{min} from N to $X_{\theta''}$ that is a semicircle. Let this portion of curve be C''.



Now parametrise C_{min} as $(x(t), y(t)), 0 \leq t \leq 1$ where $x(0) = 0$, $x(1) = d$, $y(0) = y(1) = 0$. Let $h(t) = \frac{\delta y}{\delta x}$ and let $h(t_0) = 0$ (exists by Mean Value Theorem). Let $X_{\theta'} = (x(t'), y(t'))$, $X_{\theta''} = (x(t''), y(t''))$. Clearly $0 < t' \leq t_0 \leq t'' < 1$. If $t' < t''$, then the portion of curve from t' to t'' is where both semicircles C' and C'' intersect thus both semicircles have the same centre and radii; it follows that C_{min} is a circular arc. If $t' = t''$, We have that the diameters of C' and C'' are equal and thus $d = MN = 0$. The centre lies on the normal to the curve at $t_0 = t' = t''$ thus the semicircles share the same centre too and C_{min} is a full circle.

Case III: $A < \frac{1}{8}\pi d^2$.

Let C_{arc} be a circular arc from M to N that bounds an area of A with MN. Assume C_{arc} is centred at O with radius r. Then $OM = ON = r$. Now extend MO to N' such that $ON' = r$. Now let C_1 be the curve of minimum perimeter bounding an area of $\frac{1}{2}\pi r^2$ with the line segment $MN' = 2r$. By Case I C_1 is a semicircle centred at O with radius r, thus N lies on C_1. Let C_{arc} be portion of C_1 from M to N and C_2 be portion of C_1 from N to N'. Suppose there exists C_{min} that is not a circular arc and bounds an area of A with MN with the least perimeter. Replace C_{arc} with C_{min} and let the resulting curve consisting of C_{min} and C_2 be C_1'. Now C_1' bounds an area of $\frac{1}{2}\pi r^2$ with the line segment $MN' = 2r$ thus C_1' is also a semicircle. Thus $C_{min} = C_{arc}$ is a circular arc.



2.3.2 C_{min} Basic Structure

Now let S_1 and S_2 be the curves forming the boundaries of the two unit areas respectively in C_{min}.

If S_1 and S_2 do not intersect, by Isoperimetric Inequality[1], both S_1 and S_2 are circles of radii $\frac{1}{\sqrt{\pi}}$. Consequently, the minimum perimeter achieved is $2(2\pi r) = 4\sqrt{\pi} \approx 7.090$. A numerical comparison with results obtained further below will conclude that this does not obtain minimum total perimeter.

Thus S_1 and S_2 intersect at curves I_1, I_2, ..., I_k. WLOG let them be placed consecutively. If $k > 1$, let I_1 be from points W_0 to W_1 and I_2 be from points W_2 to W_3. Let curves T_1 and T_2 be curves belonging to S_1 and S_2 respectively, both running from W_1 to W_2, such that T_1 and T_2 bound a non-zero area outside the regions of both bounded unit areas. Now remove T_1; area of S_1 consequently becomes larger while area of S_2 remains the same. Now perform a linear transformation on S_1 to reduce it to unit area and consequently reduce its perimeter. This new collection of curves have lesser total perimeter which is a contradiction.

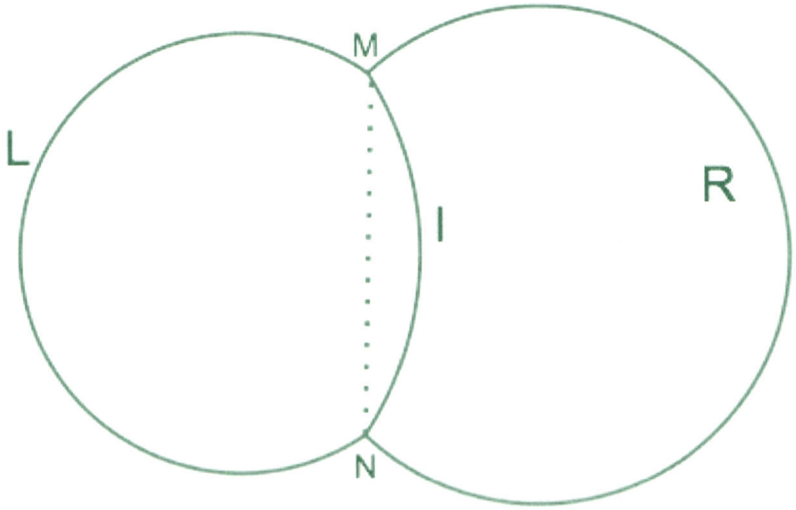
Thus $k = 1$ and S_1 and S_2 only intersect at a single curve I. Let the portions of curve solely belonging to S_1 and S_2 and not I be L and R respectively. Let I, L and R meet at points M and N, where $|MN| = d$. WLOG we have

$$A_{R,MN} - A_{I,MN} = 1 \tag{1}$$

$$A_{L,MN} + A_{I,MN} = 1 \tag{2}$$

Thus we have:

$$A_{L,MN} + A_{R,MN} = 2 \tag{3}$$



Because satisfaction of (1) and (2) is sufficient fro L, I and R to bound 2 unit areas, we find that since L, I and R form C_{min} and have total minimum perimeter, thus by Lemma 2.3., L, I and R are consequently circular arcs. It is evident that the circular arcs do not intersect since $A_{R,MN} > A_{I,MN}$ and L is on different sides of MN as I and R.

2.3.3 Case I: C_{min} symmetric

This is when I is line segment MN i.e. C_{min} is symmetric about MN.

Now since $P_I = d$ and $A_{I,MN} = 0$ thus by (1) and (2) we have $A_{L,MN} = A_{R,MN} = 1$. Note that L is a reflection of R about MN.

Let r and θ be the radius and arc angle of R. Now we have the following:

$$A_{R,MN} = \frac{1}{2}r^2(\theta - sin\theta) = 1 \tag{4}$$

$$d = 2rsin(\frac{\theta}{2}) \tag{5}$$

$$P_L = P_R = r\theta \tag{6}$$

Substituting (5) into (4) we get

$$A_{R,MN} = \frac{d^2(\theta - sin\theta)}{4(1 - cos\theta)} = 1 \tag{7}$$

Thus

$$d = \sqrt{\frac{4(1 - cos\theta)}{\theta - sin\theta}} \tag{8}$$

Substituting (5) into (6) we get

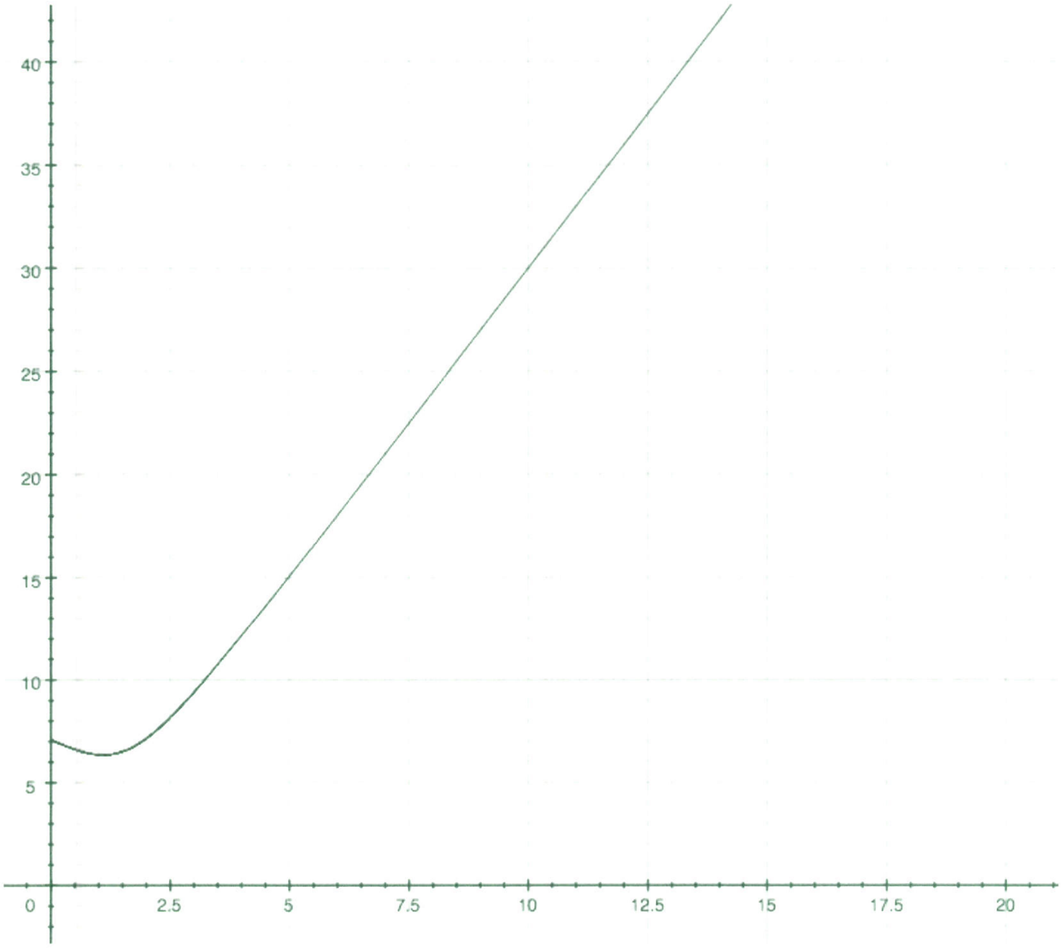
$$P_L = P_R = \frac{\theta d}{2sin(\frac{\theta}{2})} \tag{9}$$

Since $P_{total} = P_L + P_I + P_R = d + 2P_R$ thus

$$P_{total} = d(1 + \frac{\theta}{sin(\frac{\theta}{2})}) \tag{10}$$

Substituting (8) into (10) we get

$$P_{total} = (1 + \frac{\theta}{sin(\frac{\theta}{2})})\sqrt{\frac{4(1-cos\theta)}{\theta - sin\theta}} \tag{11}$$



The above is the graph of P_{total} against d. Since each value of d gives unique arcs L and R, it suffices to find the absolute minimum of P_{total} for all d.

$$\frac{\delta P_{total}}{\delta d} = \frac{\frac{\delta d}{\delta \theta}(1 + \frac{\theta}{sin(\frac{\theta}{2})}) + d \cdot \frac{\delta}{\delta \theta}(1 + \frac{\theta}{sin(\frac{\theta}{2})})}{\frac{\delta d}{\delta \theta}} \tag{12}$$

$$\frac{\delta P_{total}}{\delta d} = 1 + \frac{\theta}{sin(\frac{\theta}{2})} + \frac{(\frac{sin\frac{\theta}{2} - \frac{\theta}{2}cos\frac{\theta}{2}}{sin^2\frac{\theta}{2}})\sqrt{\frac{4(1-cos\theta)}{\theta - sin\theta}}}{\frac{2(\theta sin\theta + 2cos\theta - 2)}{(\theta - sin\theta)^2\sqrt{\frac{4(1-cos\theta)}{\theta - sin\theta}}}} \tag{13}$$

$$\frac{\delta P_{total}}{\delta d} = 1 + \frac{\theta}{sin(\frac{\theta}{2})} + \frac{2(1-cos\theta)(\theta - sin\theta)}{sin^2\frac{\theta}{2}} \cdot \frac{sin\frac{\theta}{2} - \frac{\theta}{2}cos\frac{\theta}{2}}{\theta sin\theta - 2(1-cos\theta)} \tag{14}$$

$$\frac{\delta P_{total}}{\delta d} = 1 + \frac{\theta}{sin(\frac{\theta}{2})} + \frac{2(1-cos\theta)(\theta - sin\theta)}{\frac{1}{2}(1-cos\theta)} \cdot \frac{sin\frac{\theta}{2} - \frac{\theta}{2}cos\frac{\theta}{2}}{2\theta sin\frac{\theta}{2}cos\frac{\theta}{2} - 4sin^2\frac{\theta}{2}} \tag{15}$$

$$\frac{\delta P_{total}}{\delta d} = 1 + \frac{\theta}{sin(\frac{\theta}{2})} + 4(\theta - sin\theta)\frac{-1}{4sin\frac{\theta}{2}} \tag{16}$$

$$\frac{\delta P_{total}}{\delta d} = 1 + \frac{\theta}{sin(\frac{\theta}{2})} + \frac{sin\theta - \theta}{sin\frac{\theta}{2}} = 1 + \frac{sin\theta}{sin\frac{\theta}{2}} \tag{17}$$

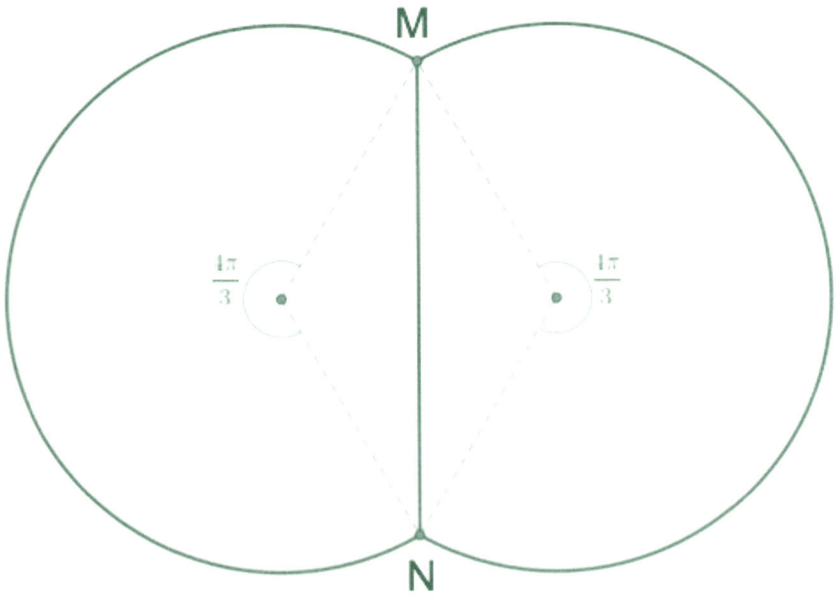
$$\frac{\delta P_{total}}{\delta d} = 1 + 2cos\frac{\theta}{2} \tag{18}$$

Since $\theta \in [0, 2\pi)$, thus $\frac{\theta}{2} \in [0, \pi)$

$$\frac{\delta P_{total}}{\delta d} = 1 + 2cos\frac{\theta}{2} = 0 \Rightarrow cos\frac{\theta}{2} = -\frac{1}{2} \Rightarrow \frac{\theta}{2} = \frac{2\pi}{3} \Rightarrow \theta = \frac{4\pi}{3} \tag{19}$$

$$\frac{\delta^2 P_{total}}{\delta d^2} = \frac{\delta}{\delta\theta}(\frac{\delta P_{total}}{\delta d}) \cdot \frac{\delta\theta}{\delta d} = -sin\frac{\theta}{2}(\frac{\sqrt{\frac{4(1-cos\theta)}{\theta - sin\theta}}(\theta - sin\theta)^2}{2(\theta sin\theta + 2cos\theta - 2)}) \tag{20}$$

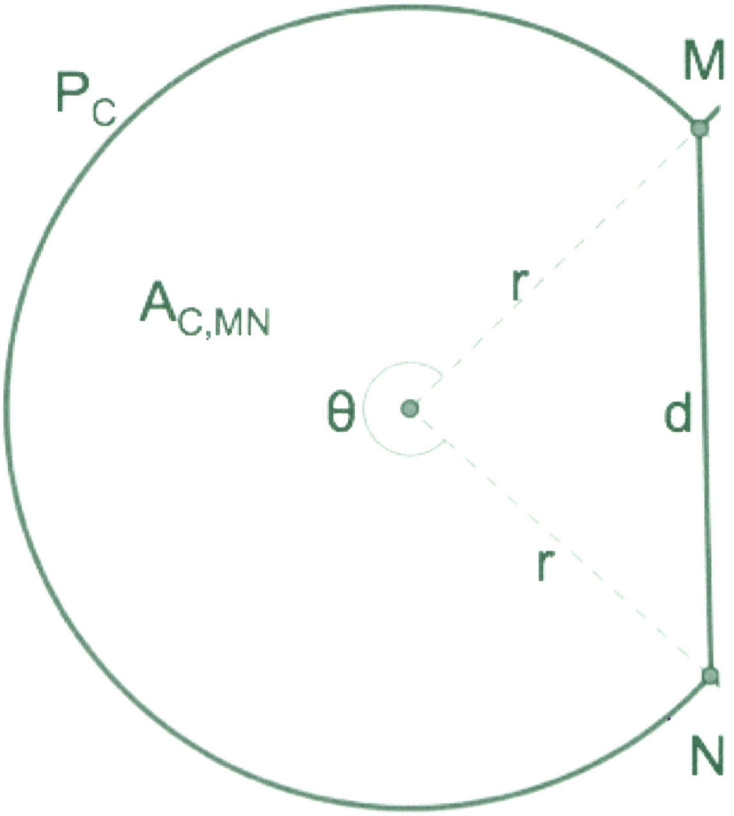
With (20), at $\theta = \frac{4\pi}{3}$, $\frac{\delta^2 P_{total}}{\delta d^2} \approx 2.35 > 0$ thus P_{total} achieves global minimum at $\theta = \frac{4\pi}{3}$.



By (11) we have that $L_{total} = \sqrt{\frac{32\pi}{3} + 4\sqrt{3}} \approx 6.35913$. By (4) and (8) we know that this is achieved with a straight edge of length $d = \sqrt{\frac{6}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}}} \approx 1.089$ and two circular arcs of radii $r = \sqrt{\frac{2}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}}} \approx 0.629$ and arc angle $\theta = \frac{4\pi}{3}$. This will be the minimum total perimeter for Case I (assuming $I = MN$ i.e. C_{min} is symmetrical about MN).

2.3.4 Case II: C_{min} asymmetric

This is when $I \neq MN$ i.e. C_{min} is not symmetrical.
Assume $|MN| = d$ is constant. For all circular arc C from M to N with radius and arc angle r and θ respectively,



$$A_{C,MN} = \frac{1}{2}r^2(\theta - sin\theta) \tag{21}$$

$$d = 2rsin(\frac{\theta}{2}) \tag{22}$$

$$P_C = r\theta \tag{23}$$

Substituting (22) into (21) and (23) we get

$$A_{C,MN} = \frac{d^2(\theta - sin\theta)}{4(1 - cos\theta)} \tag{24}$$

$$P_C = \frac{\theta d}{2sin\frac{\theta}{2}} \tag{25}$$

Now since $0 - sin0 = 0$ and $1 - cos0 = 0$ we have

$$\lim_{\theta \to 0} A_{C,MN} = \lim_{\theta \to 0} \frac{d^2 \frac{\delta}{\delta\theta}(\theta - sin\theta)}{4\frac{\delta}{\delta\theta}(1 - cos\theta)} = \lim_{\theta \to 0} \frac{d^2(1 - cos\theta)}{4(sin\theta)}; \tag{26}$$

Now since $sin0 = 0$ and $1 - cos0 = 0$ we have

$$\lim_{\theta\to 0} A_{C,MN} = \lim_{\theta\to 0} \frac{d^2\frac{\delta}{\delta\theta}(1-cos\theta)}{4\frac{\delta}{\delta\theta}(sin\theta)} = \lim_{\theta\to 0} \frac{d^2(sin\theta)}{4(cos\theta)} = \frac{d^2}{4}tan0 = 0; \tag{27}$$

Also note that

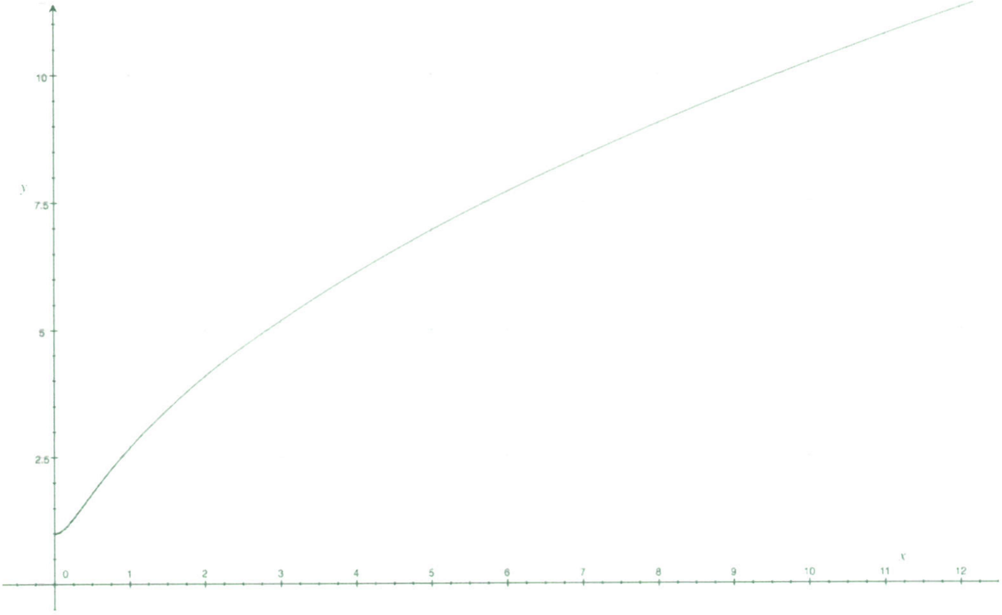
$$\lim_{\theta\to 2\pi} A_{C,MN} = \infty \tag{28}$$

And

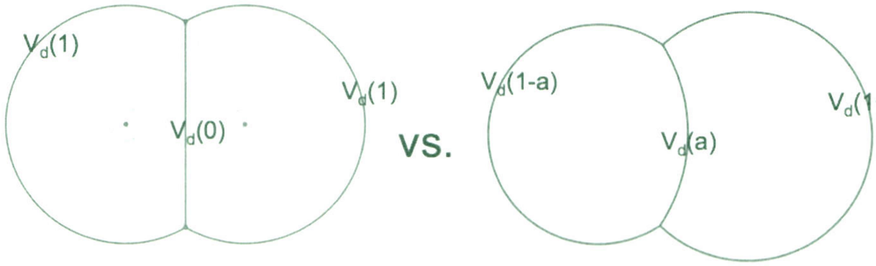
$$\frac{\delta A_{C,MN}}{\delta\theta} = \frac{d^2(2 - 2cos\theta - \theta sin\theta)}{4(1-cos\theta)^2} > 0 \tag{29}$$

Thus we can let $P_C = V_d(A_{C,MN})$ i.e. we can express P_C as a function V_d of $A_{C,MN}$. We note that V_d is a strictly increasing function. The following equation describes the convexity/concavity of V_d:

$$\frac{\delta P_C}{\delta A_{C,MN}} = \frac{2sin\frac{\theta}{2}}{d} \tag{30}$$



The above is the graph of P_C against $A_{C,MN}$ i.e. $P_C = V_d(A_{C,MN})$, for $d = 1$. Now for L, I and R, let $A_{I,MN} = a$. Now $A_{R,MN} = 1+a$, $A_{L,MN} = |1-a|$. For $a > 1$, let a=k+1. Then $L_{total} = P_L + P_I + P_R = V_d(k+1) + V_d(k+2) + V_d(k) > V_d(2) + V_d(1) + V_d(0)$. Thus it suffices to consider the case where $0 < a \leq 1$ i.e. $A_{L,MN} = 1 - a$ and $L_{total} = P_L + P_I + P_R = V_d(1+a) + V_d(1-a) + V_d(a)$.



Conjecture 2.4 *For any* $0 < a \leq 1$, $V_d(1+a) + V_d(1-a) + V_d(a) > 2V_d(1) + V_d(0)$.

The proof for $n = 2$ is still incomplete; the conjecture is yet to be proven, and is the final step to confirming that a straight edge of length $d = \sqrt{\frac{6}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}}} \approx 1.089$ and two circular arcs of radii $r = \sqrt{\frac{2}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}}} \approx 0.629$ and arc angle $\theta = \frac{4\pi}{3}$ are the curves with the least total perimeter ($L_{total} = \sqrt{\frac{32\pi}{3} + 4\sqrt{3}} \approx 6.35913$) to bound two unit areas.

3 Extensions

3.1 Consider Different Areas

A farmer has n animals. He needs to use fence to mark out n pens of areas $A_1, A_2, ..., A_n$ in the pasture. Assume the fences are infinitely thin and flexible. What is the minimum length of fence required, and the structure of fence that obtains this minimum length?

Remark 3.1 *Note that for this problem, solutions might not be symmetrical, and more complex structures may arise.*

3.2 Extend to k-Dimensions

A prison is to be designed to hold n prisoners such that each cell has volume 1. Assume the walls are infinitely thin. What is the minimum area of walls required?

3.3 Restrict Curves to Straight Edges

A farmer has n pigs, each requiring a pen of area n. Assume the fences are infinitely thin. What is the minimum length of fence required? At most 8 straight lengths of fence are permitted.

4 Acknowledgements

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5 References

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