What are Numbers and What should They Be? A Journey Backward

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A few years ago I had a discussion with my classmates on how integers could be defined. Simple as the question seemed to be, we found it hard to give a satisfactory answer. The best definition we came up with was 'a number with no decimal places', but isn't the term 'decimal place' defined as the part of a number other than the integral part!

We concluded that this was a meaningless question, since everyone knows what a whole number is.

At least this was what I believed, until I read Richard Dedekind's *Was sind und was sollen die Zahlen?* (*What are numbers and what should they be?* or *The nature and meaning of numbers*), published in 1888. It is one of the first complete and rigorous set-theoretical treatments of the foundation of arithmetic, i.e. the construction of natural numbers. I chose to read it not exactly because I was interested in natural numbers, as I thought I knew them pretty well, but rather that I was eager to understand the way of thinking of nineteenth-century mathematicians, and especially that of Dedekind, who is renowned for his sharp mathematical intuition, pursuit for rigor, and clarity in expression. Writing a review of a mathematical masterpiece is not easy for me, so in what follows I will just give a brief introduction and comment on the philosophy behind this work, as well as its significance in mathematics. Lastly, I will venture to share my personal feeling about it.

It would be suitable to call this work a journey backward: it deals with the derivation of what we believe to be the most fundamental concepts: infinity, natural numbers, induction[1], addition, multiplication, and the number of objects. These concepts are too easy and familiar to us that most of us would feel it pointless to ponder about their true nature and meaning. However, basic as they are in our understanding of the physical world, Dedekind finds that logically, they do have more basic origins, which are the set and the map. A *set* is just a collection of different things (called *elements*). A *map*, also called a *function*, is a rule of correspondence that links one thing to another thing. These two concepts are unmistakably more fundamental, as without them no thinking is possible. All those concepts mentioned above are defined in terms of sets and maps. Here I just give some examples to give a glimpse of what to expect in the book.

1. A set S is said to be *infinite*, if all elements in S can pair up in a one-to-one manner with all elements in another set which contains only some (but not all) of S's elements.

The latter set is called a *proper subset* of S. A one-to-one pairing-up is called a *bijective map*.

This sounds weird, since, for example, 5 tomatoes certainly cannot establish a one-to-one relationship with 3 tomatoes, which is a proper subset of 5 tomatoes! But this is because the set of 5 tomatoes is finite. In the case of an infinite set, we can map it bijectively to its proper subset. For example, all natural numbers can pair up with all even natural numbers: simply multiply each natural number by 2!

1 Induction can be understood by thinking of dominos: if the first domino falls, and if the falling of any particular domino guarantees the fall of the one immediately behind it, then the whole series of dominos, no matter how long, will certainly fall.

This strange relationship between 'the whole' and 'the part' is characteristic of infinite sets, and Dedekind used it as a definition.

2. There exists an infinite set. That is, we can construct a set that satisfies the above definition.

Of course the set of all natural numbers is infinite, but since at this point of the book, natural numbers are not yet defined, Dedekind proved this via a pretty amusing argument: the set which contains everything that can be possibly thought by people is infinite.

Let's call this set T, then for any thought t, (for example 'I am hungry',) we can relate it to another thought called φ(t), which is 't is contained in T', (for example, 'The thought "I am hungry" is contained in T'). Now if we use φ(T) to denote the set of all thoughts with the structure 'The thought ... is contained in T.', then it is obvious that φ(T) is only a proper subset of T, since not all thoughts are in the form of 'The thought ... is contained in T'. But it does relate in a one-to-one manner to all our thoughts. Therefore, by the definition of an infinite set, T is infinite.[2]

3. The set of natural numbers **N** = {1,2,3...} can be defined by the following construction:

a. Take an infinite set T. Then there exists a bijective map f that pairs up all elements in T with all elements in a proper subset of T. We call this proper subset $f(T)$. Since $f(T)$ is a proper subset, there exists at least one element in T that is not in $f(T)$. We pick any such element and call it 1.

b. **N** equals exactly the set $\{1, f(1), f(f(1)), ...\}$, with f and 1 explained above.[3]

This seems unfamiliar, but when we think about it, this is just an instruction of how to go step by step 'from here to infinity'. We need a starting point, which we call 1, and something to designate a single step, which is the function f. All elements in N are just positions that we get to in this long journey, so $\mathbf{N}=\{1, f(1), f(f(1)), ...\}$, where $f(1)$ means 'the position we get to after taking a step from 1', $f(f(1))$ means 'the position we get to after taking a step from $f(1)$', and so on.

Now in order to really go to infinity, there are some problems to consider:

a. How to make sure that 1 is really the starting point?

This is equivalent of making sure that in this journey, we cannot step from anywhere to 1. So we have to state 1 that $1 \neq f(n)$ for any element n in **N**. Or we say, 1 is not contained in $f(\mathbf{N})$.

b. How to make sure that we never get to the same place again (i.e. we never move in circles) but are forever marching forward?

Firstly, for reasons stated just now, we cannot step back to 1.

2 The idea of a set containing all possible human thoughts implies a set of everything, or 'a set of all sets', but further studies of set theory in early 20th Century show that this concept leads to some logical paradoxes. Therefore, this argument of Dedekind is not considered sound mathematics today. In contemporary mathematics the existence of an infinite set is simply assumed as an axiom.

3 In Die Zahlen Dedekind actually used a lengthier, but equivalent, expression, so as to lay the foundation of some of his other arguments.

Secondly, any position other than 1 has to be reached by stepping from somewhere else. Therefore, not getting back to the same position again means not stepping from two different places to the same place. That is, if $m \neq n$, then $f(m) \neq f(n)$. This is guaranteed by the bijectiveness of f, which says elements in **N** and $f(\mathbf{N})$ are paired up in a one-to-one manner.

Now, if we are lazy and simply label $f(1)$ as 2, $f(f(1))$ as 3, and so on, we are actually counting our steps! This is the nature of counting.[4]

4. A natural number m is said to be *smaller than* another number n, if the set $\{n, f(n), f(f(n)), ...\}$ (also called the chain of n) is a proper subset of the set $\{m, f(m), f(f(m)), ...\}$ (the chain of m).

For example, the chain of 2 is the set $\{2, 3, 4, 5 ...\}$, the chain of 3 is $\{3, 4, 5, ...\}$. The latter is a proper subset of the former, and therefore we say 2 is smaller than 3.

The definitions of addition, multiplication, and the concept of 'the number of elements' in a finite set are more complicated as they involve something called definition by recursion. The idea of recursion is similar to that of induction. Suppose we want to define addition $m+n$, where m and n are natural numbers, we may first define what is $m+1$, and then write a statement that relates $m+k$ and $m+f(k)$ for any natural number k. Now because $f(1)$ is 2, $f(2)$ is 3, and so on, we can understand $m+2$ from $m+1$, $m+3$ from $m+2$, and so on through this relationship, and all formulas in the form $m+n$ can be understood. The detailed definitions may be left for the reader to find out.

As it may be clearly seen, this work is immensely revolutionary. Its mission can be likened to that of Darwin's theory of origin of species - finding the origin of what people commonly think no further origin is there. Kronecker, Dedekind's contemporary mathematician, famously stated that 'God created the natural numbers, and all the rest is work of man.' *Die Zahlen*, however, challenged this by showing that even natural numbers are created by man. All concepts have the common ancestors called sets. But this analogy only refers to the theories' impacts in their respective fields, and isn't to be taken too seriously, as species are physical beings, natural numbers are, after all, abstract constructions.

But why should anyone care to define natural numbers? What is the use of doing all these? Here we must hold our surprise and examine what kind of philosophy had motivated Dedekind to overthrow our familiar, time-honoured basis of mathematics as natural numbers, but rebuild arithmetic upon set theory instead.

Firstly, the emergence of set theory, brought forward by Cantor, was itself hugely remarkable. Cantor mainly focused on how to harness infinity. He proved, simply speaking, that the number of real numbers is strictly larger than the number of natural numbers, but there are as many even numbers as integers, through the sole concepts of the set, the map, and pure logic. Dedekind was among the earliest people who understood Cantor's theory. Just the fact that such beautiful and astonishing results could be derived from so unattractive basis was extremely appealing. If properties about infinity can be determined by the study of sets, how about the number of elements in finite systems? Shouldn't it be promising that

4 You may have observed that f(n) is exactly n+1! But we cannot say so at this point, because addition has not been defined. Defining addition is not trivial!

this, and other properties concerning natural numbers, also be derived purely from sets? People (even Cantor) did not care to do this tedious derivation, as they thought the properties of natural numbers, like $m+n=n+m$, are immediately obvious. However, Dedekind wrote in the preface of *Die Zahlen* that, 'these seemingly obvious truths are never acquired by inner consciousness, but by repeated practice.' He made the analogy of people's assimilation of arithmetic theorems to the process of learning to read. Reading is based on spelling, which is itself very wearisome work for a starter, but because people read so much, as time goes by, they get practiced a lot and gradually possess the ability of recognising a word as soon as they see it, which is an instantaneous recall of spelling. However, we can't say that spelling is an easy task simply because an adult can read without difficulty. Similarly, the concepts of the natural numbers, infinity, etc, are actually based on the most fundamental basis of set theory. From the concepts of sets (just a collection of things) and map (relation between things), our brains actually had gone through a very long and tedious series of thinking to build the concepts of numbers as what they are today. However, this construction is so important to human beings that it developed when we were very young and still unconscious, perhaps even when we were still foetus, which means we were already 'accomplished readers who are not serious of spelling anymore' when we began to learn how to count. By questioning 'What are numbers and what should they be?', Dedekind wanted to 'redo the spelling seriously', to bring to light what has always been unconscious to us, and thereby to bring our familiar natural numbers to their true, solid foundation.

We are now just at the edge of understanding the spirit of 19th Century mathematics that underlies many mathematical works at Dedekind's time. Traditional viewpoint of mathematics, that it is a tool for us to understand nature's laws, which prevailed throughout Newton's physics, was losing its place. Increasing evidence (both in mathematics and in natural sciences) showed that mathematics has its own charm and peculiarity that can go beyond the role of describing the physical world. For example, mysterious properties of prime numbers, non-differentiable functions, series, and imaginary numbers cannot easily find their proper corresponding phenomena in nature, but are greatly attractive to the mind. (For example, who would expect that e, π, and i appear together in Euler's equation?) In the mid-19th Century, the traditional view of mathematics was further crushed by new achievements in mathematics. Euclid's geometry was shaken by new forms of geometry developed by Riemann, etc., and therefore, the perception of space that passed on from Galileo to Newton was known as only a special case of all possible geometries. Dedekind himself was at the heart of this revolution of mathematical philosophy. He probably learned about the discontinuity of matter (i.e. the theory of atoms and molecules) through his early studies on chemistry, and therefore realised that to serve for the room for all matters, the space actually does not need to be continuous. A continuous line could be our illusion only. By his ingenious and rigorous definition of continuity on the basis of a cut, he was further convinced that the concept of continuity and irrational numbers can be constructed purely by the human mind, without any help from the physical world or our intuition of space and time. Continuity is so, how about natural numbers? Surely, people created natural numbers for physical purposes like determining the amount of food, but do we really need the physical world? Or is it just a motivation that stimulates our brains to create them by pure logic? Encouraged by his former work on continuity (and other contemporary mathematical works), Dedekind believed that 'numbers are free creations of the human mind', which is his ultimate answer to 'what are numbers and what should they be', and it is the faith that lies in the core of the work.

Lastly I want to emphasise on the importance of Dedekind's motto in research that 'in science nothing capable of proof should be accepted without proof'. That is why as soon as he saw the possibility of founding arithmetic using set theory, he had the determination to continue working on the tiresome task until a truly rigorous system was constructed.

Die Zahlen is considered a masterpiece on the foundation of arithmetic. It was later developed into a simpler and more attractive form by Peano, which mathematics students learn today.[5] But besides this, it is also a representative work for 'structural reasoning' and 'axiomisation'. To be sure, these ideas already existed in Euclid's works, but the complete independence of our intuition of space and time, and the richness in the creation of new, helpful concepts are something for which *Die Zahlen* is unique. This style of doing mathematics influenced later mathematicians including Peano, Hilbert and Noether.

My personal feelings

The notion of a Dedekind cut from my mathematical analysis course is still echoing so powerfully in my mind, that whenever I see the word Dedekind I would still unconsciously relate it to the word continuity. Therefore, I am totally unprepared to see such excellent work when I started reading *Die Zahlen*, which exceeds *Stetigkeit und irrational Zahlen* (Dedekind's work on the cut, in which the 'natural origin' of natural numbers is assumed) in maturity, understanding of mathematics, and conciseness in expression.

I would say how much enthusiasm as well as courage is needed for a person who had already achieved so much to allow himself a 'metamorphosis', to overthrow the old system in the mind and build another upon the newly born idea of set theory! Even today few would understand why some people care about why $1<2$ instead of seeing them as insane, but Dedekind managed to convince the mathematical society at that time, due to his perfect rigor and clear expression. To me he is a symbol of great scientific ability, pure and strong passion, as well as open-mindedness, my definition of a great scientist.

So about Dedekind, remember not only the cut, but also *Die Zahlen* and perhaps more.

5 Actually, my interpretation of the creation of N above, with the help of the concept of a step-by-step journey, is inspired by Peano's formulation.