CWMI 2017

Day 1

1. Let p be a prime and n be a positive integer such that p^2 divides $\prod_{k=1}^{n}(k^2+1)$. Show that $p < 2n$.

2. Let n be a positive integer such that there exist positive integers $x_1, x_2, \ldots, x_n$ satisfying
$$x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n) = 100n.$$
Find the greatest possible value of n.

3. In triangle ABC, let D be a point on BC. Let I_1 and I_2 be the incenters of triangles ABD and ACD respectively. Let O_1 and O_2 be the circumcenters of triangles AI_1D and AI_2D respectively. Let lines I_1O_2 and I_2O_1 meet at P. Show that $PD \perp BC$.

4. Let n and k be given integers such that $n \geq k \geq 2$. Alice and Bob play a game on an n by n table with white cells. They take turns to pick a white cell and color it black. Alice moves first. The game ends as soon as there is at least one black cell in every k by k square after a player moves, who is declared the winner of the game. Who has the winning strategy?

Day 2

5. Let $a_1, a_2, \cdots, a_9$ be 9 positive integers (not necessarily distinct) satisfying: for all $1 \leq i < j < k \leq 9$, there exists $l(1 \leq l \leq 9)$ distinct from i, j and j such that $a_i + a_j + a_k + a_l = 100$. Find the number of 9-tuples $(a_1, a_2, \cdots, a_9)$ satisfying the above conditions.

6. In acute triangle ABC, let D and E be points on sides AB and AC respectively. Let segments BE and DC meet at point H. Let M and N be the midpoints of segments BD and CE respectively. Show that H is the orthocenter of triangle AMN if and only if B, C, E, D are concyclic and $BE \perp CD$.

7. Let $n = 2^{\alpha} \cdot q$ be a positive integer, where α is a nonnegative integer and q is an odd number. Show that for any positive integer m, the number of integer solutions to the equation $x_1^2 + x_2^2 + \cdots + x_n^2 = m$ is divisible by $2^{\alpha+1}$.

8. Let $a_1, a_2, \cdots, a_n > 0$ $(n \geq 2)$. Prove that
$$\sum_{i=1}^{n} max\{a_1, a_2, \cdots, a_i\} \cdot min\{a_i, a_{i+1}, \cdots, a_n\} \leq \frac{n}{2\sqrt{n-1}} \sum_{i=1}^{n} a_i^2.$$