The 39th China Mathematical Olympiad

Day 1

1. Find the smallest $\lambda \in \mathbb{R}$ such that for all $n \in \mathbb{N}_+$, there exists $x_1, x_2, \ldots, x_n$ satisfying $n = x_1 x_2 \ldots x_{2023}$, where x_i is either a prime or a positive integer not exceeding n^λ for all $i \in \{1, 2, \ldots, 2023\}$.

2. Find the largest real number c such that
$$\sum_{i=1}^{n}\sum_{j=1}^{n}(n-|i-j|)x_i x_j \geq c\sum_{j=1}^{n} x_i^2$$
for any positive integer n and any real numbers $x_1, x_2, \ldots, x_n$.

3. Let $p \geqslant 5$ be a prime and $S = \{1, 2, \ldots, p\}$. Define $r(x, y)$ as follows:
$$r(x,y) = \begin{cases} y - x & y \geqslant x \\ y - x + p & y < x \end{cases}.$$
For a nonempty proper subset A of S, let
$$f(A) = \sum_{x \in A}\sum_{y \in A}(r(x,y))^2.$$
A good subset of S is a nonempty proper subset A satisfying that for all subsets $B \subseteq S$ of the same size as A, $f(B) \geqslant f(A)$. Find the largest integer L such that there exists distinct good subsets $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_L$.

Day 2

4. Let $a_1, a_2, \ldots, a_{2023}$ be nonnegative real numbers such that $a_1 + a_2 + \ldots + a_{2023} = 100$. Let $A = \{(i, j) \mid 1 \leqslant i \leqslant j \leqslant 2023,\ a_i a_j \geqslant 1\}$. Prove that $|A| \leqslant 5050$ and determine when the equality holds.

5. In acute $\triangle ABC$, K is on the extension of segment BC. P, Q are two points such that $KP \parallel AB, BK = BP$ and $KQ \parallel AC, CK = CQ$. The circumcircle of $\triangle KPQ$ intersects AK again at T. Prove that:
   (1) $\angle BTC + \angle APB = \angle CQA$.
   (2) $AP \cdot BT \cdot CQ = AQ \cdot CT \cdot BP$.

6. Let P be a regular 99-gon. Assign integers between 1 and 99 to the vertices of P such that each integer appears exactly once. (If two assignments coincide under rotation, treat them as the same.) An operation is a swap of the integers assigned to a pair of adjacent vertices of P. Find the smallest integer n such that one can achieve every other assignment from a given one with no more than n operations.