IMO 2014 problems with selected solutions

IMO 2014
Cape Town - South Africa

Language: English
Day: 1

Tuesday, July 8, 2014

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problem 2. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Handwritten solutions for Problems 1 and 2 by SIMO team members can be found on page 30 - 33.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

IMO 2014
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Language: English
Day: 2

Wednesday, July 9, 2014

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

Handwritten solutions for Problems 4 and 5 by SIMO team members
can be found on page 34 - 35.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

We denote $a_n < \frac{a_0 + a_1 + \ldots + a_n}{n}$ as condition P, and $\frac{a_0 + a_1 + \ldots + a_n}{n} \leq a_{n+1}$ as condition Q.

It suffices to show exists a unique integer $n \geq 1$ which satisfies both conditions P and Q at the same time.

Note $\frac{a_0 + a_1}{1} > a_1$, so for $n=1$, condition P holds.

Lemma 1: If for some $i \geq 1$, condition Q fails, condition P will hold for $i+1$.

Suppose condition Q fails for some i, $\frac{a_0 + a_1 + \ldots + a_i}{i} > a_{i+1}$, then

$a_0 + a_1 + \ldots + a_i > i\, a_{i+1} \Rightarrow a_0 + a_1 + \ldots + a_i + a_{i+1} > (i+1) a_{i+1}$

$\Rightarrow \frac{a_0 + a_1 + \ldots + a_{i+1}}{i+1} > a_{i+1}$, hence condition P holds for $i+1$. Lemma proven.

Lemma 2: If for some $j \geq 1$, condition Q holds, condition P will fail for $j+1$ and condition Q will hold for $j+1$.

Suppose condition Q holds for some j, $\frac{a_0 + a_1 + \ldots + a_j}{j} \leq a_{j+1}$, then

$a_0 + a_1 + \ldots + a_j \leq j\, a_{j+1} \Rightarrow a_0 + a_1 + \ldots + a_j + a_{j+1} \leq (j+1) a_{j+1}$

$\Rightarrow \frac{a_0 + a_1 + \ldots + a_{j+1}}{j+1} \leq a_{j+1} < a_{j+2}$, so condition P fails and condition Q holds for $j+1$, lemma proven.

Let $a_0 + a_1 = k$, then as $a_0 < a_1 < \ldots$ and they are all positive integers, it follows $a_k > k$.

Hence $a_0 + a_1 + \ldots + a_{k-1} < k + (k-2) a_{k-1} < (k-1) a_k$

$\Rightarrow \frac{a_0 + a_1 + \ldots + a_{k-1}}{k-1} < a_k$

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If condition Q holds for $n=1$, by lemma 2 condition P will fail for all $n>1$ (since condition Q hold for $i \Rightarrow$ condition P fail for $i+1$, condition Q hold for $i+1$ $\Rightarrow$ condition P fail for $i+2$, condition Q hold for $i+2$ etc.)

Else, since condition Q holds for $n=k$, we know at some point condition Q must hold. (i.e. there exist n for which condition Q holds)

Consider the smallest integer m for which $n=m$ has condition Q hold.

As such, condition Q fails for $n=m-1$, by lemma 1 this means condition P holds for $n=m$.

By lemma 2, condition P fails for $n=m+1$ and condition Q holds for $n=m+1$, by induction condition P will fail for all $n>m$, and by our definition of m, condition Q fails for all $1 \leq n < m$, hence m is a unique integer such that for $n \geq 1$, $n=m$ is the only integer with that gives $a_n < \frac{a_0+a_1+\cdots+a_n}{n} \leq a_{n+1}$, that is condition P and Q both holding.

Hence we have proven the existence of a unique integer $n \geq 1$ that fulfils the conditions.

Hence done.

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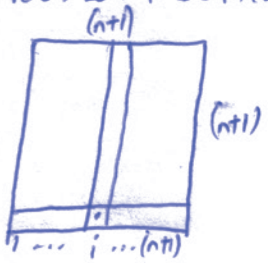
2. Let k_n = largest positive integer k for which a $k\times k$ square can always be found in a peaceful configuration of n rooks (essentially, what the question is ~~looking~~ asking for n).

Claim 1: k_n is non-decreasing.

Proof: Suppose we have ascertained k_n. Consider a peaceful configuration of $(n+1)$ rooks. Consider the rook in the bottommost row. Let it be in column i from the left.



Consider the remaining n columns (excluding column i) and the top n rows. Their intersection forms an $n\times n$ board with a peaceful configuration of n rooks, so $\exists$ a $k_n\times k_n$ square without any rooks. If this does not cross column i, we are done (since it will appear as it is on the $(n+1)\times(n+1)$ board). If it crosses column i, we are also done since we ~~in fact~~ actually obtain a $k_n\times(k_n+1)$ empty rectangle (since the top n rows of column i are empty).

$\therefore k_{n+1}\geq k_n \ \forall n\geq 2$.

Claim 2: $k_{x^2}<x$.

Proof: Label the rows $1,2,\dots,x^2$ from bottom to top and ~~to~~ label the columns $1,2,\dots,x^2$ from left to right. Denote (i,j) = row i, column j.

Place the rooks in the squares $(1,1),(2,x+1),(3,2x+1),\dots,(x,x^2-x+1)$;

$(x+1,2),(x+2,x+2),(x+3,2x+2),\dots,(2x,x^2-x+2)$;

$(2x+1,3),(2x+2,x+3),(2x+3,2x+3),\dots,(3x,x^2-x+3)$;

$\vdots$

$(x^2-x+1,x),(x^2-x+2,2x),(x^2-x+3,3x),\dots,(x^2,x^2)$.

Essentially all squares of the form $(ix+j+1,\ jx+i+1)\ \forall\ 0\leq i,j<x$. (This is clearly peaceful due to the unique expression of all numbers ~~between~~ from 1 to x^2 in the form $ix+j+1$.)

Consider an $x\times x$ square taking up rows $ax+b+1$ to $(a+1)x+b$ and columns $cx+d+1$ to $(c+1)x+d$, $0\leq a,b,c,d<x$

We have $ax+b+1\leq ix+j+1\leq(a+1)x+b \Leftrightarrow (a-i)x+b\leq j\leq(a+1-i)x+b-1$ —①

$cx+d+1\leq jx+i+1\leq(c+1)x+d \Leftrightarrow cx+(d-i)\leq jx\leq(c+1)x+(d-i-1)$ —②

If $i=a$, then ①: $b\leq j\leq x+b-1$ and ②: $cx+(d-a)\leq jx\leq(c+1)x+(d-a-1)$.

If $i=a+1$, then ①: $b-x\leq j\leq b-1$ and ②: $cx+(d-a-1)\leq jx\leq(c+1)x+(d-a-2)$.

(P.T.O.)

Contestant : Sheldon Kieran Tan
Problem : 2 (cont.)
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(cont'd)

Now if $\exists$ ~~ex~~ j with $cx+(d-a)\le jx \le (c+1)x+(d-a-2)$, choose this j. If $j\le b-1$, choose $i=a+1$ (clearly $b-x<0\le j$). If $j\ge b$, choose $i=a$ (clearly $x+b-1\ge x-1\ge j$). Hence $(ix+j+1, jx+i+1)$ is a rook in this $x\times x$ square.

If $\nexists j$ with $cx+(d-a)\le jx\le (c+1)x+(d-a-2)$, then $d-a-1=0$. If $c+1\ge b$, choose $j=c+1$, $i=a$ and we are done. If $c+1<b$, then $c\le b-1$, so choose $j=c$, $i=a+1$ and we are also done

$\therefore$ We can always find a rook $(ix+j+1, jx+i+1)$ in an $x\times x$ square $\Rightarrow k_{x^2}<x$.

Claim 3: $k_{x^2+1}\ge x$.

Proof: Label the columns $1,2,\dots,x^2+1$ from left to right. Consider a set of x columns $i+1,\dots,i+x$ $(0\le i\le x^2-x+1)$ containing the ~~bott~~ rook in the bottom row. Label the rows $1,2,\dots,x^2+1$ from bottom ~~top~~ to top (so the rook in the bottom row is in row 1). Clearly there are x rooks in $i+1,\dots,i+x$.

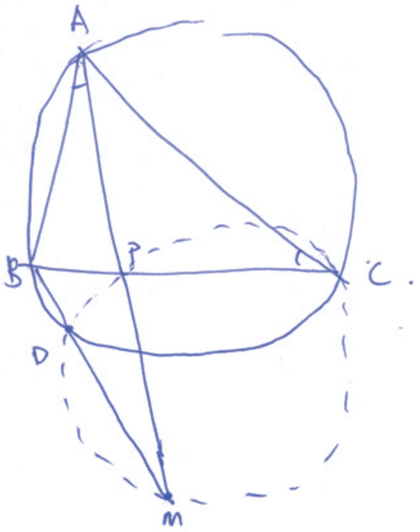
Consider the $x\times x$ squares in columns $i+1,\dots,i+x$ intersecting rows $ax+2, ax+3,\dots, ax+x+1$ for some $0\le a\le x-1$. Clearly there are x such disjoint squares (x possible values of a). However, the union of these squares only contains $(x-1)$ rooks (since there are x rooks in total but 1 of them is in row 1, which is not included), so at least one of these $x\times x$ squares is empty.

$\therefore k_{x^2+1}\ge x$.

Now from Claim 1 and 2, we have $k_i<\lceil\sqrt{i}\,\rceil$

From Claim 1 and 3, we have $k_i\ge\lceil\sqrt{i}\,\rceil-1$

$\therefore k_i=\lceil\sqrt{i}\,\rceil-1$ for all $i\Rightarrow k=\lceil\sqrt{n}\,\rceil-1$ is the desired answer (or $k=\lfloor\sqrt{n-1}\rfloor$, since $\lceil\sqrt{n}\,\rceil=\lfloor\sqrt{n-1}\rfloor+1$ for all positive integers n: obvious when $\sqrt{n},\sqrt{n-1}\notin\mathbb{Z}$. When $n=x^2$, $x=(x-1)+1$ holds, and when $n=x^2+1$, $x+1=x+1$ holds).



Let BM intersect the circumcircle of $\triangle ABC$ at D.

Since $\triangle BAP \sim \triangle BCA$:

Note $\angle CDM = \angle CAB = \angle APB = \angle MPC$

$\Rightarrow$ C, D, P, M concyclic.

$\Rightarrow$ $\triangle BPM \sim \triangle BDC$.

$\Rightarrow$ $\frac{BD}{DC} = \frac{BP}{PM} = \frac{BP}{PA} = \frac{AB}{AC}$.

Let CN intersect the circumcircle of $\triangle ABC$ at D'.

By symmetry (or similar arguments as above) we get $\frac{BD'}{D'C} = \frac{AB}{AC}$.

Since $D \neq A$ and $D' \neq A$, $D = D'$. Thus $BM \cap CN = D$, which lies on the circumcircle of $\triangle ABC$.

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5. Claim: Given a finite collection of such coins with total value at most $\frac{2n-1}{2}$, we can split it into $\leq n$ groups with each total value ≤ 1.

Proof: This is obvious when $n=1$ (total value $=\frac{1}{2}$). Suppose it is true for $n=i\geq 1$. Now consider $n=i+1$.

WLOG the total value of the coins is $\frac{2i+1}{2}$. (If not, let the total be x. Let $\frac{2i+1}{2}-x=\frac{p}{q}$, where $p,q\in\mathbb{Z}$. We can always add p coins of denomination $\frac{1}{q}$, and if we can split this into $\leq i+1$ groups, we can split the original set too.)

If $\exists$ a subset of coins with sum $=1$, we are trivially done by induction (just remove this subset). So suppose no subset of coins has sum $=1$. In particular, this implies that there are $<k$ coins with denomination $\frac{1}{k}$ $\forall k\in\mathbb{N}$.

Also, if there are 2 coins with denomination $\frac{1}{2k}$, we can just treat them as 1 coin of denomination $\frac{1}{k}$. So WLOG there is at most 1 coin with denomination $\frac{1}{2k}$ $\forall k\in\mathbb{N}$.

Hence the sum of all coins with denominations $\geq\frac{1}{2i}$ (i.e. denominator is $\leq 2i$) is

$$\leq \frac{1}{2}+\underbrace{\frac{2}{3}+\frac{1}{4}}_{<1}+\underbrace{\frac{4}{5}+\frac{1}{6}}_{<1}+\underbrace{\frac{6}{7}+\frac{1}{8}}_{<1}+\cdots+\underbrace{\frac{2i-2}{2i-1}+\frac{1}{2i}}_{<1}<\frac{1}{2}+(i-1)=\frac{2i-1}{2}$$

$\Rightarrow$ the sum of all coins with denominations $<\frac{1}{2i}$ (i.e. denominator is $>2i$) is more than 1.

Now we list all the coins in order from smallest to largest (i.e. largest denominator to smallest denominator), and start taking coins in increasing order and stop once the sum of the coins chosen just exceeds 1. From above, the denomination of the largest coin chosen is $<\frac{1}{2i}$ (otherwise the sum of all coins with denominations $<\frac{1}{2i}$ is less than 1).

Remove this set of coins. Now the sum of the remaining coins is $<\frac{2i-1}{2}$, so by induction, we can place these into i groups with sum ≤ 1 in each group.

Note that the smallest group will have sum $<\frac{\frac{2i-1}{2}}{i}=\frac{2i-1}{2i}$, so we can place the largest chosen coin (in the set that was removed) into this group (since new sum $<\frac{2i-1}{2i}+\frac{1}{2i}=1$).

Now by the definition of the subset we removed earlier, the sum of the remaining coins (without the largest coin) is ≤ 1. So letting this be our $(i+1)$th group, we are done.

$\therefore$ Our induction is complete and the question statement immediately follows from $n=100$.

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