Singapore International Mathematical Olympiad Committee

2014 National Team Selection Test

Day 1

Time allowed: 4 hours

1. Let $\mathbb{Z}^+$ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

Soln. The answer is $f(n) = n$. Setting $m = n = 2$ tells us that $4 + f(2) \mid 2f(2) + 2$. Since $2f(2) + 2 < 2(4 + f(2))$, we must have $2f(2) + 2 = 4 + f(2)$, so $f(2) = 2$. Plugging in $m = 2$ then tells us that $4 + f(n) \mid 4 + n$, which implies that $f(n) \leq n$ for all n.

Setting $m = n$ gives $n^2 + f(n) \mid nf(n) + n$, so $nf(n) + n \geq n^2 + f(n)$ which we rewrite as $(n - 1)(f(n) - n) \geq 0$. Therefore $f(n) \geq n$ for all $n \geq 2$. This is trivially true for $n = 1$ also. It follows that $f(n) = n$ for all n. This function obviously satisfies the desired property.

2. In a mathematical competition, there are 4 multiple-choice questions and each question has 3 choices. Among any group of 3 contestants, there is at least 1 question such that all 3 contestants give a different answer. Determine the maximum number of contestants.

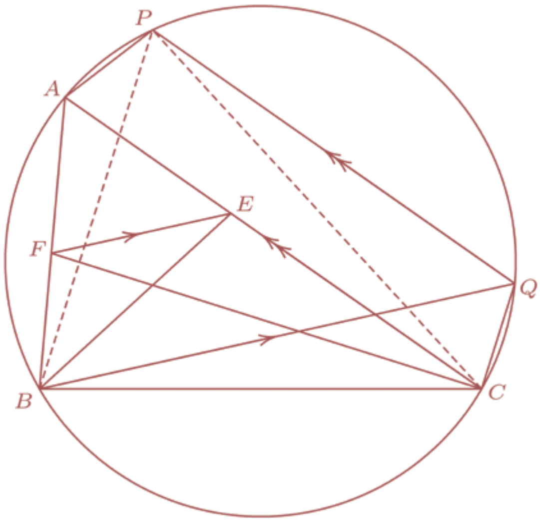
Soln. Let $f(n)$ denote the maximum number of contestants satisfying the above conditions where there are n questions. We have $f(1) = 3$. If 3 persons give different answer to any one question, then this question can be used to satisfy the given conditions for this group of 3 persons. Using the pigeonhole principle on the first question, there is one particular answer that will be selected by at most $\lfloor f(n)/3 \rfloor$ number of contestants. This means that the remaining $f(n) - \lfloor f(n)/3 \rfloor$ only choose the other 2 answers. So any group of 3 persons from the remaining $f(n) - \lfloor f(n)/3 \rfloor$ cannot use the first question to satisfy the given conditions. The question that is answered differently by all 3 contestants must come from one of the remaining $n - 1$ questions. Hence $f(n - 1) \geq f(n) - \lfloor f(n)/3 \rfloor \geq 2f(n)/3$. Thus, $f(2) \leq 4$, $f(3) \leq 6$ and $f(4) \leq 9$.

There can be 9 contestants from the following configuration.

$$\begin{array}{cccc} & A & B & C \\ Q1 : & \{1,2,3\}, & \{4,5,6\}, & \{7,8,9\} \\ Q2 : & \{1,4,7\}, & \{2,6,8\}, & \{3,5,9\} \\ Q3 : & \{1,5,8\}, & \{2,4,9\}, & \{3,6,7\} \\ Q4 : & \{1,6,9\}, & \{2,5,7\}, & \{3,4,8\} \end{array}$$

3. In an acute triangle ABC, $AC > AB$, E, F are points on AC, AB respectively such that BE, CF bisect $\angle B$, $\angle C$ respectively. Points P and Q are on the minor arc AC of the circumcircle of the triangle ABC such that AC is parallel to PQ and BQ is parallel to FE . Show that $PA + PB = PC$.

Soln.



Since AC is parallel to PQ, $ACQP$ is an isosceles trapezium so that $AP = CQ$. Thus $\angle ABP = \angle CBQ$ so that $\angle ABQ = \angle PBC$. It follows that $\angle AFE = \angle PBC$. Since $\angle BAC = \angle BPC$, we have the triangles AFE and PBC are similar. Let $BC = a$, $AC = b$ and $AB = c$. By angle bisector theorem, $AE = \frac{bc}{a+c}$ and $AF = \frac{cb}{a+b}$. It follows that $\frac{PB}{PC} = \frac{AF}{AE} = \frac{a+c}{a+b}$. We are to show $PA + PB = PC$. Hence, it suffices to show that $PA+PB = PB\cdot\frac{a+b}{a+c}$, or $PA = PB\cdot\frac{b-c}{a+c}$. Ptolemy's theorem implies $a\cdot PA+c\cdot PC = b\cdot PB$. Hence, $a \cdot PA + \frac{a+b}{a+c} \cdot c \cdot PB = b \cdot PB$, which gives $a \cdot PA = (b - \frac{a+b}{a+c} \cdot c) \cdot PB = a \cdot \frac{b-c}{a+c} \cdot PB$. This completes the proof.

Second solution. Let $\angle QBC = \angle PBA = \angle PCA = \theta$, and let the circumradius of the triangle ABC be R. Then $PA = 2R\sin\theta$, $PB = 2R\sin(C+\theta)$ and $PC = 2R\sin(B-\theta)$. Thus it suffices to verify that $\sin\theta + \sin(C + \theta) = \sin(B - \theta)$. Let EF intersect BC at M. Then AM is the external angle bisector of $\angle A$. Direct calculation using angle bisector theorem give $CE = ab/(a + c), MC = ab/(b - c), AE = bc/(a + c), AF = bc/(a + b)$. Using sine rule on the triangles MCE and AFE, we obtain $\sin\theta/\sin(C+\theta) = CE/MC = (b - c)/(a + c)$, and $\sin(B - \theta)/\sin(C + \theta) = AE/AF = (a + b)/(a + c)$. Consequently, $\sin\theta+\sin(C+\theta) = [(b-c)/(a+c)+1]\sin(C+\theta) = [(a+b)/(a+c)]\sin(C+\theta) = \sin(B-\theta)$.

4. Prove that if $m \geq 1$, and C is a subset of $\{0, 1, \ldots, m\}$ such that

$$|C| \geq \frac{m}{2} + 1,$$

then some power of 2 is either an element of C or the sum of two distinct elements of C.

Soln. The proof is by induction on m. It is easy to check that the result is true for $m = 1, 2, 3, 4$. Let $m > 4$ and assume that the result holds for all integers $m' < m$. Choose $s \geq 2$ such that

$$2^s \leq m < 2^{s+1}.$$

Let $r = m - 2^s$ and

$$C' = C \cap \{0, \ldots, 2^s - r - 1\}$$

and

$$C'' = C \cap \{2^s - r, \ldots, 2^s + r\}.$$

Then C is the disjoint union of C' and C'', and

$$|C| = |C'| + |C''|.$$

Suppose the result is false for C. Then $|C| \geq m/2 + 1$, but no power of 2 either belongs to C or is the sum of two distinct elements of C. It follows that $2^s \notin C''$ and for each $i = 1, ..., r$, the set C'' contains at most one of the two integers $2^s - i, 2^s + i$. Therefore,

$$|C''| \leq r.$$

If $m = 2^{s+1} - 1$, then $r = 2^s - 1$ and $C' \subseteq \{0\}$; thus

$$|C'| \leq 1.$$

It follows that

$$\frac{m}{2} + 1 \leq |C| \leq 1 + r = 2^s = \frac{m+1}{2},$$

which is impossible.

Similarly, if $2^s \leq m < 2^{s+1} - 1$, then $0 \leq r < 2^s - 1$ and $m' = 2^s - r - 1 \geq 1$. Since the set C contains C', it follows that no power of 2 either belongs to C' or is the sum of distinct elements of C'. By the induction hypothesis, we have

$$|C'| < \frac{m'}{2} + 1 = \frac{2^s - r - 1}{2} + 1,$$

and so

$$\frac{m}{2} + 1 \leq |C| = |C'| + |C''| < \frac{2^s - r - 1}{2} + 1 + r = \frac{m+1}{2},$$

which is also impossible.

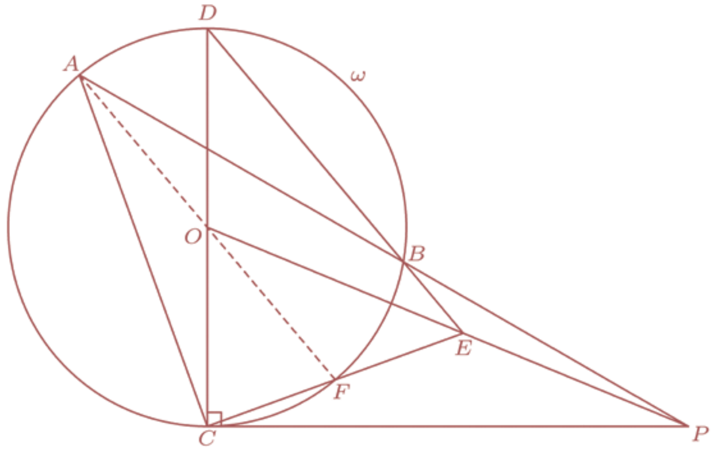
Remark: The source of this problem is the book: Additive Number Theory: Inverse Problems and the Geometry of Sumsets (pp 31-33). This is actually a lemma used to prove the following result: If $S \subset \{1, \ldots, n\}$ and $|S| > \frac{n}{3}$, then it is possible to select at most four, not necessarily distinct elements from S, whose sum is a power of 2.

Day 2

Time allowed: 4.5 hours

5. Let CD be a diameter of a circle ω centered at O. Points A and B on ω are on opposite sides of the line CD such that the tangent to ω at C intersects the line AB at P. The lines DB and OP intersect at E. Prove that $\angle ACE = 90°$.

Soln.



Solution 1. Let CE intersect ω at F. It suffices to show that A, O, F are collinear. Consider the hexagon $ABDCCF$, we have AB intersects the tangent at C at the point P, BD intersects CF at E and DC intersects FA at O'. By Pascal's theorem, P, E, O' are collinear. This implies $O' = O$ so that A, O, F are collinear.

Solution 2. Let ω be the circle $x^2 + y^2 = 1$. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$, where $a_1^2 + a_2^2 = 1$ and $b_1^2 + b_2^2 = 1$. Then $P = (\frac{a_2b_1 - a_1b_2 - a_1 + b_1}{a_2 - b_2}, -1)$. Using the relation $b_1^2 + b_2^2 = 1$, we obtain $E = (\frac{a_1b_2 - a_2b_1 + a_1 - b_1}{a_1b_1 + a_2b_2 - 1}, \frac{a_2 - b_2}{a_1b_1 + a_2b_2 - 1})$. Thus

$$\overline{CE} = \langle \frac{a_1b_2 - a_2b_1 + a_1 - b_1}{a_1b_1 + a_2b_2 - 1}, 1 + \frac{a_2 - b_2}{a_1b_1 + a_2b_2 - 1} \rangle \quad \text{and} \quad \overline{CA} = \langle a_1, a_2 + 1 \rangle.$$

It follows that $\overline{CE} \cdot \overline{CA} = \frac{(a_1^2 + a_2^2 - 1)(1 + b_2)}{a_1b_1 + a_2b_2 - 1} = 0$. That is $\angle ACE = 90°$.

6. Prove that in any set S of 2000 distinct real numbers there exist two distinct pairs (a, b), (c, d) so that

$$\left| \frac{a - b}{c - d} - 1 \right| < \frac{1}{100000}.$$

Soln. Let $D_1 \leq D_2 \leq \cdots \leq D_m$ be the distances between them, displayed with their multiplicities. Hence $m = 1000 \cdot 1999$. By rescaling, we may assume that $D_1 = 1 = x - y$ where $x, y \in S$. Evidently, $D_m = v - u$ where v is the largest and u the smallest number in S.

If $D_{i+1}/D_i < 1+10^{-5}$ for some $i = 1, 2, \ldots, m-1$, then the required inequality holds because $0 \le \frac{D_{i+1}}{D_i} - 1 < 10^{-5}$. Thus we may assume that

$$D_{i+1}/D_i \ge 1 + 10^{-5} \quad \text{for all } i = 1, 2, \ldots, m-1.$$

Therefore

$$v - u = D_m = \frac{D_m}{D_1} = \frac{D_m}{D_{m-1}} \cdots \frac{D_3}{D_2}\frac{D_2}{D_1} \ge \left(1 + \frac{1}{10^5}\right)^{m-1}.$$

From $m-1 = 1000 \cdot 1999 - 1 > 19 \cdot 10^5$ and the fact that for $n \ge 1$, $(1+\frac{1}{n})^n \ge 1 + \binom{n}{1}\frac{1}{n} = 2$, we get

$$\left(1 + \frac{1}{10^5}\right)^{19 \cdot 10^5} = \left(\left(1 + \frac{1}{10^5}\right)^{10^5}\right)^{19} \ge 2^{19} = 2^9 2^{10} > 500 \cdot 1000 > 2 \cdot 10^5,$$

and so $v - u = D_m > 2 \cdot 10^5$.

Thus either

$$\text{(i)} \quad v - x > 10^5 \quad \text{or (ii)} \quad x - u > 10^5.$$

In case (i), we have $v > x > y$. The result holds with $a = v$, $b = y$, $c = v$ and $d = x$.

In case (ii), we have $x > y > u$. The result holds with $a = y$, $b = u$, $c = x$ and $d = u$.

7. Let n be a positive integer and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence so that $a_{n+i} = a_i$ for all $i \ge 1$. If

$$a_1 \le a_2 \le \cdots \le a_n \le a_1 + n$$

and

$$a_{a_i} \le n + i - 1 \quad \text{for } i = 1, 2, \ldots, n,$$

prove that

$$a_1 + a_2 + \cdots + a_n \le n^2.$$

Soln. In the coordinate plane draw the bar chart where in column i, there is bar of height a_i. Let the configuration obtained be P. Reflect P about the line $x = y$ and then translate to the left by n. Call the image Q. We claim that P and Q do not intersect.

Suppose on the contrary that P and Q intersect in a cell (i, j).

If $a_i \le n$, then $j = a_i$ and thus $a_{a_i} \ge n + i$. But this gives a contradiction as $a_{a_i} \le n + i - 1$.

If $a_i \ge n + 1$, then $j = n$ as column i certainly intersects row n. Thus $a_n \ge n + i$. From $a_n \le a_1 + n$, we get $a_1 \ge i$. Thus $a_{a_1} \ge a_i \ge n + 1$. But $a_{a_1} \le n + 1 - 1 = n$ and we have a contradiction.

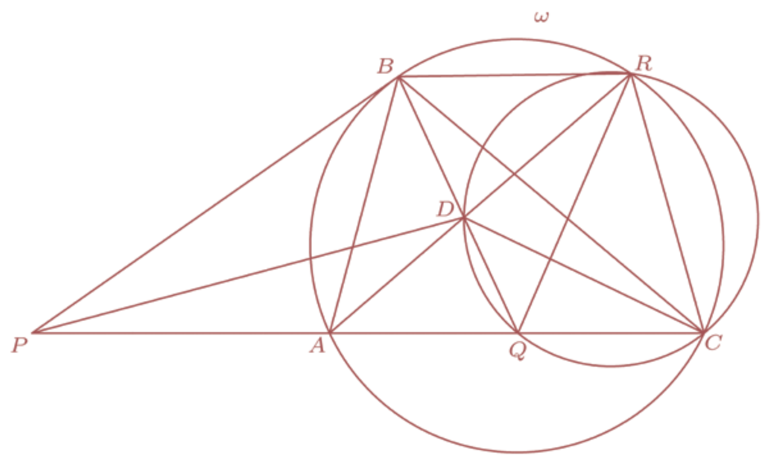
Consider the $n \times n$ square S in the first quadrant. The portion of P that is outside S is congruent to the portion of Q inside S. Thus the area of P which is $\sum a_i \le n^2$.

Day 3

Time allowed: 4.5 hours

8. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of the segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

Soln.



Denote by ω the circumcircle of the triangle ABC, and let $\angle ACB = \gamma$. Note that the condition $\gamma < \angle CBA$ implies that $\gamma < 90°$. Since $\angle PBA = \gamma$, the line PB is tangent to ω, so $PA \cdot PC = PB^2 = PD^2$. By $PA/PD = PD/PC$, the triangle PAD and PDC are similar, and $\angle ADP = \angle DCP$.

Next, since $\angle ABQ = \angle ACB$, the triangles ABC and AQB are also similar. Then $\angle AQB = \angle ABC = \angle ARC$, which means that the points D, R, C, and Q are concyclic. Therefore $\angle DRQ = \angle DCQ = \angle ADP$.

Now from $\angle ARB = \angle ACB = \gamma$ and $\angle PDB = \angle PBD = 2\gamma$, we get

$\angle QBR = \angle ADB - \angle ARB = \angle ADP + \angle PDB - \angle ARB = \angle DRQ + \gamma = \angle QRB$, so the triangle QRB is isosceles, which yields $QB = QR$.

9. Find an explicit formula for the least number $f(n)$ of distinct points in the plane such that for each $k = 1, 2, \ldots, n$, there exists a straight line containing exactly k of these points.

Soln. Suppose there is a set of S of points such that, for $1 \leq k \leq m$, there is a straight line ℓ_k containing exactly k points of S.

Now suppose that $m = 2n$. Now remove points from $\ell_n, \ell_{n+1}, \ldots, \ell_{2n}$ in that order. Since ℓ_{n+j}, $1 \leq j \leq n$, intersects $\ell_n, \ell_{n+1}, \ldots, \ell_{n+j-1}$ in at most j points, when the points on line ℓ_{n+j} are removed, at most j points had been removed previously. Thus at least n points are removed. Hence the total number of points is at least $n(n+1)$. Thus $f(2n) \geq n(n+1)$.

If $m = 2n+1$, remove points from $\ell_{n+1}, \ell_{n+2}, \ldots, \ell_{2n+1}$ in that that order. Now at each step, the number of points removed is at least $n+1$. Therefore $f(2n+1) \geq (n+1)^2$.

For $m = 2n$, we now construct a configuration of $n(n+1)$ points that satisfies the given condition. Take a set of $2n+1$ lines $\ell_0, \ell_1, \ell_2, \ldots, \ell_{2n}$ with no 2 parallel and no 3 concurrent. For each pair (u, v) with $n \leq u < v \leq 2n$, place a point at the intersection of ℓ_u and ℓ_v. This yields $\binom{n+1}{2} = n(n+1)/2$ points and now each of the lines $\ell_n, \ldots, \ell_{2n}$ contains n points. For each j with $1 \leq j \leq n-1$, place a point at the intersection of ℓ_j with $\ell_{2n+1-j}, \ldots, \ell_{2n}$. Finally, place a point at the intersection of ℓ_0 with $\ell_{n+1}, \ldots, \ell_{2n}$. This yields another $1 + 2 + \cdots + n = n(n+1)/2$ points giving a total of $n(n+1)$ points. Also for each $j = 1, 2, \ldots, 2n$, ℓ_j has exactly j points.

For $m = 2n+1$, we do a similar construction, with a set of $2n+2$ lines $\ell_0, \ldots, \ell_{2n+1}$. Here we first place a point at the intersection of ℓ_u, ℓ_v with $n+1 \leq u < v \leq 2n+1$. Then we do similar thing to $\ell_0, \ldots, \ell_n$.

Thus $f(2n) = n(n+1)$ and $f(2n+1) = (n+1)^2$.

10. Fix an integer $k \geq 2$. Two players, called Ana and Banana, play the following game of numbers: Initially, some integer $n \geq k$ gets written on the blackboard. Then they take turns to move, with Anna making the first move. A player making a move erases the number m that has just been written and writes a number m', with $k \leq m' < m$, that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geq k$ is called *good* if the initial number is n and Banana has a winning strategy. Otherwise it is *bad*.

Prove that two integers n and n' are either both good or both bad if for any prime $p \leq k$, $p \mid n$ iff $p \mid n'$.

Soln. We shall only consider integers $\geq k$. For two integers m, n, we say $m \sim n$ if the prime factors of m and n which are $\leq k$ coincide. Also we define $f(n)$ to be the smallest integer such that $n \sim f(n)$.

Note that in a move a good number can only be replaced by a bad number. Thus we have the following two lemmas.

Lemma 1. For any two integers m, n, if $\gcd(m, n) = 1$ then at least one of them is bad.

Lemma 2. Given an integer n, if for all $m \leq n$ with $\gcd(m, n) = 1$, m is bad, then n is good.

The third lemma is crucial.

Lemma 3. For any n, the prime factors of $f(n)$ are $\leq k$.

Proof. If suffices to prove that there is an integer $x \leq n$ such that $x \sim n$ and x has no prime factors $> k$. If n contains no prime factor $> k$, we can take $x = n$.

Otherwise n has a prime factor $q > k$. Let a be the product of all the prime factors of n which are $\leq k$ and let p be one of these prime factors. If $a \geq k$, we can take $x = a$. Otherwise $a < k$. Let j be the smallest power such that $p^j a \geq k$. Then $j \geq 1$ and $p^{j-1}a < k < q$. Thus $p^j a < pq \leq n$. Thus we can take $x = p^j a$.

The required result then follows from the next lemma.

Lemma 4. For any integer n, n is good iff $f(n)$ is good.

Proof. We shall prove by induction on n. The base case $n = k$ is trivially true as $k = f(k)$. Let n be an integer $> k$. Now we suppose that the result holds for all integers less than n. Suppose n is good. Let $m < f(n)$ be such that $\gcd(m, f(n)) = 1$. Then, by Lemma 3, $\gcd(f(m), n) = 1$. Since n is good, $f(m)$ is bad by Lemma 1. By the induction hypothesis, m is also bad. Then by Lemma 2, $f(n)$ is good.

Conversely, suppose that $f(n)$ is good. For any integer $m < n$ with $\gcd(m, n) = 1$, we have $\gcd(m, f(n)) = 1$ by Lemma 3. Thus m is bad by Lemma 1. Hence n is good.

The selection and training of the Singapore team to the International Mathematical Olympiad is the responsibility of Singapore International Mathematical Olympiad Committee (SIMO). The national team is selected through the Singapore Mathematical Olympiad (Open section). These students undergo rigorous training from October to April. The final six members of the national team are selected based on the results of the National Team Selection Tests. The 2014 version of the tests is published in this issue. See next page for the flow chart of the selection process.

Selection Procedure of IMO participants