Singapore Mathematical Project Festival 2014

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Projects presented at the Festival Congress

This year, 10 teams from secondary schools are selected to present their projects at the SMPF 2014 Festival Congress on March 22, 2014 at Nanyang Girls' High School. Below is a compilation of the summaries of the projects submitted by the teams.

(Anticlockwise from top)
1. Opening speech by SMS Vice President Prof Tan Victor.
2. The oral presentation was held in Lecture Theatre 1 at NYGH.
3. Junior section judges Dr Toh Pee Choon, Dr Teo Kok Ming, Prof Zhao Dongsheng.
4. Interview session by Senior section judges Dr Ku Cheng Yeaw, Prof Tay Tiong Seng, Prof Yang Yue.

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Junior Section

Project 1: Conway's Game of Life

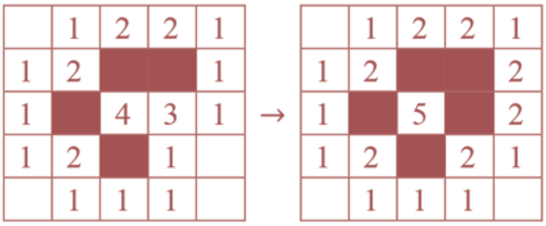
- Team members: Wang An Aloysius, Koh Shang Hui
- Mentor: Wang Qian
- School: Hwa Chong Institution (High School)

- Award: Gold & Foo Kean Pew Memorial Prize

Summary

The universe of the Game of Life is a grid of square cells, each of which is either alive or dead. Every cell reacts with the 8 surrounding cells. The rules of the game are that:

1. Any live cells with fewer than two live neighbors dies by under-population
2. Any live cells with two or three live neighbors lives on to the next generation
3. Any live cell with more than three live neighbors dies by overcrowding
4. Any dead cell with exactly three live neighbors becomes a live cell by reproduction.



These rules would be applied in each generation until the cells die out, or becomes stable. Stable life comprises of two different components. Firstly, an oscillator is a pattern that returns to its original state in the same orientation and position after a finite number of generations. Next, spaceships are patterns that reappear after a certain number of generations in the same orientation but in a different position.

Conway's Game of Life is mainly focused around square tessellations, perhaps because Conway felt that other tessellations of shapes would not be able to sustain life as there are too little neighboring cells to interact with. Therefore, our group wants to experiment with hexagonal tessellations.

One of the conditions for Conway's Game of Life is that the game must satisfy class 4 complexity. As mentioned in *"Cellular Automata: A Discrete Universe"*, in class 4 complexity, nearly all initial patterns evolve into structures that interact in complex ways, with formation of local structures that are able to survive for many generations. With this in mind, our project seeks to identify a rule for Hexagonal-Life, such that class 4 complexity can be achieved. By comparing number of live cells and bounding box of the expansion after 20 generations, we aim to find the situation of hex-life that is closest to Conway's Game of Life and therefore, deduce the best set of rules to ensure continuous growth. (Hex-life here refers to life with hexagonal tessellations. In the conventional Conway's game of life, each cell has 8 neighboring cell, whereas in hex-life, each cell only have 6 cells. We are trying to investigate if the rules for the hex-life are the same as those of the original Conway's game of life, but scaled by a factor of 0.75.) We would also like to investigate fractal properties in Hex-life using the Minkowski-Bouligand dimension.

Project 2: Optimal Strategy Usage via Geometric Figures

- Team members: Clarence Chew Xuan Da
- Mentor: Chai Ming Huang
- School: NUS High School of Mathematics and Science

- Award: Gold

Summary

This experimental project is to serve as exploration when combinatorics and a bit of geometry is combined. With area as the main variable as a part of a regular polygon, as well as the introduction of 2 players, Alice and Bob, playing a game where each play has theoretically infinitely many moves, this project aims to see how optimal players would behave in a game where moves span a continuum. An integer n □ 3 is chosen and a regular n-gon with area of 1 is formed. Alice and Bob are going to play a game on this regular n-gon, where Alice is the first player. On every turn, each player chooses an unchosen side and picks a point on that line segment. This point includes the endpoints.

Only by observing how the game works does one appreciate it. My mentor has provided me with notation, real numbers spanning from 0 to 1 (0: no area, 1: maximal area) defining the area which is formed by the convex hull of all points picked. At this rate, this game terminates with the first player attempting to maximise the area of the final convex hull of the picked points while the second player minimises the area of the hull. Distance between points appears to affect the final area although this is not rigorously proven yet.

The game for $n = 3$ terminates with an area of 1/4, shy of the area formed with $n = 4$ as 1/2. Nine as n would be a problem too hard, truth is when $n = 6$ the game behaves in 6-dimensional space which makes it too hard for my thoughts.

To code this down, my computer has to have a program with a tree-output algorithm and an algorithm that accepts a tree as input. The algorithm for tree output is given in my report.
With enough time and good coding, the final algorithm will unlock the secrets to this game.

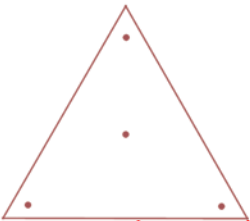
Project 3: Winning Strategy of Game of Criss-Cross

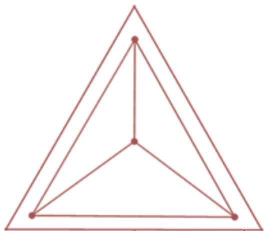
- Team members: See Xiaomin, Kwak Yoon Joo, Chen Qi Jia, Geraldine Zhang Fang
- Mentor: Chan Sock Har
- School: Singapore Chinese Girls' School

- Award: Silver

Summary

The Game of Criss-Cross is played by two players. The game board is created by drawing three points at the vertices of a large equilateral triangle, additional points can be drawn anywhere in the interior of the triangle. Figure 1 shows an example of one additional dot inside the equilateral triangle.





Two Players (Player 1 and Player 2) take their turns to draw a single straight line segment joining any two points, as long as the segment does not pass through any other points or segments already appearing on the game board. The winner is the last player able to make a legal move. Figure 2 shows an example of line segments drawn by Player 1 and 2. The dotted lines are representing the line segments drawn by Player 1 and the solid lines are representing the line segments drawn by Player 2. Both players have drawn 3 line segments. Since Player 2 is the last one to make the legal move, Player 2 is the winner.

We want to find out the winning strategies of the Game of Criss-Cross. From our observation, it seems that Number of Line Segments = 3 × [Number of dots □ 2].

To test out the validity of this hypothesis, we try to find out the relationship between Vertices (V) Edges (E) and Faces (F). We continuously play the game by varying the number of dots drawn. It is observed that 2E = 3F. As we know from Euler's Formula: V – E + F = 2, this gives E = 3(V – 2).

Hence, it is concluded the winning strategies for the Game of Criss-Cross is such that when the number of dots (V) is even, the number of edges (E) is also even, then Player 2 is the winner. When the number of dots (V) is odd, the number of edges (E) is also odd, then Player 1 is the winner.

The extension of the problem is to investigate that if the additional point(s) lie on the line segments joining the 3 dots at the corners of the equilateral triangles, will the winning strategies still work?

We carried out investigation on the extension of the problem using additional 3 dots. It was found that instead of Player 2 wins (by the winning strategies), Player 1 is the winner. Hence, it shows that the strategies do not work for this case.

Project 4: Josephus Problem: History, Extensions, Generalizations

- Team members: Chan Wayde, Chen Pang Yen Byron, Quek Sze Long
- Mentor: Jiang Zewei
- School: NUS High School of Mathematics and Science

- Award: Silver

Summary

Our problem is based from a Jewish historian, Josephus Flavius. The problem involves n number of people sitting in a circle. Thereafter, every second person, i.e. 2, 4, 6 and so on, will be killed until the last person left. Our aim is to find that person's position, y, in the circle. The original Josephus problem had the third person killed instead of the second but our group have done the latter as it is simpler than the original. First, we plotted a table of n and y using the results when n = 1, ..., 10. By doing so, we found a pattern: when the value of n can be expressed as a power of 2, i.e. 2, 4, 8 and so on, the value of y is 1. Otherwise, with each successive case where n is not a power of two, the value of y increases by 2. Therefore, the solution to finding y is:

$$n = 2^m + l,\ y = 2l + 1$$

where m, l are integers and m is the greatest number possible while $2^m < n$ is true. We then explored the two possible cases: $l < 2^m$ and $l = 2^m$, the former resulting in an increase of the value of y by 2 and the latter resets the value of y back to 1, therefore proving the solution by induction. After solving the original problem, we extended it: instead of finding just the value of y of the last person, we find a way to find the value of y of the x^{th} person where $x < n$.

We first plotted an answer triangle that shows the sequence in which the people are eliminated. From the answer triangle, we see that there are two distinct parts of the sequence: when x £ n/2 and $x > n/2$. For the first half, it is simply all the even numbers. Thus when x £ n/2, $y = 2x$. For $x > n/2$, we see a similar pattern to that of the original problem, which is actually a subset of the extension where $x = n$, for value of x relative to n. For $x = n$, when n can be expressed as 2^m where m is an integer, $y = 1$. For the case of $x = n - 1$, $y = 1$ when n can be expressed perfectly as $3(2^m)$ where m is an integer. For the case of $x = n - 2$, $y = 1$ when n can be expressed perfectly as $5(2^m)$ where m is an integer. Otherwise, for each successive case the value of y increases by 2. This pattern can be seen for all values of x when $x > n/2$, hence the general solution for y for the second half is:

$$n = (2n - 2x + 1)(2^m) + l,\ y = 2l + 1.$$

Once again, there are two cases: $l < (2n - 2x + 1)(2^m)$ and $l = (2n + 2x + 1)(2^m)$, the former case resulting in an increase of 2 for the value of y while the latter resets it to 1. By expanding the equations, we resulted in these two possible end results and proven our solution correct by induction once again.

Project 5: The Chicken McNugget Theorem

- Team members: Matthew Fan Xin Yu
- Mentor: Chai Ming Huang
- School: NUS High School of Mathematics and Science
- Award: Bronze

Summary

In the earliest time, chicken nuggets are sold at McDonald's in packages of 9 and 20. People had wondered what was the largest amount of chicken nuggets they could never buy assuming they bought only using the packages of 9 and 20 given and did not remove any.

A number is "expressible in (m, n)" if it can be expressed in the form am + bn where a, b are positive integers. The Chicken McNugget Theorem states that: For relatively prime integers m and n, the largest number which is inexpressible in (m, n) is mn – m – n.

If McDonald's only offer nuggets in packages of 9 and 20 and I have 7 friends who wants 15 nuggets each, can I buy 7 □ 15 = 105 nuggets? We know that mn – m – n is the largest which is inexpressible in (m, n). So is there a way to determine whether a number k smaller than mn – m – n is expressible in (m, n). A possible method is to use the following result: Exactly one element in the set {k, mn – m – n – k} is expressible in (m, n). This helps to cut down the work of guess and check to "see whether you can buy a certain amount of nuggets". Now, if we know that we can buy 105 nuggets given packages of 9 and 20 nuggets, how do we actually determine the number of 9-piece and 20-piece packages each to buy, instead of going through guess and check? We also provide answer to determine what package-types to select when buying nuggets. Another problem studied in this project is the Classical Water Pouring question:

If we have an infinite source of water and a 5-liter jug and 3-liter jug, we know that by Chicken McNugget Theorem, we would be able to get any amount of water for above 7 liters. So how do we get, for example, 4 litres? Note that we can pour away water too, in this case, so this is an "extension" of the Chicken McNugget Theorem. By using Bezout's Lemma, we have the following statement: For relatively prime integers m and n, all positive integers can be expressed as am + bn, where a, b are integers. So, contrary to the Chicken McNugget Theorem, all amounts of water can be obtained here.

We also discuss the optimal method to find the least amount of pouring. We need to find the values of x, y satisfying ax + by = k such that |x| + |y| is minimized.

Senior Section

Project 1: Tromino Covering

- Team members: Liu Hang, Surya Mathialagan,
- Pek Yu-Xuan Sean
- Mentor: Goh Shu Liang
- School: NUS High School of Mathematics and Science
- Award: Gold & Foo Kean Pew Memorial Prize

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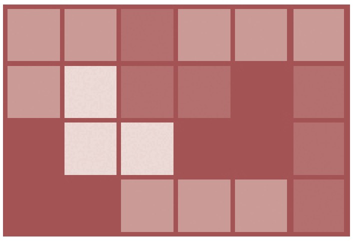
Summary

The problem of tiling originated from antiquity, with the Ostomachion attributed to Archimedes. The basic premise of the problem is where a completed board is partitioned into smaller pieces. The specific problem of tiling a rectangular board with polyominoes is more recent, with earliest occurrences in Japanese Tatami mats covering.

More recently, the advent of the computer has changed the way the problem has been tackled. The Dancing Links algorithm has been invented by Hiroshi Hitotsumatsu and Kohei Noshita in 1979. It has been used to tackle the ways to tile a finite alphabet of blocks into a rectangular grid. More generally, when the alphabet is not restricted to a finite set, a zero-suppressed binary decision diagram by Shin-ichi Minato can be used to solve the problem. However, these algorithms are both exponential in the size of the input. Hence, the need of a fast algorithm is required if we wish to solve the problem efficiently.



In method X, we consider a right-aligned, horizontally-convex board with row vector {a1, a2, ..., an} and maximum point p. Note that in order to obtain a tiling of the right-aligned, horizontally-convex (RAHC) board, a few trominoes can be chosen to be removed at each stage such that the top-left cell of the RAHC board is removed, while maintaining the property that the resulting board is RAHC. Hence, by considering the possible e□ective values of the board, a set of rules to recursively remove the top-left cell of a RAHC board has been derived.
Method M uses the rules of Method X to partition a RAHC board, but in this method we aim to come up with a recursive relationship by considering all possible states of a RAHC Board. Each state can only move to another fixed states of the board. Therefore, based on this, we will be able to obtain a 25 degree recurrence relationship.
In conclusion, this paper illustrates a method in which a closed form for the kth term of the tiling of a $4 \times 3k$ board can be determined. This method can be trivially generalised to higher dimensional boards or di□erent alphabets.

The full report of this project is featured in this issue of Medley (see page 10 – 25).

Project 2: Generalization for Computing Bishop's Polynomial Derived From the Conventional Rook's Polynomial

- Team members: Sim Ming Hui Melodies, Qu Siyang
- Mentor: Samuel Lee Tzi Yew
- School: Raffles Girls' School (Secondary)
- Award: Silver

Summary

The rook polynomial is a powerful tool in the field of restricted permutations of combinatorial Mathematics. Ever since it was first studied in the 1946 by Kaplansky and Riordan many researches and works by mathematicians such as Goldman et al., expanded the pool of knowledge of rook polynomial, notably matching polynomials in graph theory, and calculating the permanent of matrices, which is elaborated in the Application section of the report. Specifically, our project is central to the bishop polynomial, which is a subset of the relatively better established rook polynomial. Unlike the rook polynomial which has various generalisations being formulated already, the bishop polynomial remains foreign and relatively fresh to many people researching in the fields of Mathematics. In the course of our report, we present an original generalization of the bishop polynomial, named Qu-Sim Theorem (for Even and Odd boards) in hopes of raising the awareness of bishop polynomial as well as to add on the gaps of knowledge in the field, enabling other prominent mathematicians to step up and develop the polynomial to greater heights.

Here, we acknowledged that our work will not be possible without the works and theorems (presented as follows) done and produced by previous researchers.

The rook polynomial $R_{m,n}(x)$ by Riordan et al. is a generating polynomial of the number of ways to place k non-attacking rooks on to a $m \times n$ chessboard:

$$R_{m,n}(x) = \sum_{k=0}^{\min(m,n)} C(m,k)P(n,k)\,x^k$$

The notations $C(m,k) = \frac{m!}{k!(m-k)!}$ denote choosing k objects from m objects, and $P(n,k) = \frac{n!}{(n-k)!}$ denotes permuating k objects from n objects.

Similarly, the bishop polynomial is defined as a generating polynomial of the number of ways to place k non-attacking bishops on to a $m \times n$ chessboard.

Due to the fact that the rook polynomial is better established, we translated the problem in terms of the bishop polynomial to the rook polynomial and seek a generalization for all boards from there. Developing the bishop polynomial brought forward a greater insight into more complication patterns in the field of restricted combinatorics, as well as allow for further expansion and study of the polynomial. In particular, applications of bishop polynomial in the design of spotlights and city lights render it applicable and relevant in the real-world context.

Project 3: The Car Park Problem

- Team members: Chen Yankang, Sun Lu, Hu Jingjie, Huang Shimeng
- Mentor: Lye Wai Leng
- School: National Junior College

- Award: Bronze

Summary

The purpose of this project is to find out the most appropriate layout plan and the optimum parking angle for minimising parking area and to explore different arrangements of cars under various situations possible in a car park.

To facilitate our discussion, it is assumed that all parking stalls are of the same size, which is sufficient for the parking of the biggest cars that are allowed to park in this car park. The parking stalls are assumed to be rectangular which includes the space for door-opening. All drivers are assumed to be able to drive cars perfectly as we calculate (i.e. they are able to park the cars at a certain angle using minimum area based on our calculation). In addition, the car park discussed in this section is assumed to be a most simplified rectangular space without any auxiliary facilities that may appear in real life, such as pillars and fireplugs.

In our discussion, angle parking is adopted as opposed to perpendicular parking, because it effectively reduces the requirement for the width of the lanes. In perpendicular parking, the car has to make a full turn in order to get into the parking stall, and this requires the lane to be at least wider than the length of the car. However, when the parking stalls are slightly tilted, the car only has to turn over a much smaller angle to get into the stall, and the lane thus does not have to accommodate the full length of the car. As such, the width of lane is significantly reduced. Another key structure in the ideal layout is the herringbone structure, which allows two cars to be parked directly head-to-head. Nevertheless, such structure can only be utilised in the middle section of the car park. In a rectangular car park, at least two rows of parking stalls are to be arranged against the walls without being paired up with another row to form the herringbone structure.

Angle parking effectively reduces the requirement for the width of the lane. However, it wastes more space (the empty triangle-shaped space) in each parking stall as compared to perpendicular parking. Thus, we need to find the optimum parking angle, with which minimum area is wasted and the number of cars in a car park is maximised. To maximise the number of cars in a car park, the average area occupied by a car must be minimised. Hence, we will establish a function about the parking angle through the average area occupied by a car. We applied the concept of differentiation to find out the optimum angle of parking,

In our extension part, we calculated out the value of the parking angle by substituting statistics from real life and obtained the value of the optimum parking angle, 77.2°.

As in real life context, for convenience's sake, construction companies tend to use special angles when constructing car parks, we have also calculated out the appropriate width of the lanes for safety concerns.

Project 4: Projectile Motion: The Math behind Angry Birds

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- Team members: Charmaine Kho Ling Wei, Chen Mei Zhu,
- Jere Low Wenn, Nicole Ong Wen Pei
- Mentor: Tan Chik Leng
- School: Nanyang Girls' High School

- Award: Bronze

Summary

In this project, we will be investigating the math behind Angry Birds - projectile motion and how a structure collapses or breaks. These two concepts are the main elements of the game. We have also included calculations of the energy required to topple the structure if the game was translated to real-life. Essentially, we have created an 'Angry Birds Problem' based loosely on the 'basic' structure of the game - a structure of wooden planks with alternating rows of horizontal and vertical planks and with every other row having 1 less plank as compared to the previous layer. We will be exploring the most ideal location to strike the structure, as well as the angle of catapulting the projectile, so as to obtain a full collapse of the structure - the most ideal situation in the game.

In summary, we have found the most ideal locations – bottom left supporting slab and top most slab – to strike the structure based on the two different types of structures, non-supported and supported respectively.

We have also calculated the amount of energy possessed by a projectile – 350 J – based on the measurements we have set and discovered that the maximum number of levels of an unsupported structure that can be destroyed by a projectile at once is 6. We have also discovered that the ideal angle to launch the projectile to strike the bottom leftmost block is 94.3 degrees.

Also, we have obtained a general formula for the amount of force acting on the bottom vertical block of a supported structure: $3 + (n-1)(n+4)$, where n is the number of levels the structure has.
Throughout this project, while discovering the math and patterns found in Angry Birds, we had to apply math concepts previously learnt in school into figuring out how angry bird worked. Besides existing topics that we had already been thought though, such as trigonometry and forces, we also had to learn new topics to be able to fully understand the math in Angry Birds. Without a doubt, having to apply the math concepts we learnt in school into solving the math behind Angry Birds really did help improve our understanding of those concepts as we had to think further about the concepts before applying them in our calculations.

At first, it seemed difficult to understand the math behind the toppling or breaking of blocks in Angry Birds. However, with patient guidance from our teacher-mentor, we realized that most of the math required for our calculations were actually already taught to us by our teachers, we just had to break down the problem into a few parts and tackle those parts individually and this was another learning point for us. All in all, this project has really given us a chance to deepen our understanding in multiple math concepts and we are very grateful for the opportunity to participate in this.

Project 5: PIN: The More, The Better?

- Team members: Xin Jiawen, Wang Zhenghao, Qian Hongyu
- Mentor: Zong Lixing
- School: Hwa Chong Institution (High School)
- Award: Bronze

Summary

PIN is short for Personal Identification Number. It refers to the secret passwords shared between a user and a system that can be used to authenticate the user to the system. Typically, the user is required to provide a non-confidential user identifier or token (the user ID) and a confidential PIN to gain access to the system. Upon receiving the user ID and PIN, the system looks up the PIN based upon the user ID and compares it with the received one. The user is granted access only when the number entered matches the number stored in the system. For this reason, PIN has been commonly seen as a very good method to allow only ourselves to access our accounts. However, the PIN we use to protect our accounts has also become the exact tool people use to access them. While we are keying in our passwords, our fingerprints are left on the keyboards. These fingerprints have given a very important clue for others who want to access our accounts since they give those people all the possible combinations of our passwords, which will make it significantly easier for them to then work out our passwords. Hence, our research aims to investigate how to increase the safety of our PIN despite the detectable fingerprints one may leave behind while typing the password. The intended value of our research lies in the above-stated aim - the result of this research will provide the vast majority who use PIN with a useful method to further increase the difficulty of breaking it, thus making sure that their accounts are protected under even better conditions.

However, the value of this research is also limited since our research is based on the following assumptions. First, when the person keys in his password, he types in every number correctly and does not accidentally touch or type other numbers. Second, every single fingerprint he leaves behind is perfectly detectable and there are no other fingerprints that may belong to other users. Third, for those who intend to break a person's PIN, they do that only using the method of working out all the possible combinations and no other methods are used. Hence, our research will only provide a general way of increasing the safety of our PIN. The two basic research problems are as follows. First, given the length of a PIN, how many digits should be used so that we can obtain the most number of combinations. Second, is there a general formula describing the relationship between the length of the PIN and the digits used. In order to answer the first question, we look into a trivial case where the length of the PIN is six. The result is that five digits should be used to obtain the maximum number of combinations. For the second question, we try to derive the general formula given that the length of the PIN is n. The general formula is that to make an n-digit PIN the safest, n-k numbers should be used, with $= \left\lfloor \frac{2n-1}{7} \right\rfloor$.