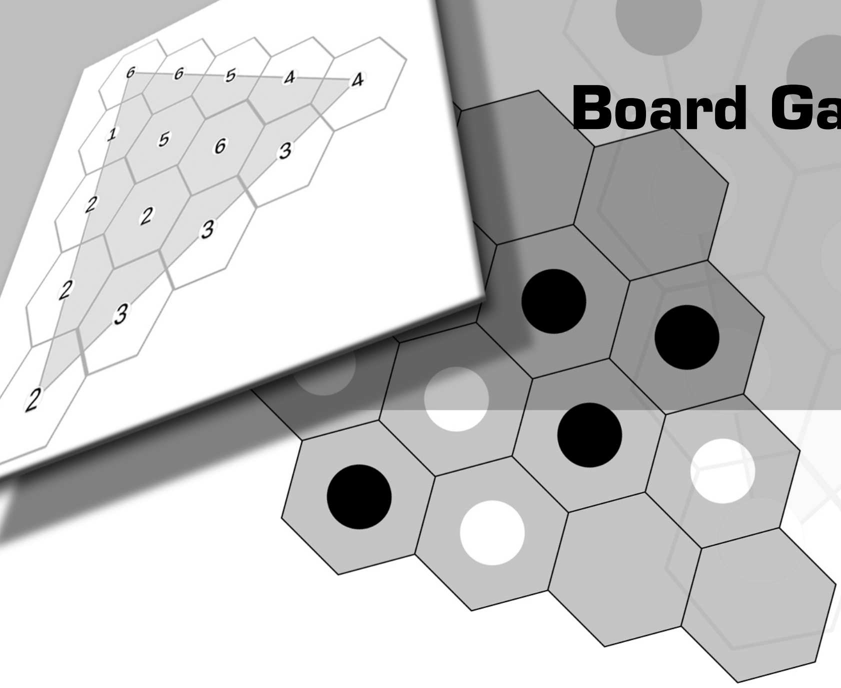

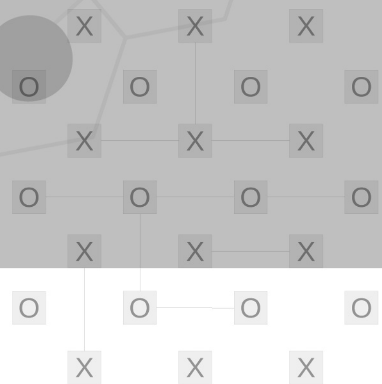

Board Games and Economics

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Abstract. We are going to introduce the Brouwer Fixed Point Theorem, explain how it is related to mathematical economics, and give a proof of the Fixed Point Theorem based on a special type of board games where the objective is to connect two sides of the board with a continuous path. The original proof was published by David Gale in 1979; our article is a refined version of his ideas adapted for a reader who doesn't have a university degree in mathematics. The article is quite informal and all the technical details are skipped. No special mathematics knowledge beyond year 1 of junior college is needed to understand this paper.

1. Economy models

Equilibrium is an important concept in mathematical economics. In a simple model, a number of market players possess some goods. Each player wants to trade his goods with something of more value, at least from his own point of view. An equilibrium is a distribution of all the existing goods among the market players so that everyone is satisfied with what he's got and doesn't need to trade any more.

For example, consider the situation when there are only two individuals, Winnie-the-Pooh and Piglet, and two resources, honey and acorns. If originally Winnie has a ton of acorns and Piglet has a ton of honey, all they have to do is to trade honey for acorns. Once it's done, the economy is in the equilibrium state. On the other hand, if both of them had a ton of honey and nothing else, then Piglet couldn't get what he needs. There is no equilibrium in the second model, which means it's not a good model — in real life there would probably be more players and someone would have the missing resource.

Thus an equilibrium state describes what is actually going to happen on the market. If there is no equilibrium, our model is probably not very realistic. An important question you should ask yourself when coming up with a mathematical model of an economy is whether we can prove a theorem that says that an equilibrium state always exists. One of the simplest such models is called the *pure exchange economy*. We are not going to describe it here in full detail — the interested reader is referred to [1] or to any of plenty books on economics. Let's just mention that this model deals with $n+1$ resources whose essential characteristics is the price vector

$$(P_0, P_1, \ldots, P_n),$$

where $P_0, P_1, \ldots, P_n$ are nonnegative real numbers. It is assumed that market players in pure exchange economy exchange goods for other goods. There is no money in this model and hence only the relative price is important (if all the goods suddenly become twice as expensive, it doesn't change anything). Thus the price vector is assumed to be normalised, that is,

$$P_0 + P_1 + \cdots + P_n = 1.$$

In particular, the set of all possible price vectors for three goods (the smallest number making the model interesting) is a triangle in the 3D space. Indeed, identifying P_0 with x, P_1 with y, and P_2 with z, we get the following conditions on the relative prices:

$$x \geq 0, \ y \geq 0, \ z \geq 0, \ x + y + z = 1. \tag{1}$$

Exercise 1. *Explain why the set of points in the 3D space given by* (1) *is an equilateral triangle and draw this triangle.*

Just like coordinates in the 3D space are 3-component vectors (x, y, z), an $n+1$-component vector $P = (P_0, P_1, \ldots, P_n)$ is an element of the multi-dimensional space $\mathbb{R}^{n+1}$ (here, $\mathbb{R}$ is the usual notation for the set of real numbers and the superscript $n+1$ shows the number of coordinates).

Definition 1. *Consider* $\mathbb{R}^{n+1}$ *with coordinates* $X_0, X_1, \ldots, X_n$. *The set of all points* $(X_0, X_1, \ldots, X_n)$ *satisfying*

$$X_0 \geq 0, \ X_1 \geq 0, \ \ldots, \ X_n \geq 0, \ X_0 + X_1 + \cdots + X_n = 1$$

is called the standard n-simplex *in* $\mathbb{R}^{n+1}$. *We'll denote the* n*-simplex* Δ^n.

In particular, the standard 2-simplex is an equilateral triangle and the standard 3-simplex is a regular tetrahedron (in the 4D-space). Thus all possible price vectors of an economy with $n+1$ goods form the n-simplex Δ^n.

There are a lot of players on the market — companies and individuals looking for different goods or resources. Some need oil, some need computer parts, some need technologies etc. Complicated behavior of the market is roughly described by a special function $B : \Delta^n \to \Delta^n$. This notation means that the function B assigns a price vector $B(P) \in \Delta^n$ to every possible price vector $P = (P_0, P_1, \ldots, P_n) \in \Delta^n$. The idea is that if P represents the actual price vector, then $B(P)$ would be the potential price vector the market is driven to by demand and supply. Specifically, if $B(P)_i > P_i$ holds, then the demand for the ith good exceeds its supply and hence its price is going to rise. Conversely, if $B(P)_i < P_i$, then the price for the good i is going to fall. In other words, the sign of the difference $B(P)_i - P_i$ shows if the price for the ith resource is going to increase or decrease and the magnitude indicates how fast it will do so.

Definition 2. *An* equilibrium state *of a pure exchange economy is a price vector* $P \in \Delta^n$ *such that* $B(P) = P$.

As we'll later see, a pure exchange economy always has an equilibrium state.

2. Fixed points

Mathematics deals with sets and functions all the time. Functions used to describe processes in nature are usually *continuous* — it means that a small change of the input results in a small change of the output. For instance, speeds and coordinates of moving objects are continuous functions of the time, which means that in a small time interval an object can only travel a small distance.

The reasonable assumption about the market behavior function $B : \Delta^n \to \Delta^n$ is that it is continuous. Indeed, if prices of all the goods change a little, then the demand will not change drastically, will it?

There is a special area of mathematics called *topology*. It studies continuous functions; and the situation when we have a set X, a continuous function $f : X \to X$, and a point $p \in X$ such that $f(p) = p$ is of particular importance.

Definition 3. *Let X be a set and let $f : X \to X$ be a function. A point $p \in X$ is called a* fixed point *of the function f if $f(p) = p$.*

Thus from the mathematical point of view an equilibrium state of a pure exchange economy is nothing else but a fixed point of a market behavior function $B : \Delta^n \to \Delta^n$.

Example 1. *Let $X = [0, 1]$ be the closed unit interval and let $f(x) = \frac{x^2+1}{3}$. Observe that*

$$0 \le x \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow 1 \le x^2 + 1 \le 2 \Rightarrow \frac{1}{3} \le \frac{x^2+1}{3} \le \frac{2}{3}$$

and hence for each $x \in [0, 1]$, the value $f(x) \in [1/3, 2/3] \subset [0, 1]$. A fixed point of the function f is a solution of the equation $f(p) = p$, that is,

$$\frac{p^2+1}{3} = p \Leftrightarrow p^2 - 3p + 1 = 0.$$

The solutions are $\frac{3\pm\sqrt{5}}{2}$. Since we are looking for points of the interval $[0, 1]$, the only fixed point is $\frac{3-\sqrt{5}}{2}$.

Example 2. *Let X be the unit circle in the plane, i.e., the set of points (x, y) satisfying the equation $x^2 + y^2 = 1$. Let $f : X \to X$ be the symmetry about the y-axis, which means that $f(x, y) = (-x, y)$. The function f has two fixed points — $(0, 1)$ and $(0, -1)$.*

Example 3. *Let again X be the unit circle in the plane and let now f be the central symmetry. It means that $f(x, y) = (-x, -y)$. The function f doesn't have any fixed points at all. Indeed, if we had $(x, y) = (-x, -y)$, then it would imply $x = y = 0$, which is not a point of the unit circle.*

As we see, sometimes there is a unique fixed point, sometimes there are several and occasionally there is none. The Brouwer Fixed Point Theorem is an old and very powerful general result about fixed points:

Theorem 1 (L. E. G. Brouwer)**.** *Let $n \ge 1$ be an integer. A continuous function from the n-simplex Δ^n to itself always has a fixed point.*

As we see, it precisely means that there is always an equilibrium state in a pure exchange economy. The theorem was proved in 1910 by Jacques Hadamard and Luitzen Egbertus Jan Brouwer as a result of two decades of collective work of themselves and other mathematicians. The original proof involved revolutionary ideas that led to the birth of a new mathematical discipline — algebraic topology. We are going to present a different, simple and beautiful approach via a certain type of board games.

First, let's make some observations. If $n = 1$, what is the 1-simplex? According to the definition, the 1-simplex is the set of points (X_0, X_1) such that

$$X_0 \geq 0,\ X_1 \geq 0,\ X_0 + X_1 = 1.$$

Notice that (X_0, X_1) is just a point on the plane, the inequalities $X_0 \geq 0$ and $X_1 \geq 0$ together describe the fact that the point (X_0, X_1) belongs to the first coordinate quadrant, and $X_0 + X_1 = 1$ is a straight line. Thus the 1-simplex is the part of the line $X_0 + X_1 = 1$ that lies in the first quadrant — obviously, the line segment whose endpoints are $(1, 0)$ and $(0, 1)$. But of course, all line segments are the same in a way and the closed interval $[0, 1]$ is also a 1-simplex. Thus the theorem for $n = 1$ says that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. Then x is a fixed point if $f(x) = x$, that is, $f(x) - x = 0$. Let now $g(x) = f(x) - x$. Observe that $g(0) = f(0) - 0 = f(0) \geq 0$ since $f(0) \in [0, 1]$. At the same time, $g(1) = f(1) - 1 \leq 0$. But then by the famous Intermediate Value Theorem, there is x such that $g(x) = 0$ — the Fixed Point Theorem for $n = 1$ is a trivial implication of the Intermediate Value Theorem.

However, the situation for $n = 2$ is already much more complicated. There is no 2-dimensional analogue of the Intermediate Value Theorem and the reason is simple — there is no such thing as comparison $<$ or $>$ for 2-component vectors. Indeed, which is bigger, $(-3, 7)$ or $(-12, 10)$? We are going to show how Hex and similar games can be used to prove the Brouwer Fixed Point Theorem for $n = 2$.

3. Board games

Hex. The game of Hex was invented independently by Piet Hein in 1942 and by John Nash in 1948. It is played by two players on a $k \times k$ hexagonal grid. The white player aims to join the top and the bottom of the Hex board by a continuous path of white tiles while the black player's objective is to join the left and the right as shown in Figure 1.

Of course, white and black cannot win at the same time. The crucial fact for us is that a tie in Hex is impossible. The following theorem was proved by John Nash in 1952:

Theorem 2 (Hex Theorem). *A tie in the Game of Hex is impossible. Moreover, if the whole $k \times k$ Hex board is filled randomly with white and black tiles, then there is either a white path joining the top and the bottom sides or a black path joining the left and the right sides.*

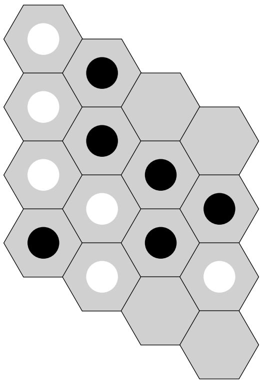

FIGURE 1. A position in the 4×4 Hex. The white player won.

Exercise 2. *Prove the Hex Theorem. First, try to find a few small configurations where neither players wins and think about why you could not do it. Coming up with a general rigorous proof of the Hex Theorem might be still very hard. Don't be discouraged if you cannot do it yourself — just google and try to understand the proof by John Nash.*

Exercise 3. *Prove that the first player always has a winning strategy in Hex.*

Bridg-It. The game of Bridg-It was invented by David Gale. Again, the goal of each player is to connect his two sides with a continuous path. There are two overlapping grids of dots of dimensions $k \times (k+1)$ and $(k+1) \times k$. We call this a $k \times k$ Bridg-It. They are labelled by O and X in Figure 2 respectively. The actual game positions are spots between grids and each move should connect two neighboring dots of the same label, either O or X.

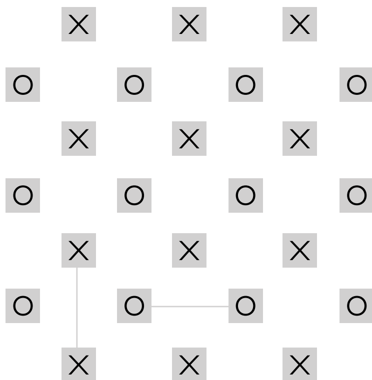

FIGURE 2. A position in the 3×3 Bridg-It. The O-player won.

Exercise 4. *Prove that a Bridg-It game cannot end in a draw, just like Hex.*

Exercise 5. *Is the Bridg-It game equivalent to the game of Hex? In other words, is it possible to construct a natural bijective correspondence between $k \times k$ Bridg-It positions and $k \times k$ Hex positions?*

Tripod. The game of Tripod is played on a triangular board with a hexagonal grid. Both players try to mutually join all the three sides of the triangle, i.e., create a configuration where each side has a tile of the player's colour and there is a continuous path of the same colour between every two of the three tiles.

As we see, Tripod is substantially different from Hex or Bridg-It in the sense that both players have the same objective while in Hex and Bridg-It the two players have different objectives — joining either top to bottom or left to right. However, the same main result is true about Tripod, although it is less intuitive.

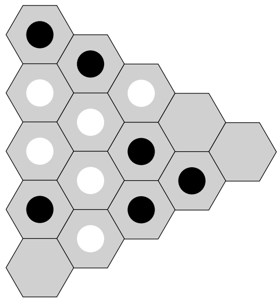

FIGURE 3. A position in the 5-Tripod. The white player won.

Exercise 6. *Prove that there is one and only one winner in any Tripod game. In other words, show that if the triangular board is filled with white and black tiles randomly, then there is either[1] white or black winning configuration.*

4. Proof of the Brouwer Theorem

Recall that the Brouwer Fixed Point Theorem says that any continuous function $f : \Delta^n \to \Delta^n$ has a fixed point. Here, Δ^n is the n-simplex. We are going to prove it for $n = 2$, leaving the general case as an exercise for the reader.

Again, let $f : \Delta^2 \to \Delta^2$ be a continuous function from the equilateral triangle Δ^2 to itself. We'll prove the Fixed Point Theorem by contradiction. Assume, on the contrary, that there is no fixed point. It means that the condition $f(X) \neq X$ holds for any point $X \in \Delta^2$.

Our triangle Δ^2 lies in the plane $X_0+X_1+X_2 = 1$ in the 3D space with the coordinates X_0, X_1, X_2. We'll introduce a polar coordinate system on this plane. Since $f(X) \neq X$ for any X, it means that $f(X) - X$ is a nonzero vector. Let $\varphi(X)$ be the polar angle of the vector $f(X) - X$. In other words, $\varphi(X)$ is the angle measured in the counter-clockwise direction from the x-axis to the vector $f(X) - X$ in the Cartesian plane.

Now let's make some observations[2] about continuity. To say that f is continuous means that $f(X)$ is near $f(Y)$ whenever X is near Y. Of course, it would also mean

[1]not both, not neither

[2]Our speculations about continuity in this paragraph are not rigorous, but they can easily be made rigorous if one uses the machinery developed in real analysis, i.e., the notion of uniform continuity. Since focusing on technical details is not our goal, we'll skip them here.

that $f(X) - X$ is close to $f(Y) - Y$ whenever X is near Y and hence the polar angle $\varphi(X)$ is close to $\varphi(Y)$ if the distance between points X and Y is sufficiently small.

Now we are able to present the essential step of our proof. We'll draw a grid for the Tripod game on our triangle Δ^2 and play the game where the moves are decided by the function f in a clever manner. What is the size k of the Tripod board? It's going to be clear very soon; let's just describe the construction first.

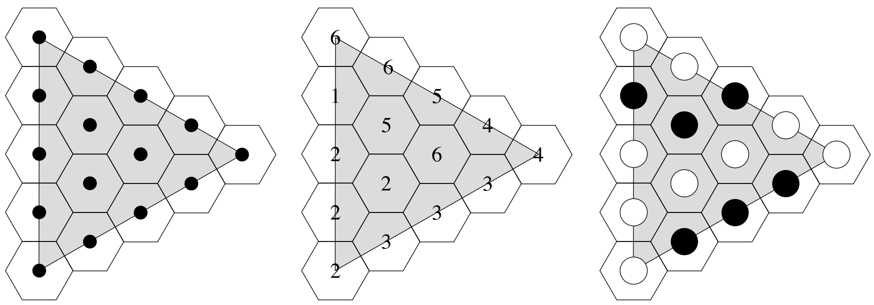

FIGURE 4. Tripod game constructed from a function f.

The grid is shown on the left of Figure 4 — it's very simple. Further, we'll label the $\frac{k(k+1)}{2}$ game positions according to the following rule: for each game position $X \in \Delta^2$, we look at which circular sector the angle $\varphi(X)$ belongs to. Then by definition we say that $l(X)$ is one of the numbers $1, 2, 3, 4, 5, 6$ assigned as shown in Figure 5.

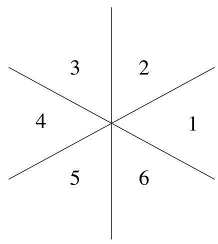

FIGURE 5. Construction of the function $\varphi(X)$ in the proof of Brouwer's Theorem.

For instance, if $\frac{\pi}{6} < \varphi(X) \leq \frac{\pi}{2}$, then X is labelled 2; if $\frac{7\pi}{6} < \varphi(X) \leq \frac{3\pi}{2}$, then X is labelled 5 etc. After assigning labels, we get the Tripod board where each of the positions has a number between 1 and 6 written on it as shown in the middle diagram of Figure 4.

Exercise 7. *Explain why the three corner positions are always given the numbers* 2, 4, *and* 6 *as in Figure 4 for any function* f. *Also, explain why the left side only has labels* 1, 2, *or* 6 *and cannot have any of the other three labels. Similarly, the right top side can only have labels* 4, 5, *or* 6. *Finally, the right bottom side can only have labels* 2, 3, *or* 4.

Finally, we'll play the game of Tripod according to the very simple rule — we'll put white tiles on positions with even numbers and black tiles on positions with odd numbers.

According to the Tripod Theorem (Exercise 6), either white or black player wins the game. We'll explain why it leads to a contradiction if it's the white one — the case of the black player is similar. If the white player wins the game, then there is a continuous white path joining the three sides of the triangle. Since it's white, the labels on this path can only be even ones — 2, 4, or 6. Since it's got to join all the three sides, they cannot all be the same — there must be a mixture of 2, 4, and 6. Then it means that somewhere 2 is next to 4, 4 is next to 6, or 6 is next to 2. But if the grid is sufficiently fine and the neighbour Tripod positions are close to each other, then it contradicts the continuity of the function f — indeed, such a situation would imply that we have found two points X and Y — neighbour Tripod positions such that the angles $\varphi(X)$ and $\varphi(Y)$ differ by at least $\frac{\pi}{3}$, or 60°. It's impossible for continuous functions. This contradiction concludes the proof of the 2-dimensional Fixed Point Theorem.

5. Conclusion

We have used the Tripod game to prove the Brouwer Fixed Point Theorem in dimension 2. Can we use Hex or Bridg-It for the same purpose? Obviously, the shape of the game board effects the set used in the Fixed Point Theorem. Hex and Bridg-It allow us to prove the Fixed Point Theorem for a parallelogram (rectangle, square).

Exercise 8. *Use the Hex and the Bridg-It games to prove the alternative version of the Fixed Point Theorem: any continuous function $f : A \to A$, where A is a parallelogram, has a fixed point.*

Continuous functions and fixed points are studied in the area of topology. Amazingly, the shapes of triangle, parallelogram, square, or even a circle are considered to be indistinguishable in topology. The reason is that if our triangle was made of clay, we could continuously deform it into a square. Such a deformation is called a *homeomorphism* in mathematics. Homeomorphic shapes are considered equal in topology in the same way as congruent shapes are equal in geometry. Moreover, if a shape A possesses the property that any continuous function $f : A \to A$ has a fixed point, then any shape homeomorphic to A also has the same property. Thus the Brouwer Fixed Point Theorem can be formulated for a square or a circle instead of a simplex.

Finally, we'll propose some directions for further research. If you're actually interested in a project like this, you can contact the author of this article, Dr Fedor Duzhin (`fduzhin@gmail.com`) to help you with it.

All the above machinery of board games provides means to prove the 2-dimensional Brouwer Theorem. Can you think of a way to construct multi-dimensional analogues of these games to prove the n-dimensional statement? You'll have to think of game rules that ensure that no draw is possible. Can you come up with your own, different board games with the objective of building paths that can be used to produce more proofs of the Fixed Point Theorem?

References

[1] Colin Adams, Robert Franzosa *Introduction to Topology: Pure and Applied*, Pearson 2007.
[2] David Gale *The Game of Hex and the Brouwer Fixed-Point Theorem*, The American Mathematical Monthly, Vol. 86, No. 10. (Dec., 1979), pp. 818-827.

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