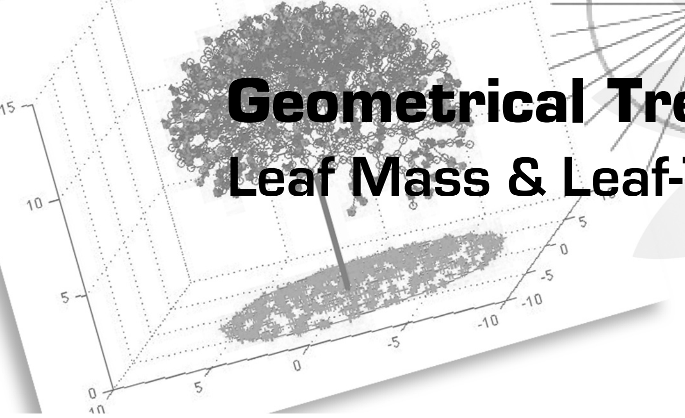

Geometrical Tree: Leaf Mass & Leaf-Tree Relationship (Part 2)

Xu Wenji, Lu Jingyi & Zhang Jing

Winner in the Outstanding Winner category at the
International Mathematics Contest in Modeling 2012

Part 1 of this article appeared in Volume 39 No. 1 of Mathematical Medley.

5. Leaf Mass of a Tree

In this section, we will use the vector tree model and sunlight model [14] to estimate the total leaf mass of a tree. The general formula to calculate total leaf mass of a tree is

$$\text{Total leaf mass} = \text{weighted leaf mass per area ratio(LMA)} \times \text{total leaf area} \quad (10)$$

The computation of LMA and leaf area will be introduced.

5.1. **Leaf Mass per Area Ratio (LMA).** Leaf Mass per Area (LMA) is defined as the ratio of leaf mass over its area (gm^{-2}).

In this section, we describe a method to calculate the average LMA for the tree. Since leaves on a tree have different thicknesses due to different photosynthesis rate determined by the sunlight irradiance at the leaf [10], they tend to have different LMA. Thus, we need to calculate LMA for each leaf based on their photosynthesis rate and then calculate the weighted average of all the individual LMAs.

5.1.1. *Photosynthesis Rate.* We adopted the model in [7] which relates the internal gross photosynthetic rate P with sunlight irradiance I_l , internal CO_2 concentration of the plant C_i and the CO_2 concentration in the air C_a.

$$P = \frac{\frac{\alpha I_l C_i}{r_x}}{\alpha I_l + \frac{C_i}{r_x}} \quad (11)$$

where α is known as the photochemical efficiency ($kgCO_2J^{-1}$) and r_x as the carboxylation resistance (sm^{-1}). Next, denote P_n as the net photosynthetic rate, and the following equation is satisfied

$$P_n = P - R_d \quad (12)$$

and

$$P_n = \frac{C_a - C_i}{r_d} \quad (13)$$

where R_d is the constant dart respiration rate, r_d is the diffusion rate. Use (5.11) (5.12) and (5.13) to solve for P and P_n, we get

$$0 = P_n^2 r_d - P_n[\alpha I_l(r_x + r_d) + C_a - R_d r_d] + \alpha I_l C_a - R_d(\alpha I_l r_x + C_a) \quad (14)$$

Therefore we get

$$P_n = \frac{-[\alpha I_l(r_x + r_d) + C_a - R_d r_d] \pm \sqrt{[\alpha I_l(r_x + r_d) + C_a - R_d r_d]^2 + 4 r_d R_d(\alpha I_l r_x + C_a)}}{2r_d} \tag{15}$$

However, by [7] only the negative root is biologically valid, thus we can solve for P_n. In our model, I_l is obtained from the sunlight model in section 3 [14], C_a, r_x, r_d, R_d are all assumed to be constant at any position in the tree crown.

5.1.2. *Leaf Mass Per Area Ratio (LMA) Affected by Venation Networks.* In modern ecology, LMA is an important measure in classifying different kinds of leaves. By observing the venation network of a given leaf area, one can also find a surprisingly correlated relationship between three venation functional traits and LMA [1]. We follow the modeling process of the derivation of LMA with respect to the following functional traits:

- Density σ : total path length of veins in the area of interest (ROI) divided by the ROI area;
- Distance d : the mean diameter of the largest circular masks that fit in each closed loop;
- Loopiness ξ : the mean number of closed loops in ROI

In [1] the derivation of the relationship between LMA and the functional traits shows the following relationship:

$$\delta = k_0 d \tag{16}$$

$$LMA = \pi r_v^2(\rho_v - \rho_l)\sigma + \frac{2\rho_l d}{k_0} \tag{17}$$

where

- ρ_v is the inner radius of the veins in the ROI
- r_v is the mass density of the terminal veins
- ρ_l is the mass density of the lamina
- δ is the thickness of the leaf at ROI
- k_0 is a constant which relates d and δ

5.1.3. *Peak Carbon Assimilation Rate Per Mass.* In [1], a detailed model is given to derive the Peak Carbon Assimilation Rate per Mass (A_m)

$$A_m = \frac{[\pi c_0(1-h)k_0 n_s a_s WUE]\sigma}{[2\rho_l D + K_0\pi r_v^2(\rho_v - \rho_l)\sigma][(\pi t_s + \sqrt{\pi a_s})\sigma + 2a_s n_s log(\frac{D}{k_0 r_v})]} \tag{18}$$

where a_s is the maximum aperture of a stomate, n_s is the number density of open stomata, t_s is the thickness of a stomatal pore, D is the temperature and pressure dependent diffusion constant of water in air, c_0 is the temperature and pressure dependent saturation vapor concentration of water in air, WUE is water-use efficiency and h is the relative humidity [3] [9].

5.1.4. *Derivation of LMA.* Now we use the results in the above sections to derive LMA. One observation we make is $P_n \leq A_m$, since A_m is the peak photosynthetic rate. Therefore, we use μ which accounts for environmental factor such that

$$\mu P_n = A_m \tag{19}$$

Therefore, by assumption, we can solve for LMA combining equations (5.16) - (5.19) even if the thickness of the leaf δ is unknown. Then we can get the desired LMA. By applying the above method for each leaf, we will be able to get the average LMA for the tree.

5.2. **Total Leaf Area.** We will calculate the total leaf area of a tree by the following formula:

$$\text{Total leaf area} = \text{Leaf Area Index(LAI)} \times \text{ground area covered by the tree crown} \tag{20}$$

In the following sections, we will introduce the concept of LAI and the method of getting the ground area covered by the tree crown from our vector tree model.

5.2.1. *Leaf Area Index.* Leaf Area Index (LAI) is defined as the one sided green leaf area per unit ground area in broadleaf canopies, or as the projected needle leaf area per unit ground area in needle canopies.

It is an indispensable parameter in studying plant physiology, since vegetation surface is an important determinant of various plant functions such as photosynthesis and transpiration. Methods used for measuring LAI in hardwood forests include destructive sampling, allometric equations, litter fall, and light interception based techniques [13].

A data set for LAI has been compiled containing 1008 records of worldwide data on leaf area index for the time period 1945-2000. [11]

5.2.2. *Ground area of a tree.* We can derive the ground area of a tree from the vector tree model introduced in section 2.2 in [14]. By computer simulation, we project the tree crown onto the ground, and plot the margin of the projection. Hence calculate the ground surface area according to the projected polygon hull in the computer. A simulated tree and its crown projection are shown in Figure 5.1. Once we know the ground area covered by the tree crown, we multiply it with LAI to get the total leaf area of the tree. And then, we multiply it to the LMA ratio to get the total leaf mass.

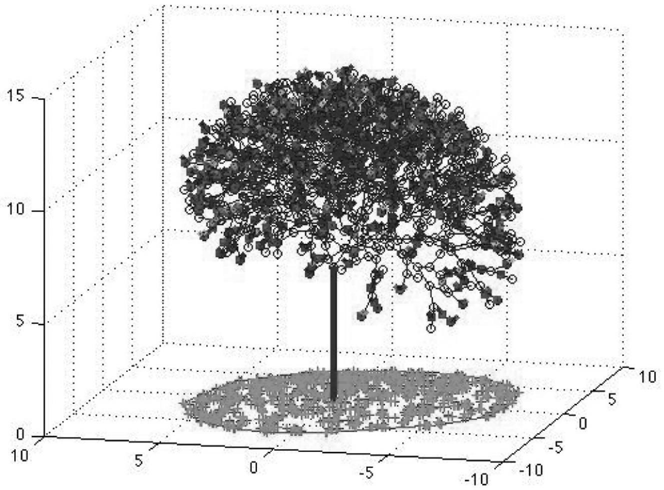

Figure 5.1

Figure 5.2

5.3. **Case study: a real tree Cinnamomum camphora.** We would like to test our model by studying a real tree (Cinnamomum camphora) and estimate its total leaf mass. A picture of the tree is shown in Figure 5.2.

5.3.1. *Simulation of the tree in computer.* In order to simulate the branching structure of this tree, we need to know the angles $(\theta_1, \phi_1, \theta_2, \phi_2)$ of transformation stated in model 2.2 in [14]. Since there are no empirical data for these parameters, we need to conduct a field measurement. The major difficulty is to determine the direction of our imaginary x-axis and y-axis: a wrong direction of coordinates will result in wrong angles and hence a twisted shape of tree profile.

In our measurement, we first set the trunk to be the z-axis, and the origin is at where the z-axis touches the ground (our xy-plane). We equally divide the ground using 32 outgoing rays. (See in Figure 5.3)

Setting each outgoing ray as x-axis, we then have 32 possible coordinate systems.

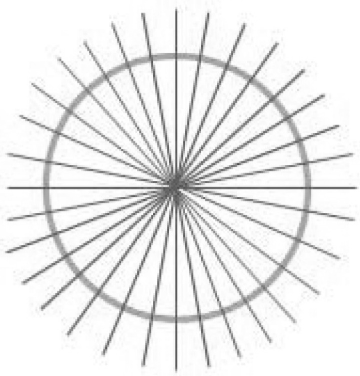

Figure 5.3

In each coordinate system, we can directly measure and calculate a data set of $(\theta_1, \phi_1, \theta_2, \phi_2)$ by measuring the angle of the branches. We simulate trees using each of these 32 data sets; compare the simulated vector tree with the real tree profile. The one that matches the real tree most closely will be chosen. The data we get from this field measurement and simulation is: $\theta_1 = \frac{\pi}{12.5}, \phi_1 = \frac{\pi}{3.7}, \theta_2 = -\frac{\pi}{6}, \phi_2 = \frac{\pi}{4.8}$.

The total number of branches of our simulated tree is 29, and we assume that the function $d(I)$ defined in section 2.2.3.2 in [14] which determines the spacing of leaves follows the Gaussian distribution with parameters $\mu = 0.1, \sigma = 0.1$.

We classify the shape of this tree leaf as Elliptic, and its average transverse diameter is around 5cm. For simplicity of calculation, we represent the leaves using rhombuses with average longer diagonal 0.25 in our computer simulation. Our simulated tree is shown in Figure 5.4.

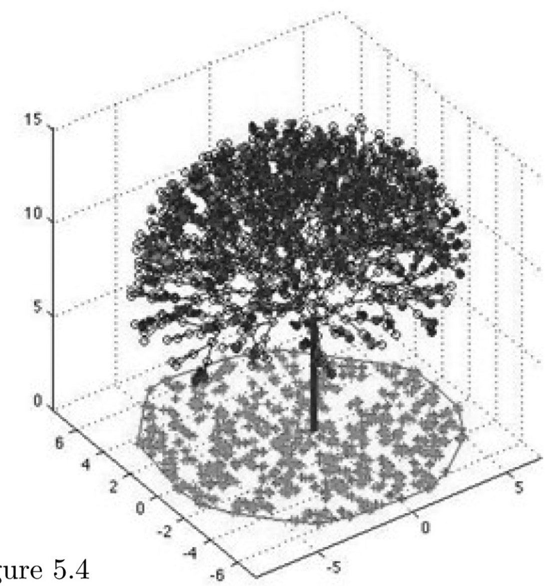

Figure 5.4

5.3.2. *Estimating the total leaf mass of the tree.* Using the model introduced in section 5.1, our computer simulates the effect of light intensity on the Leaf Mass Area (LMA) of leaves in different parts of the tree crown and hence derive the average LMA of the tree to be 177.74 gm^{-2}.

The ground surface area can be calculated by the computer as introduced in section 4 [14].

The output of the polygon ground surface area is 150.0359. Converting it to scale of the tree, the ground surface area is calculated to be around 6 m^2 (actual number 6.001436). This data matches the actual ground area of this tree quite closely.

According to the empirical data [2], Cinnamomum as evergreen broadleaved plant usually has a rather high leaf area index (LAI) above 10. We take the LAI of this tree to be 12. Hence the total leaf area of this tree is around 6 x 12=72 m^2.

Therefore, the leaf mass of this tree is estimated as

$$177.47gm^{-2} \times 72m^2 = 12777.84g. \tag{21}$$

5.4. **Correlation between the leaf mass and the size characteristics of the tree.** In this section, we investigate the correlation between leaf mass and the size characteristics of trees, specifically the height of trees.

We use our vector tree model [14] to simulate trees with 3 different profiles (see Figure 5.5) each at height varying from 5m to 10m (with 0.5m intervals), and calculate their total leaf area using the method introduced above.

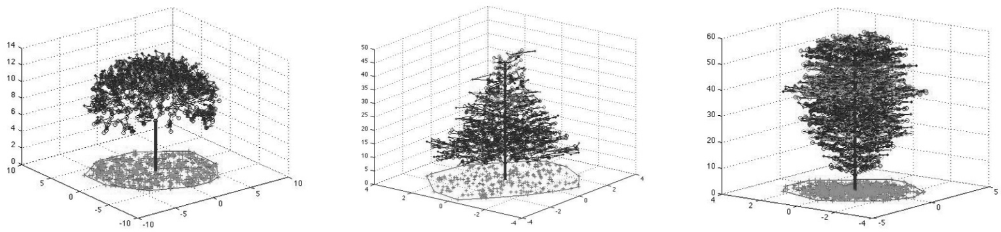

Figure 5.5: Profile 1: Semi-sphere; Profile 2: Cone; Profile 3: Ellipsoid

We conduct a regression analysis for the total 30 data and for each of the 3 profiles separately, to study the correlation between total leaf mass and the size characteristics of trees. The data is shown in Figure 5.6.

Hence we may conclude that for trees in general, we can hardly say that there is obvious correlation between the leaf mass and the size characteristics of trees. However, when considering trees with similar crown profile, it appears that the leaf mass may be related with the height of the tree—the total leaf mass tends to

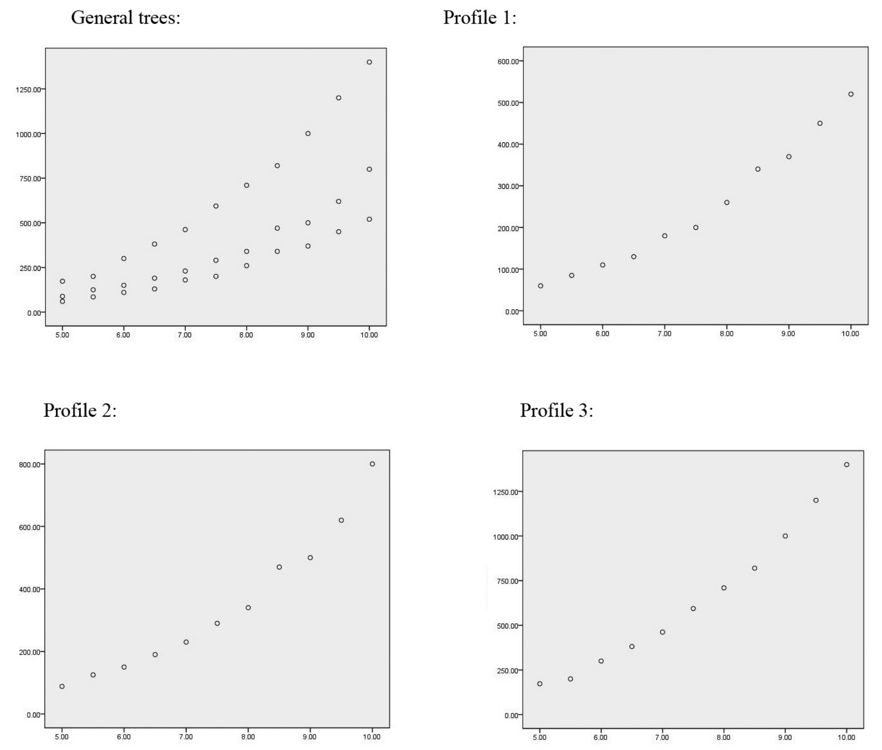

Figure 5.6 x-axis: height of the tree, y-axis: total leaf mass

increase cubically as tree height increases. Since we only have limited data set, the actual correlation may be determined and tested through further investigation.

6. Improving the model

6.1. **Leaf classification.** The basic building block of our model is the leaf model introduced in section 2.1 [14], in which we classify leaves based on their shapes. We can improve this by taking into account other biological features of the leaves, such as their photosynthesis characteristics (e.g. C_3 v.s C_4 plants), transpiration patterns, or vein structure.

6.2. **Exposure area of leaves.** In section 3 [14], when calculating the exposure area of leaves to the sunlight, we implement the model by discretizing the x-y plane into small grids and determine whether the leaves block each grid. This algorithm works satisfactorily for small leaf size. However, as we have seen in the Japanese banana tree example, when leaf size is big, this algorithm tends to generate similar results for different leaf shapes. Thus, we may improve our algorithm to calculate leaf shadows for big size leafs.

6.3. **Determinants of photosynthesis rate.** In section 5.1, we assume that only sunlight irradiance affect the photosynthesis rate. Other factors relating to photosynthesis such as concentration of CO_2, water and N_2 may also be incorporated into the model.

7. Conclusion

The problem of investigating the relationship between leaf shapes and branching structures tend to be nontrivial given the biological complexity of the plant structure. Our approach tackles the problem geometrically since many previous studies in the field of plant physiology have shown geometric uniformity. This approach is proved to be appropriate and reveals the fact that the leaf shape relates with branching structure in that the shape tends to maximize sunlight exposure under the given tree profile.

The model can be extended to estimate the total leaf mass of a tree using the relationship between photosynthesis rate and leaf mass per area (LMA). Empirical test of our model on cinnamomum camphora tree has shown that the method gives realistic estimation of the leaf mass.

References

[1] Benjamin BlonderViolle, Lisa Patrick Bentley and Brian J. EnquistCyrille, (2011), *Venation networks and the origin of the leaf economics spectrum*, Ecology Letters, 14, 91100.

[2] Beth EllisC. Daly, Leo J. Hickey, Kirk R. Johnson, John D. Mitchell, Peter Wilf, and Scott L. WingDouglas, (2009), *Manual of Leaf Archetecture*, The New York Botanic Garden.

[3] Buck A.L. (1981), *New equations for computing vapor pressure and enhancement factor*, J. Appl. Meteor., 20, 15271532.

[4] Fisher B. and Honda, HJ. (1977), *Computer simulation of branching pattern and geometry in Terminalia (combretaceae), a tropical tree*, Botanical Gazette, 138, 377-384.

[5] Fisher B. and Honda, HJ. (1979), *Branch geometry and effective leaf area: a study of Terminalia-branching pattern*, American Journal of Botany, Vol. 66, No. 6, 633-644.

[6] http://www.math.smith.edu/phyllo//About/Classification.html.

[7] Johnson H.M. Thornley and Ian R.John. (1990), *Plant and Crop Modelling*, Oxford Science Publications.

[8] Masami Monsi, Toshiro Saeki. (1953), *On the Factor Light in Plant Communities and its Importance for Matter Production*, Annals of Botany, Volume 95, Issue 3, 549-567.

[9] Nobel P. (1999), *Physicochemical and Environmental Plant Physiology*

[10] SARAH J. COOKSON1VAN LIJSEBETTENS2 & CHRISTINE GRANIERMIEKE. (2005). *Correlation between leaf growth variables suggest intrinsic and early controls of leaf size in Arabidopsis thaliana*, Plant, Cell and Environment, 28, 13551366.

[11] Scurlock M. O., G. P. Asner, and S. T. GowerJ. (2001). *Worldwide Historical Estimates and Bibliography of Leaf Area Index, 1932-2000*, ORNL Technical Memorandum TM-2001/268, Oak Ridge National Laboratory.

[12] Tsukaya Hirokazu. (2006). *Mechanism of Leaf-Shape Determination*, The Annual Review of Plant Biology.

[13] Vose, J. M., Sullivan, N. H., Clinton, B. D., and Bolstad, P. V., 1995. *Vertical leaf area distribution, light transmittance, and the application of the Beer-Lambert Law in four mature hardwood stands in the southern Appalachians*, Canadian Journal of Forest Research, 25, 10361043.

[14] Wenji Xu, Jing Zhang, and Jingyi Lu, 2013. *Geometrical Tree: Leaf Mass & Leaf-Tree Relationship (Part 1)*, Mathematical Medley, Vol. 39 No. 1, 2013.

The authors are currently final year undergraducate students in the Department of Mathematics at the National University of Singapore.