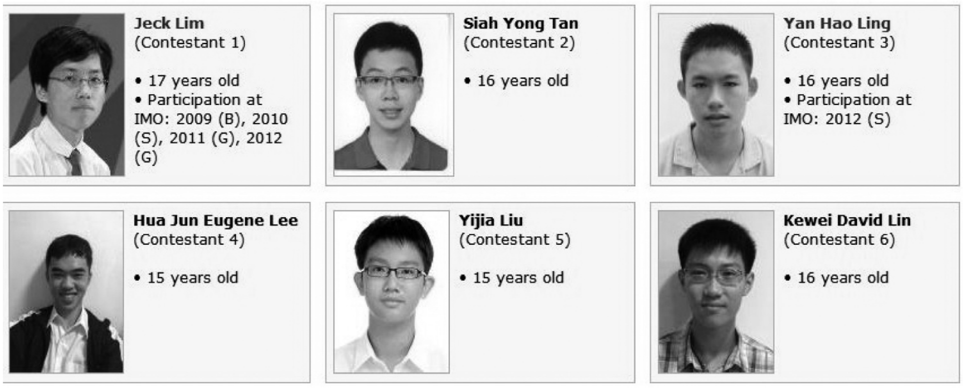
Report on the
54th International Mathematical Olympiad
Santa Marta, Columbia, July 18 - 28, 2013

Wong Yan Loi

The Singapore National Team to the 54th International Mathematical Olympiad in Santa Marta, Columbia consisted of the following members:

Team Leader	: Wong Yan Loi	(National University of Singapore)
Deputy Leader	: Wang Haibin	(NUS High School of Mathematics and Science)
Contestants	: Lim Jeck	(NUS High School of Mathematics and Science)
	Lin Kewei David	(Raffles Institution)
	Tan Siah Yong	(Raffles Institution)
	Liu Yijia	(Raffles Institution)
	Lee Hua Jun Eugene	(Raffles Institution)
	Ling Yan Hao	(NUS High School of Mathematics and Science)
Observers	: Tay Tiong Seng	(National University of Singapore)
	Teo Teck Kian Thomas	(Raffles Institution)
	Loh Yao Chen Ivan	(Observer sponsored by MOE)
	Wong Ying Lin Gabriel	(Observer sponsored by SMS)



Jury meetings. Leaders arrived Barranquilla of Columbia on 18 July and stayed until the first day of exam on 23 July. During these 6 days, leaders held several rounds of meetings to discuss, select, translate the problems as well as to discuss the marking schemes of the solutions. In the beginning we were given a booklet of shortlisted problems consisting of 6 algebra problems (A1-A6), 8 combinatorics problems (C1-C8), 6 geometry problems (G1-G6), 7 number theory problems (N1-N7). The problems A5, G3, and N3 were discarded due to their similarity with problems that had appeared in other country's competitions.

A new procedure. This year, the jury was trying out a new procedure to select the 6 competition problems. The aim was to set a more balanced paper on all 4 topics. The first step was to select the best easy and the best medium problems in each of the 4 topics. This would result in selecting 4 easy problems and 4 medium problems. Among the "4 choose 2" pairs of best easy problems, the leaders voted for the best pair as problems 1 and 4. After that, the leaders selected the best pair of medium problems as problems 2 and 5. It turned out N2 was chosen as the same best easy and best medium problem, meaning it would definitely be chosen to be one of the competition problems. Next, the leader selected the pair of hard problems. In the end the pair (C7,G5) were chosen. However it was later reported that problems similar to G5 had appeared before in USA and Polish Olympiads. Thus G5 was replaced by the next favourable problem G6. The chosen problems were Day 1: N2 proposed by Japan, C2 proposed by Australia, G6 proposed by Russia; Day 2: G1 proposed by Thailand, A3 proposed by Bulgaria, C7 proposed by Russia.

The competition. After the opening ceremony, leaders went for a short excursion to the city of Cartagena. Then the asian pacific countries had a meeting on APMO. Singapore was appointed to be moderating country for APMO 2014 and 2015. For the first day of competition, the leaders spent the first half hour to answer students' questions. It was followed by a meeting to discuss the marking schemes. We received the day 1 solution scripts of our students in the early morning of the second day of competition. An initial assessment showed that all solved problems 1 and 2, except that David could only solve problem 2 partially. Lim Jeck and David solved problem 3. David produced a succinct solution to problem 3. At noon, leaders and observers A were transferred to Santa Marta where students and deputy leaders were staying. We met our deputy leader Mr Wang Haibin, our observers B, Mr Thomas Teo, Ivan Loh and Gabriel Wong. At night we received the answer scripts of the second day's competition. Basically, all our students solved problems 4 and 5, but only Lim Jeck could partially solved problem 6 which was the hardest problem in this year's IMO. We went through the answer scripts carefully.

Coordination. The next 2 days were for coordination. The coordination was done rather efficiently. In fact it is the fastest and easiest coordination that I had ever gone through. We basically got the marks we wanted. All of our students got full marks for problems 1, 2, 4, 5, except that David got 2 points for problem 2. Lim Jeck and David got full marks for problem 3. Lim Jeck got 5 points for problem 6. Our total score was 182 which was a record high. Eugene's problem 5 relied on a result that the sequence of the fraction parts of the nth power of a number greater than 1 is uniformly distributed† in [0,1). In fact he only needed that 0 is an accumulation point of such a sequence. It turns out that this statement is an open problem. The coordinators were fairly generous on this and did not penalize countries using this assertion.

Cut-off and Result. The guidelines for awarding the prizes are: about 1/2 of the total contestants will get a medal; and the number of gold, silver and bronze medals will

† For a sequence of real numbers $(x_n)_{n=1}^{\infty} \subset [0,1]$, let $A(a,b,N)$ be the number of terms x_n of the sequence up to index N such that $a \le x \le b$. A sequence $(x_n)_{n=1}^{\infty} \subset [0,1]$ is said to be "*uniformly distributed*" if

$$\lim_{N\to\infty} \frac{A(a,b,N)}{N} = b - a,$$

for every $0 \le a < b \le 1$.

be approximately in the ratio 1:2:3. Based on these guidelines, it was proposed and approved that the cut-off for gold was at 31, silver at 24 and bronze at 15. Singapore got 1 gold, 5 silvers, and ranked 6th among 97 countries. There were a total of 45 golds, 92 silvers, 141 bronzes and 141 honorable mentions awarded to 527 contestants (52 female contestants). Detailed results can be found at the end of this report and at the IMO official website. `http://www.imo-official.org/`

Singapore Team's Results

	1	2	3	4	5	6	Total	Rank	Rank(%)	Award
Lim Jeck	7	7	7	7	7	5	40	3	99.62	Gold
Lin Kewei David	7	2	7	7	7	0	30	46	91.44	Silver
Tan Siah Yong	7	7	0	7	7	0	28	61	88.59	Silver
Liu Yijia	7	7	0	7	7	0	28	61	88.59	Silver
Lee Hua Jun Eugene	7	7	0	7	7	0	28	61	88.59	Silver
Ling Yan Hao	7	7	0	7	7	0	28	61	88.59	Silver
	42	37	14	42	42	5	182	6	94.79	1 G 5 S

There are 527 contestants from 97 countries.

Conclusion. Singapore achieved the highest score of 182 ever. It was the first time that everyone attained a silver medal or above. It was a marvelous achievement. The students had done their best and their efforts paid off. They made no mistakes in the easy and medium problems and managed to solve them. However it is necessary to put in more hard work in the hard problems. The team this year consisted of one Year 3, three Year 4, one Year 5 and one Year 6 students, which was relatively young. Hopefully, they can achieve even better result in the future. Finally during the IMO advisory board meeting, it was announced that Romania will host the IMO 2018 for the sixth time.

Team photos (from left to right): Dr Wong Yan Loi (leader), Loh Yao Chen Ivan (observer), Lee Hua Jun Eugene, Tan Siah Yong, Lim Jeck, Lin Kewei David, Liu Yijia, Ling Yan Hao, Wong Ying Lin Gabriel (observer), Wang Haibin (deputy leader).

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Language: **English**

Competition Problems

Day: **1**

Tuesday, July 23, 2013

Problem 1. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1+\frac{2^k-1}{n} = \left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

Problem 2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

- no line passes through any point of the configuration;
- no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

The excircle *of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.*

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Language: **English**

Competition Problems

Day: **2**

Wednesday, July 24, 2013

Problem 4. Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;

(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 6. Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

The solution of the competition can be found at the IMO website: http://imomath.com/index.php?options=785