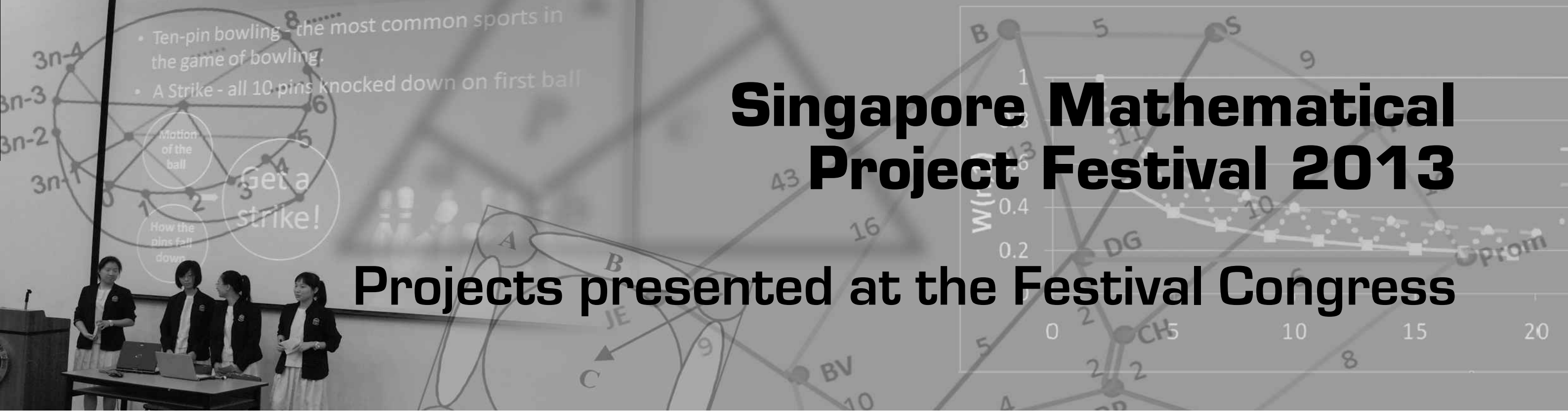


Singapore Mathematical Project Festival 2013

Projects presented at the Festival Congress

This year, 11 teams are selected to present their projects at the SMPF 2013 Festival Congress on March 23, 2013 at Commonwealth Secondary School. Below is a compilation of the abstracts of the projects submitted by the teams.

Junior Section

Project 1: Pseudo-Randomness

- Team members: Tin Jun Hao, Yew Tze Yong, Yip Seng Yeun, Kow Hong Xuan
- Mentor: Ang Lai Chiang
- School: Hwa Chong Institution
- Award: Silver

Abstract

Our project aims to investigate the two main mathematical algorithms for generating pseudo-random numbers, which are the Linear Congruential Generator (LCG) and the Middle-Square Method (MSM). They are also called Pseudo-Random Number Generators (PRNG). We tested both methods and collected statistics from both generators to analyze the pros and cons of each method. As the numbers are generated using mathematical algorithms, there must be some flaws in the methods. True random number generator, however, is flawless as the numbers are generated through natural phenomena.

The LCG involves taking a random seed n, multiply it by a pre-set number a, add it by constant c and finally modulo by m to get the next random number. This process is repeated with the next random number taking the place of n in the formula above. The MSM involves taking a random seed n, squaring it and taking the number of digits in the random seed n from the middle of the square number of n. If the number of digits in the square of n is an odd number, we add a zero in front so that it's valid to take the middle number. This process is repeated with the middle number taking the place of the next n. However the random numbers are the middle numbers of the seeds and not the whole seed.

By comparing both these PRNG's, we found out that LCG is more suitable for long term consistent generation as the frequency of the numbers generated are balanced with no number appearing more than other numbers. However, this might make certain games of chance appear dull if chances were equal. Also, the numbers repeat after $(m - 1)$ times which makes it predictable. For the MSM, the frequency of the numbers generated is quite imbalanced and therefore is only suitable for short term generation. The advantage of MSM compared to LCG is that it is harder to predict as the numbers appear randomly and you wouldn't know when the pattern will repeat itself.

Project 2: Marion Walter's Theorem

- Team members: Seow Ling Ern, Fu Wan Ying, Oan Jia Xuan
- Mentor: Tan Chik Leng
- School: Nanyang Girls' High School
- Award: Silver

Abstract

Marion Walter's theorem states that the area of the central hexagonal region determined by trisection of each side of a triangle by connecting the corresponding points with the opposite vertex is given by 1/10 the area of the original triangle. The theorem, its authenticity previously proven by the ninth-grade student Ryan Morgan in 1994, has prompted various institutions to further seek possible proofs

through investigation using geometrical software and tools. Ryan divided a side of a triangle into n congruent segments, dubbing the process "n-secting". Using Geometer's Sketchpad, Ryan experimented with different n-sections and was later invited to present his conjecture at a special mathematics colloquium at Towson State University in 1994. Propelled by his findings, our group has decided to explore, through in-depth research and scrutiny, the possibility of proving this theorem using simple mathematical concepts in-line with the current Mathematics syllabus in Singapore. The main goal of our project is to hopefully gain a deeper level of understanding of mathematical concepts through various methods of applications, while working out our own way of proving the theorem and other related conjectures.

Conceivable as it seemed, it was necessary for us to apply these concepts with a level of discernment. Under the guidance of our teacher mentor, we were able to apply the necessary skills and knowledge we had, albeit some setbacks. Utilising the topic on Similarity & Congruency, we identified and observed various similar triangles in the original triangle, otherwise known as S. Our methodology included the use of online resources (websites, articles etc.) to broaden our understanding of the subject. Through these sources were we able to siphon out relevant points which we took into consideration, particularly from the reports of those who had managed to prove the theorem using similar individually-proposed methodology. By physically sketching out the triangle S, and making multiple copies of this standard triangle, we were able to picture and highlight the various similar triangles and derive ratios pertaining to its divisors.

Observing that the height of the selected triangles was crucial in determining similarity, we labelled each height and each triangle involved to avoid any miscalculation or erroneous deduction. An organised, gradual approach towards the task was vital in ensuring that we would not digress from our target. By experimentation through substituting different values for others, we formulated some equations which permitted us to derive further equations and arrive at various conclusions. Having a clear understanding and record of each and every step assisted us greatly in realising that a certain calculation could be achieved by substituting or tweaking previous methods. Underneath the seeming complexity of this problem were feasible solutions, some even formed by logical deduction. Minor setbacks included inaccurate assumption, misconceptions and ambiguous graphical representation when we were studying the figures. Upon arriving at our first proof – for the area of the central hexagonal region being indeed, 1/10 of the original triangle, we were stimulated to continue or research and prove other related conjectures, namely figures being 1/16, 1/25 and 7/100 of the triangle, S. Aware of the successful methodology we had previously used, we attempted to use similar methods to prove the following problems. We did an extensive amount of partitioning shapes into smaller ones, and needed a lot of willpower to work with these small tedious areas. Familiar with the standard triangle and its components (similar triangles, lines etc.) we were able to get more acclimatised to using this essential concept of Similarity to aid in our proofs. Notice that each of the shapes are contained in or wrapping around the hexagon, which, as proven previously, is 1/10 of S. This knowledge greatly contributed to our further proofs as we were able to dissect the hexagon into smaller parts and tap on our findings from the original theorem. Through inculcating this progressive element into our investigation, we were able to successfully prove all the targeted related problems.

Menelaus' Theorem, named for Menelaus of Alexandria, is a theorem about triangles in plane geometry. It states that if a line meets the sides BC, CA, and AB of a triangle in the points D, E, and F then the products of the ratios AE/EC × CD/DB × BF/FA = 1. Inspired by online resources, we decided to look into the possibility of another triangle within the triangle S, which would be equal to 1/7 of S. The idea was promising enough, but it required the application of this new foreign theorem – Menelaus' Theorem. We discovered that the theorem could be applied on the new triangle, though in

slightly unexpected areas which required a deeper level of analysis and thinking. Nonetheless, we were eventually able to prove the validity of the conjecture.
Our results and findings indeed show how simple mathematical concepts can be applied to solve complex problems, and how there are possibly more than one way to prove single theorem. Upon identifying the liaising factor that enabled these related conjectures to be formed, we were able to perceive the relation between the various conjectures in order to prove them. An extension of the project, prompted by Ryan Morgan's findings, was also attempted for our interest. Recognising that the shaped formed within the triangle, as well as the ratio of the shape to the triangle differed with the number of congruent segments the triangle S was partitioned into, we sought to prove Morgan's conjecture. It showed that the ratio of the area of the shape inside a triangle which was partitioned into 5 congruent segments to the whole triangle was 1/28. Already familiar with the concept of similarity, we once again applied more or less the same type of approach and managed to prove it genuine. However, this method of approach which we adopted was a slightly more complex version of the one which we used to prove the Marion Walter' theorem. This was due to the fact that more unknown sections, which were not present in the 1/10 proof, had to be calculated. Despite the more complicated unknowns, it was not too much of a problem for us, as the same concepts were still eligible to be applied. Hence, it is safe to assume the authenticity of his general conjecture: For n odd, if the central n-section points of the sides of any triangle are connected to the opposite vertices, the ratio of the area of the original triangle to the area of the resulting hexagon is $(9n^2 -1)/8 :1$.

Project 3: Shortest Travelling Time in MRT

- Team members: Sonia Esaki Kanthimathi Nainar Arumuga, Madhumita Narayanan, Srinithi
- Mentor: Chia Vui Leong
- School: NUS High School
- Award: Bronze and Best Overall Presentation

Abstract

<u>Introduction and Motivation</u>

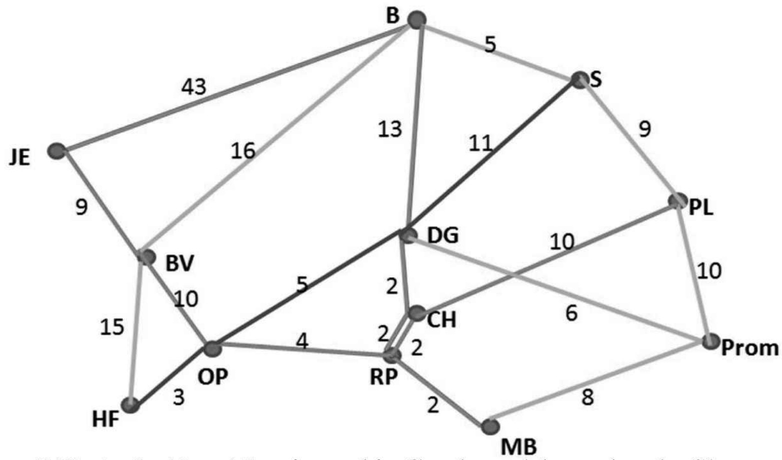
Travelling faster and more efficiently is rapidly becoming important in today's world. The MRT system today consists of 4 intersecting lines (North-South, East-West, North-East, Circle) and there are many possible routes that can be used to get from one station to another. Additionally, the time taken between stations differs so judging based on the number of stations leads to inaccurate conclusions. This makes it daunting to find the fastest route. Currently, only the timings between stations within the same line are available at MRT stations. Thus, we chose to further explore this problem and aimed to find the fastest possible route between any two stations on the MRT for easier reference and to make travelling a much faster experience. We decided not to take waiting time or transfer time between lines into consideration to begin with as they can be included later.
We noted that when 2 stations are on different lines, it is necessary to go through at least 1 interchange to get from one station to the other. Even in the case when both stations are on the same line, there is also a need to pass through at least 1 interchange when we do not travel by the straight-forward route using the same line. Thus, we thought it is logical to determine the fastest route between two interchanges first.

<u>Method 1 – Method of Exhaustion</u>

To start off, we came up with all possible pairings of interchanges and listed them down. For each pair of interchanges, we listed down all possible routes by tracing the MRT map in a systematic manner. At every interchange, we explored all possible paths to ensure our list was complete. We also double-checked that the routes were unique, in other words, no station was visited twice in any route. Once we had determined the list of routes, we used the timings between adjacent stations along the same line to calculate the time taken for each of the routes. We could then conclude which route takes the shortest amount of time. Thus, we managed to find the fastest path between any two interchanges. This method was very tedious so we looked for an alternative method to find out the fastest path between 2 interchanges.

Method 2 – Graph Theory (Shortest Path Problem)

We found out that the MRT system could be represented by a Graph Model where the interchanges serve as vertices such that two vertices (interchanges) i and j in $V(G)$ are adjacent if and only if there is a direct way to travel from i to j (or vice versa) without having to pass by another interchange. The weights of the graph are the time taken between 2 interchanges when travelling along the edge that connects them.



B: Bishan
BV: Buona Vista
CH: City Hall
DG: Dhoby Ghaut
HF: Harbourfront
JE: Jurong East
MB: Marina Bay
OP: Outram Park
PL: Paya Lebar
Pr: Promenade
RP: Raffles Place
S: Serangoon

Dijkstra's Algorithm is used in Graph model to solve the Shortest Path Problem, which is to find the smallest sum of the weights of the edges connecting two vertices. Hence, for our model, the solution of the Shortest Path Problem will be the fastest route between the two interchanges.

The working principle of Dijkstra's Algorithm is to find the shortest path between a source vertex and every other vertex. First, every vertex in the graph model is marked with infinity to show that it had not yet been visited. At first, the 'current intersection' will be the starting point. The distance to it, which is its label, will hence be zero. Subsequently, the current intersection will be the closest unvisited intersection to the starting point. From the current intersection, the distance to every unvisited intersection that is directly connected to it is updated by determining the sum of the distance between an unvisited intersection and the value of the current intersection's label, and relabeling the unvisited intersection with this value if it is less than its current value. As a result, the intersection is relabelled if the path to it through the current intersection is shorter than the previously known paths. By continuing to update the neighbouring intersections with the shortest distances, all the vertices will eventually be labelled with the shortest distances. Once all the vertices are labelled with the shortest distances we determined the shortest path to every vertex from the starting point by tracing the way back.

We used the Dijkstra's Algorithm to find the least time-consuming path from an interchange to every other interchange. When repeated for all the interchanges, we will find the fastest path between any two interchanges. The fastest paths obtained by Graph Theory is the same as the paths obtained by the Method of Exhaustion, hence we can ensure our results are reliable.

Final Result

To find out the shortest path between any two stations, we categorised the stations into 3 types. First type: both stations are interchanges; the second type: only one station is interchanges and the last type: neither of the station is interchange. We introduce the following notation in our analysis. Suppose that A and B are two interchanges. Let t_{AB} be the shortest travelling time from interchange A to interchange B.

First type: Interchange to interchange

The fastest path between any two interchanges can be obtained from the tabulated data in the report.

Second type: Interchange X to station Y (non-interchange)

Suppose that the station Y is adjacent to interchange P and interchange Q and the travelling time from station Y to interchange P and interchange Q are t_P and t_Q respectively. If interchange X is neither interchange P nor interchange Q, then the shortest travelling time will be the smaller between $\{t_{XP} + t_P, t_{XQ} + t_Q\}$. Suppose interchange X is the same as interchange P, then the shortest travelling time will be the smaller between $\{t_P, t_{XQ} + t_Q\}$.

Third type: Station X (non-interchange) to station Y (non-interchange)

Suppose that the station X (Y) is adjacent to interchange A (P) and interchange B (Q) and the travelling time from station X (Y) to interchange A (P) and interchange B (Q) are t_A (t_P) and t_B (t_Q) respectively. If all four interchanges are mutually distinct, then the shortest travelling time will be the smallest among $\{t_A + t_{AP} + t_P, t_A + t_{AQ} + t_Q, t_B + t_{BP} + t_P, t_B + t_{BQ} + t_Q\}$. If a pair of interchanges are the same, supposedly interchange B and interchange P, then the shortest travelling time will be the smallest among $\{t_A + t_{AP} + t_P, t_A + t_{AQ} + t_Q, t_B + t_P, t_B + t_{BQ} + t_Q\}$. For the last situation, suppose interchange A is the same as interchange P and interchange B is the same as interchange Q, which means that the two stations lie on the same line. The shortest travelling time will be the smallest among $\{t_A + t_{AQ} + t_Q, t_B + t_{BP} + t_P\}$.

Future Possible Research

We could take LRT lines into consideration and/or apply the algorithm to future (and more complicated) MRT systems. We are also planning to take waiting and transfer times at interchanges into consideration. Furthermore, we could try to develop a computerised system where one can enter the names of two stations and the fastest path would be displayed.

Project 4: Blind Spots in Cars

- Team members: See Wan Yi Faith, Deborah Chin Jia Min, Thng Hui Min Felicia
- Mentor: Ruth Tan
- School: Methodist Girls' School
- Award: Bronze

Abstract

This project is about the blind spot mirrors of cars.

Our purpose is to create a solution to minimize the number of accidents due to blind spots of cars that we will eventually hope to apply to most, if not, all the cars in Singapore. Many a time in our daily lives, cars are found all around us, with various issues like cars hogging lanes and getting into both major and minor accidents occasionally. In addition, these accidents may not only necessarily only involve car drivers, but also motorcyclists. Many drivers rely on the side view mirrors of their cars alone, thinking that they are able to see sufficient of the road for safe driving, but we feel that this is when blind spot mirrors come in.

Our hypothesis is that attaching a blind spot mirror that is elevated at an angle of 16.5° as compared to a flat blind spot mirror to the side view mirror facing outwards on the top left for the left mirror and top right for the right (driver's) mirror would reduce blind spots effectively thus preventing accidents from occurring. Our problem statement is that blind spots in cars are the cause for most car accidents that are related to changing lanes and crashing into other motorcyclists. Also, our aim is to reduce the car accidents that happen due to blind spots when changing lanes.

During the process, we made use of Pythagoras' Theorem and Trigonometry to calculate our measurements. All our measurements are manual, but the calculations include the use of the two concepts mentioned before.

We measured the area viewable by the blind spot mirrors at various locations, also with the use of different types of blind spot mirrors. We placed the blind spot mirrors at four different locations: top right corner, top left corner, bottom right corner, bottom left corner. These were the locations of the blind spot mirrors for both the right side view mirror and the left one. In addition, we used multiple types of blind spot mirrors: flat blind spot mirror, elevated blind spot mirror at 11.5° (the one sold in the market), and another blind spot mirror which we made where we manually adjusted it according to the different angles we wanted to measure.

When we measured beyond a certain angle of 14.5°, we realised that the area covered was decreasing. Therefore, we concluded that the largest area covered by that particular angle is the best angle for a blind spot mirror.

Hence, we concluded that attaching an elevated blind spot mirror to the side view mirror facing outwards as compared to a flat blind spot mirror would reduce the amount of blind spots one has. Moreover, we realized that the blind spot mirror angled at 14.5° actually provides the most viewable area and the mirror angled at more than 14.5° did not provide more viewable area. Also, we concluded that the best position to attach a blind spot mirror was on the top right hand for both left and right mirrors unlike our original hypothesis.

Project 5: Blokus Winning Strategies

- Team members: Hsiao I Ann, Yeo Wan Jin
- Mentor: Chai Ming Huang
- School: NUS High School
- Award: Bronze

Abstract

Blokus is a classic strategic board game for four players. Each of the players has twenty-one pieces of Tetris-like blocks to be placed corner-to-corner on the twenty-by-twenty board. The sides of the pieces of the same player must not touch, but there are no rules regarding the placement of the pieces of different players.

Our aim of this project is to find the optimal winning strategy to play Blokus. Before working on the strategies, we have worked out a logical analysis on the pieces, in terms of the number of basic square units it consisted of, the number of corner units the player is able to put some more of his pieces to expand his territory, the distance between the furthest expandable corners of a piece that determines the furthest diagonal distance it can cover within one piece, and many other factors.

After the analysis, we deduced the best pieces to be placed under different situations. We then looked into the best direction of movement, which would greatly affect the amount of territory available for the player in the later part of the game. Through detailed reasoning and numerous trials, we have found that the optimal expanding direction is linear towards the centre as shown below in Figure 1.

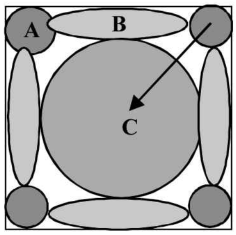


Figure 1

We can decide the starting pieces when expanding towards the center based on the following strategies:

1. The Expanding Strategy, whereby we choose pieces in terms of their diagonal distances and corresponding angles, so as to reach the centre, the optimal destination, faster.
2. The Expandability Strategy, whereby we choose pieces based on their expandability so that more territory is obtained. Moreover, there is a lesser chance that opponents will block out all of your expandable corners in few turns due to the abundance of expandable corners. This increases survivability throughout the game.

After analyzing on how to optimally expand towards the center with the above strategies, we have come up with an optimal opening by considering both strategies as shown below in Figure 2.

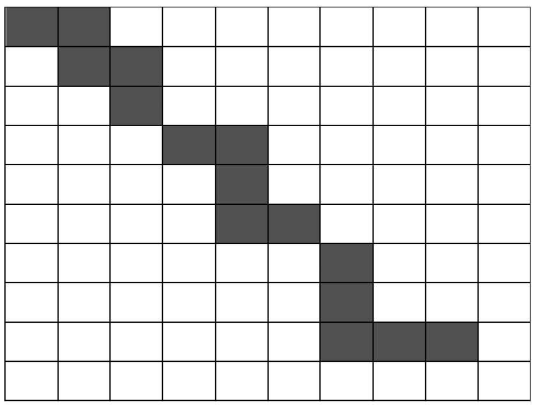


Figure 2

We also considered which pieces to choose upon reaching the center. We have decided to calculate the value per unit of each piece to have a gauge on how optimal each piece is to be placed on the board at the center. This is calculated by:

(total number of expandable corners + total number of wasted spaces up to that point of the game) ÷ (total number of tiles occupied)

The values are obtained based on our optimal opening as seen in Figure 2 and using a scale of +1 for each expandable corner and -1 for each wasted space. Pieces with the highest value per unit will preferably be placed down first.

The "Dent" Strategy is a defensive strategy that can be used at any point of the game. The concept is to create holes between two connected pieces (with *compatible dents*) such that opponents will have to use their smallest pieces to penetrate through the dent. Because smaller pieces are easier to place on a crowded board at later gameplay plus placing the 1 unit piece last gains you bonus points, opponents will have a harder time progressing. The chances of them attacking will also be lower due to these reasons. This is an original strategy that we have come up with.

Hence, with the above strategies, one has a higher chance of winning Blokus.

Senior Section

Project 1: Computer Solution to Convex 7-Gon Happy Ending Problem via Graph-to-Matrix Transformation

- Team members: Liu Changshuo, Philip Ong Zheng Yang, Gao Yuan
- Mentor: Jin Chenyuan
- School: Anglo-Chinese School

- Award: Gold and Foo Kean Pew Memorial Prize

Abstract

The happy ending problem aims to identify the minimum number of points $f(N)$ required on a plane - of which no three points are colinear - to form a convex N-gon, where the points can be in any orientation and act as vertices for the polygon. The closest bound for $f(N)$ is

$$1+2^{N-2} \le f(N) \le \binom{2N-5}{N-2} - 1.$$

It has been proven that $f(3) = 3, f(4) = 5, f(5) = 9, f(6) = 17$. However, no solution for $f(7)$ has been discovered.

In our research, we proposed a Graph-to-matrix (GTM) Transformation that transforms a graph with n vertices into an $n \times n$ matrix. For a simple graph with k points such as in Figure 1, which contains 5 points, we arbitrarily define k polar coordinates with parallel main-axes and each polar coordinate passing through one of k points in the graph. Another graph is generated identical to this graph, where point 0 is considered as the origin and the polar coordinate intersecting point 0 overlaps the y-axis. The position of the other points can be related to point 0 through polar coordinates, as (r, θ), with $0 \le \theta \le 2\pi$; and r is the length between a point and the origin. Suppose a vector starts at y-axis and spins in an anti-clockwise direction until it overlaps the vector going through origin and the point. The angle covered by the sweeping is θ. This process is repeated with points 1 through $k-1$ being considered as the origin.

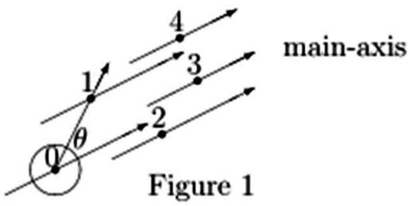


Figure 1

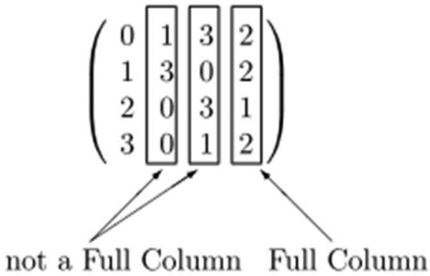
For each point considered as the origin, arrange the other points in the increasing order of θ. For instance, for point 0, the increasing order of the other points is (3, 4, 1, 2). For point 1, the increasing order is (4, 0, 2, 3). By combining these orders together, we can get a matrix as shown below.

$$\begin{pmatrix} 0 & \longrightarrow & 3 & 4 & 1 & 2 \\ 1 & \longrightarrow & 4 & 0 & 2 & 3 \\ 2 & \longrightarrow & 3 & 4 & 1 & 0 \\ 3 & \longrightarrow & 4 & 1 & 0 & 2 \\ 4 & \longrightarrow & 1 & 0 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 3 & 4 & 1 & 2 \\ 1 & 4 & 0 & 2 & 3 \\ 2 & 3 & 4 & 1 & 0 \\ 3 & 4 & 1 & 0 & 2 \\ 4 & 1 & 0 & 2 & 3 \end{pmatrix}$$

Any graph can be transformed into a matrix by GTM transformation. A matrix generated through this process is considered a GTM matrix. Such matrices are always valid as they can be represented graphically.

Now we shall define convex and concave graphs. Let a convex graph be a large finite set of vertices $V(G)$ on a plane, for which every edge in its complete graph remains inside or on the boundary of the polygon. Let a concave graph be a large finite set of vertices $V(G)$ on a plane that is not a convex graph.

Certain properties of the 4×4 matrix generated from GTM Transformation allow us to determine the convex polygon with maximum vertices in an $n \times n$ matrix from a graph with n vertices. In a 4×4 GTM matrix, we call a column Full Column when three of the four points in that column are the same.



Theorem 1. For any concave graph with 4 vertices, the matrix generated must contain one and only one Full Column.

Theorem 2. For any convex graph with 4 vertices, the matrix generated must contain either 0 or 2 Full Columns.

The theorems we proved differentiates a concave graph from a convex graph by the number of Full Columns in a GTM matrix. In addition it is easy to prove the lemma below.

Lemma 1. For a convex N-gon S, every combination of 4 vertices in S form a convex 4-gon X. Conversely, if every combination of 4 vertices in an N-gon S forms a convex 4-gon X, the polygon S must be convex.

Lemma 1 shows that for any graph, if we know whether every combination of 4 vertices is a concave graph or a convex graph, we can identify whether a convex 7-gon exists within the polygon. A convex 7-gon exists if every combination of 4 of 7 vertices generates a convex 4-gon and the 4×4 matrices contain either 0 or 2 Full Columns.

Lemma 2. For any $N \times N$ GTM matrix, a particular point m, $(m \leq N - 1)$, can be removed, so that a valid $(N - 1) \times (N - 1)$ matrix can be obtained from the original graph. This is completed through the elimination of the point m in every row, and the row generated when point m was the origin during the GTM Transformation process.

The process to extract every combination of 4×4 matrices from an $N \times N$ matrix is given in lemma 2. Hence, it allows us to computationally prove that all graphs with 33 vertices contain at least one convex 7-gon. We postulate that all 33×33 matrices must contain 7 points for which every matrix generated by any four of the seven points is a convex 4-gon (Lemma 2).

We have identified an exhaustive list of all valid 25 non-repeated 4×4 matrices that have a graphical representation. These 25 matrices are formatted as fixed matrices, matrices that have their first row and columns ordered in increasing order, so as to reduce the number of permutations.

There are $32!^{32} \approx 2{:}79 \times 10^{1133}$ different matrices that need to be analysed if 33×33 matrices are generated by permutation. Instead, we generate matrices using the set of 4×4 matrices by combining two matrices that have the same subset matrices together to form a 5×5 matrix. The 5×5 matrices are then verified by generating all possible 4×4 matrices from it, converting them to fixed matrices and comparing them to the original 25 matrices. These processes are explained in greater detail in the report. This method is then repeated till 33×33 matrices are produced. Since only valid matrices generate the next set of matrices, verification is much quicker as many invalid matrices are not even considered. This shortens computation time significantly

Project 2: Skewness of Some Generalized Petersen Graphs

- Team members: Zhu Shiyao, Lim Yu Chen, Mok Bingwei, Maurice
- Mentor: Lee Chan Lye
- School: NUS High School
- Award: Silver

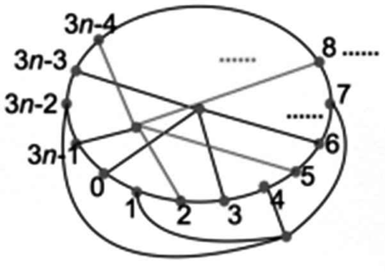
Abstract

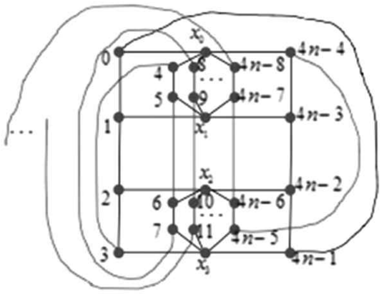
A graph is planar if it can be drawn in the plane without any edges crossing. The skewness of a graph G, denoted sk(G) is the minimum number of edges in G whose removal results in a planar graph. The graph $Q_n(k)$ is obtained by contracting all edges of the k n-cycles in the generalized Petersen graph $P(nk, k)$ with vertex set $\{u_i, v_i, 0 \le i \le nk-1\}$ to a single vertex. In our project, we investigated the skewness of the graphs $Q_n(3)$, $Q_n(4)$ and P(20,5).

<u>Theorem</u>. Let G be a simple connected graph with v vertices, e edges and having girth g. Then

$$\mathrm{sk}(G) \ge \left\lceil e - \frac{g}{g-2}(v-2) \right\rceil$$

Using the above theorem and the following drawings of $Q_n(3)$ and $Q_n(4)$,





we obtained the next two Theorems:

<u>Theorem</u>. sk($Q_n(3)$) = n - 1

<u>Theorem</u>. sk($Q_n(4)$) = $2n$ - 3

By using the following three known results:

- sk($P(4k, k)$) = k +1 if $k \ge 4$ is even.
- sk($Q_4(k)$) = k +1 if $k \ge 4$.
- $k+1 \le$ sk($P(4k, k)$) $\le k+2$ if $k \ge 5$ is odd.

we worked on the smallest case for which sk($P(4k, k)$) is unsolved, namely sk(P(20, 5)).

Let $SQ_k(i)$ represent the subgraph of $P(4k, k)$ formed by the set of vertices $\{ v_i , v_{i+k} , v_{i+2k} , v_{i+3k} \}$, where $i = 0, 1, 2, \cdots, k-1$; R(G) be a set of sk(G) edges in a graph G whose removal results in a planar graph; H_k be a planar graph formed by $P(4k, k)$ – R($P(4k, k)$).

We have obtained the following results:

- $|\, R(P(4k, k) \cap SQ_k(i)) \,| \le 1$ for all $i = 0, 1, 2, \cdots, k-1$, where $k \ge 5$ is odd.
- $|\, R(P(4k, k) \cap SQ_k(i)) \,| = 0$ for all $i = 0, 1, 2, \cdots, k-1$, where $k \ge 5$ is odd.
- H_5 is a planar graph with five 4-faces and eleven 8-faces.
- R(P(20,5)) is a union of independent rim edges of P(20,5).
- Of the eleven 8-faces in H_5, two share an edge with a 4-face while nine share two edges with two 4-faces.

Using these results, we can show the following theorem.

<u>Theorem</u>. sk(P(20,5)) = 7.

Project 3: Let's Go Bowling!—the Mathematics behind Getting a Strike in Bowling

- Team members: Wang Qian, Qu Wenqin, Pu Xijin, Chen Liu
- Mentor: Tan Chik Leng
- School: Nanyang Girls' High School
- Award: Silver

Abstract

Our study is primarily concerned with the game of ten-pin bowling, which is one of the most common forms in the game of bowling. Each game consists of ten frames, each of which starts with a full rack of ten pins. In each frame, a player has two deliveries of ball and is to knock down as many of the ten pins as the player can.

The rules of scoring are as follow:

- The base score for a frame is the total pinfall (number of pins knocked down);
- In the case of a strike (all 10 pins knocked down on first ball) the pinfalls of the next two balls are added to the score for the frame;
- In the case of a spare (all remaining pins knocked down on second ball) the pinfall of the next ball is added to the score for the frame;
- These bonus pinfalls can lead to up to two additional frames when a spare or strike occurs in the tenth frame.

A perfect game consists of twelve consecutive strikes, yielding a score of 30 for each frame and a total score of 300. The highest possible score without any strikes results from ten frames of 9-1 spares followed by an additional 9 in the eleventh frame, corresponding to a pinfall of 19 each frame and a total score of 190. As seen, it is very crucial for bowling ball players to get as many strikes as possible in order to win a game.

On the other hand, getting a strike in bowling is never an easy task. While for professional bowling players it is believed that skills can only be learnt from practice, our group is interested in seeking a systematic way to get a strike in bowling by tracing the motion of the bowling ball and by studying the mathematics behind the game. In our study, we try to find answers for the following questions: What are the conditions to get a strike? What are the factors that have impacts on the motion of the ball? At what speed should the ball be released in order for it to knock down the ten pins at the end of its motion?

We start conducting our project mainly by studying a mathematical model of the motion of the ball. In consideration of the different kinds of bowling balls and the inner structure, we decided to build a mathematical model which is a perfectly-circle and has uniform density for the convenience of our investigation. Using this model, we try to determine the theoretical direction at which the ball should go and by investigating the falling of the ten pins, to calculate theoretical results of the speeds of the ball. We have also discussed the practical factors that may make the actual results diverge from the theoretical ones. Moreover, we calculate that the angle at which the ball comes into contact with the first pin has an optimum value.

Through analysis and calculations, we draw a conclusion that the strategy to get a strike is much related to the ideal angle at which the ball knocks the first pin and the speed at which the ball is released.

Project 4: Creative Ruler

- Team members: Jovin Chua Jun Wen, Wu Xin Yun Natalie, Yoong Sean Young Edmund, Zhang Jiaheng
- Mentor: Toh Pui Yhing
- School: Clementi Town Secondary School
- Award: Bronze

Abstract

The objective of our project is to look for an effective way of identifying the minimum number of markings on the ruler so that the function of the ruler is still achieved. This particular ruler is therefore defined as a creative ruler.

In chapter 1, background information, objectives and rationale of the project, Creative Ruler, is provided. In addition, a list of mathematical functions and units used for this whole project is also included.

In chapter 2, an exploration of the lower bound, which is defined as the greatest value of the minimum number, of markings on a ruler of n units without compromising the functions of the ruler and still able to find all values from 1 to n is described.

In chapter 3, an exploration of how the upper bound, which is defined as the least value of maximum number, of markings needed on a ruler of n units length such that it will also be able to measure all values from 1 to n is described.

In this final chapter the project is included with an example on how to effectively mark a ruler of 100 unit length, how the creative ruler could be applied in real-life scenarios and the effectiveness of the creative ruler.

Project 5: The Broken Stick Problem

- Team members: Ng Chong Yi, Chua Kee Han Daryl
- Mentor: Lee Chan Lye
- School: NUS High School
- Award: Bronze

Abstract

Given a stick of length x, two points are randomly chosen on the stick and the stick is broken into three pieces at those two points. What is the probability of the three pieces forming a triangle end to end? To simplify things, take x to be 1. The first step to solving the problem would be to consider the circumstances where the three pieces would form a triangle. To do this, we make use of the triangle inequality: the sum of the two smaller lengths must be larger than the third length.

<u>Extension</u>

What is the probability of forming an acute triangle or a triangle whereby all the angles are larger than, say 30°?

We aim to solve such interesting cases which would involve the use of trigonometry such as the cosine rule, and also find possible generalizations for the cases by adjusting the previously described methods to fit the cases.

<u>Geometrical approach</u>

Using an equilateral triangle of height 1, point P is randomly selected within and the perpendicular distance from each side of the triangle is labeled a, b and c. By the altitude theorem, $a + b + c = 1$. Thus a, b, and c represent each of the pieces from the problem, and instead of choosing two points, this method allows the problem to be viewed as choosing only one point (figure 2.1).

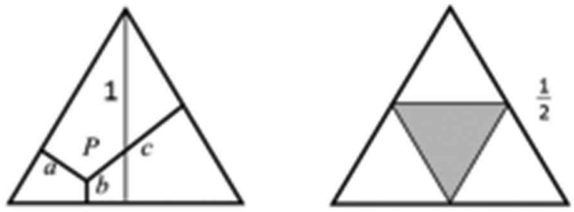


Figure 2.1 Figure 2.2

Using the triangle inequality, we conclude that in order for a, b and c to form a triangle:

$$a+b > c \text{ or } a+c > b \text{ or } b+c > a.$$

Since $a + b + c = 1$, a, b and c must each always be less than half for a triangle to form. This creates a border in the center of the triangle as shown in figure 2.2. By calculation, we find the shaded area to be a quarter of the triangle. Thus there is a 25% chance that a stick randomly broken into three pieces would form a triangle.

Graphical approach

Taking the two cut points to be x and y on the stick and two of the lengths formed to be X and Y (figure 3.1). These lengths can be mapped onto a unit square of a 2-D graph (figure 3.2). Next, we have to find conditions which the lengths can form a triangle.

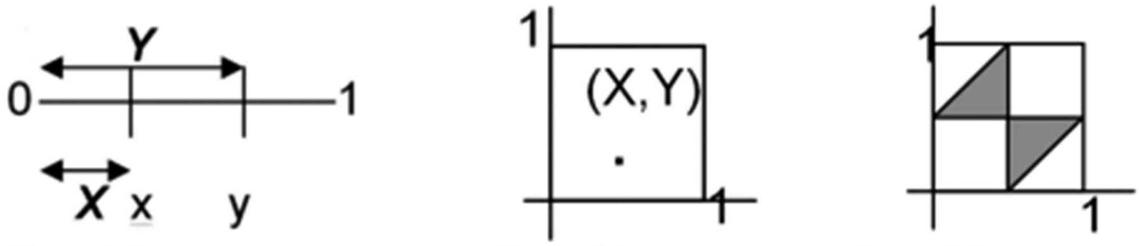


Figure 3.1 Figure 3.2 Figure 3.3

The different cases or scenarios are as listed:

Case (i): $0 < X < Y < 1$.

The stick segments after breaking the stick would have length X, $Y - X$, and $1 - Y$. For a triangle to form, the sum of the lengths of any 2 stick segments has to be larger than the last one. Hence in terms of X and Y, this simplifies to:

$X + (Y - X) > 1 - Y \implies X < \frac{1}{2}, Y > \frac{1}{2}$

$(Y - X) + (1 - Y) > X \implies Y - X < \frac{1}{2}$

$(1 - Y) + X > Y - X$

Plotting the inequalities we get the shaded triangle above the line $Y = X$ as $X < Y$ (figure 3.3). This area is $\frac{1}{8}$ the area of the unit square. Hence the probability of forming a triangle where $X < Y$ is $\frac{1}{8}$.

Case (ii): $0 < Y < X < 1$

By symmetry, this case yields another area of $\frac{1}{8}$, the shaded triangle below the line $Y = X$ as $Y < X$ (figure 3.3).

Case (iii): $0 < X = Y < 1$

This case is omitted as the probability that X and Y are equal is 0, although such a case could happen. This happens because graphically, $Y = X$ is a line that has no bounded area; therefore, the probability of it occurring is zero. This is one of the important fundamentals in probability which is worth taking note of. Thus by calculation, we once again conclude that there is a 25% chance that a stick randomly broken into three pieces would form a triangle.

Project 6: Master Mafia

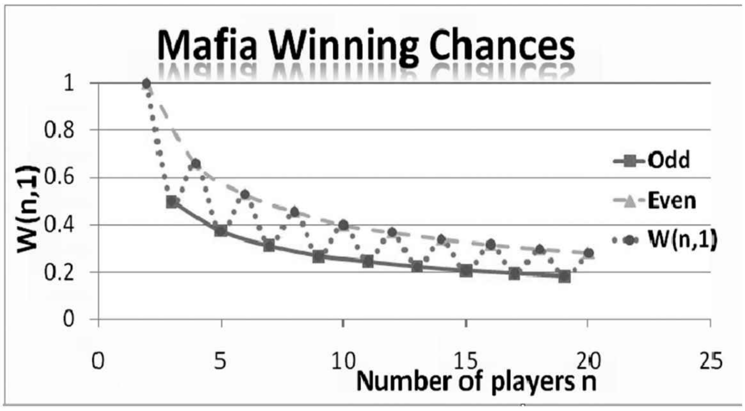
- Team members: Lang Yanbin, Liu Siyu, Xia Nan
- Mentor: Shaun Yong
- School: Hwa Chong Institution
- Award: Bronze

Abstract

The mafia game consists of four key roles: citizens, mafias, detectives and sometimes doctors. Last year we found out the general formula to calculate mafia's winning probability $w(n, 1)$, where n represents the number of players, and the number "1" shows that only one mafia is present:

$$w(n,1) = \frac{(n-2)!!}{(n-1)!!}.$$

By applying this conclusion, we found out the trend of how mafia's winning chances will change when more and more citizens participate in the game. The general relationship can thus be expressed as: $w(2k+(2m+1),1) < w(2k+(2m+2),1) < w(2k+2m,1)$.



Then we found out the following general formula to calculate the mafia's winning chances when there is a detective:

$$w'(n,1) = w(n,1) \times \prod_{i=2}^{t}\left(1-\frac{1}{n-(2i-1)}\right) = \frac{(n-2)!!}{(n-1)!!} \times \prod_{i=2}^{t}\left(1-\frac{1}{n-(2i-1)}\right)$$

where t represents the number of the rounds the detective has lived through.

By using the knowledge of series, we found out the following general formula to calculate the winning probabilities of the mafias, when there are 2 mafias among n players which is a more complicated situation:

$$w(n,2) = \frac{4-(n \bmod 2)}{6-(n \bmod 2)} + \frac{2}{n}\sum_{i=2}^{\frac{n-(n \bmod 2)}{2}} \frac{(2i-2-(n \bmod 2))!!}{(2i-1-(n \bmod 2))!!}$$

Lastly, we wanted to investigate the situations including a doctor. We built a format that when there is one mafia, one doctor and x citizens (x is a positive integer) in each round, the chances that mafia will not be eliminated can be expressed by the following expression: $\frac{1}{x+1} \times \frac{x+1}{x+2} + \frac{x}{x+1} \times \frac{x-1}{x} \times \frac{x-1}{x}$, and we call it “one, one, x” format.

This year, we proved the conjecture that mafia’s winning chances would decrease with the presence of a doctor by looking at the doctor’s function: heal a player, kill a citizen or kill a mafia. By comparing the change of mafia’s winning probability when different situations happen, we conclude that mafia’s winning chances would decrease with the presence of a doctor.

Then, we continued with our case study with 3 mafias in the game and came out with another general formula:

$$w(n,3)=\frac{(n-4)!!\times(3+(n \bmod 2))!!}{(n-1)!!}w(4+(n \bmod 2),3)$$
$$+\frac{3}{n-1}\sum_{m=0}^{k-2-(n \bmod 2)}\frac{(n-4)!!}{(n-3)!!}\frac{(n-3-2m)!!}{(n-4-2m)!!}w(2(k-m)-(n \bmod 2),2)$$

With all the case studies above, we tried to find out $w(n, m)$ where there are m mafias among n players and found out the generating formula to express $w(n, m)$ as a recurrence sequences.

We would like to simplify our results and hopefully come out with a final formula. We found the direction to derive $w(n, m)$:

$$w(n,\ m)\ =\ 1\ -\ P_0(t)$$

$$F(t,x)=\sum_{k=0}^{\infty}P_k(t)x^k$$

$$F(t+1,x)=F(t,x)-\frac{x-1}{n-2t}\frac{\partial F(t,x)}{\partial x}$$

where $P_k(t)$ denotes the probability that k mafias are left after t rounds.