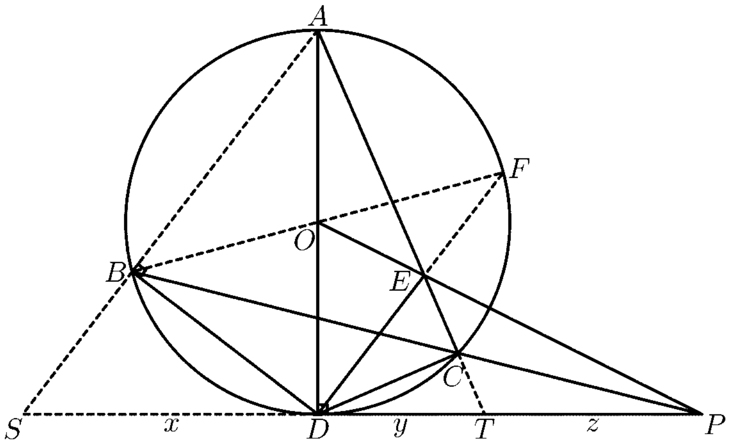
Singapore International Mathematical Olympiad Committee

2013 National Team Selection Test

Problems and Solutions

1. Let AD be a diameter of a circle with centre O. Let P be a point such that DP is a tangent to the circle, and let B, C be on the circumference of the circle such that BCP is a straight line. If OP intersects AC at E, show that BA is parallel to DE.

Solution 1. Extend AB and AC to meet the tangent line DP at S and T respectively. To show $BA \parallel DE$, it suffices to show that $\frac{AE}{ET} = \frac{SD}{DT}$. Now from $\triangle ADT$, we have $\frac{AE}{ET}\frac{DO}{OA}\frac{TP}{PD} = 1$ which implies $\frac{AE}{ET} = \frac{PD}{TP}$. Hence it suffices to show that $\frac{SD}{DT} = \frac{PD}{TP}$.



Let $SD = x$, $DT = y$ and $TP = z$. we want to show that $\frac{x}{y} = \frac{y+z}{z}$. From $\triangle AST$, we have $\frac{AB}{BS}\frac{SP}{PT}\frac{TC}{CA} = 1$. Also since $AD \perp SD$, $BD \perp AS$, we have $\frac{AB}{BS} = \frac{DA^2}{DS^2}$ and $\frac{TC}{CA} = \frac{DT^2}{DA^2}$.

Combining the above result, we get $\frac{DT^2}{DS^2} \cdot \frac{SP}{PT} = 1$. Hence $\frac{y^2}{x^2} \cdot \frac{x+y+z}{z} = 1$ which implies $z = \frac{y^2}{x-y}$ so that $\frac{x}{y} = \frac{y+z}{z}$.

Solution 2. Let the extension of DE intersect the circle at F. Let AD intersect FB at O'. We have that CA intersects DF at E, and the tangent at D intersects BC at P. By Pascal's theorem applied to the (degenerated) hexagon $CADDFB$, we have E, O', P are collinear. This means $O' = O$, so that BF is a diameter. Hence $\angle BDF = 90°$. Therefore, $\angle BAD = 90° - \angle BDA = \angle ADE$. This shows that AB is parallel to DE.

2. Let $n \geq 4$ be a positive integer. Exactly one integer is written in each cell of an $n \times n$ square such that the sum of all these integers is positive, while the sum of the integers in **any** 3×3 square is negative. Find all the values of n for this to be possible.

Solution All n that is not divisible by 3.

Firstly, if $n = 3k$, the $n \times n$ square can be divided into exactly k^2 disjoint 3×3 squares, and since each of these sums is negative, the sum of all numbers in the $n \times n$ square will be negative too, which violates the condition given.

Now for $n = 3k+1$ or $3k+2$, we just need to provide a construction that works. Let $C(i,j)$ denote the number in the cell of the ith row and the jth column.

The case $n = 3k + 1$. Let $C(3p + 2, 3q + 2) = C(3p + 2, 3q + 3) = C(3p + 3, 3q + 2) = C(3p + 3, 3q + 3) = -b$ for all $p, q = 0, 1, 2, \ldots, k - 1$, and all other entries $C(i,j) = a$.

Clearly, the sum of all integers in any 3×3 squares is $5a - 4b$, and the sum of all integers in the $n \times n$ squares equals $[(3k+1)^2 - 4k^2]a - 4k^2b$. It suffices to show that there exist integers a and b such that $5a < 4b$ and $(5k^2 + 6k + 1)a > 4k^2b$. We need $\frac{5}{4}a < b < \frac{5}{4}a + \frac{6k+1}{4k^2}a$, which means we can just choose an integer a such that $\frac{6k+1}{4k^2}a > 1$. This makes the difference between the two numbers $\frac{5}{4}a$ and $\frac{5}{4}a + \frac{6k+1}{4k^2}a$ to be greater than 1 so there exists an integer b between them. In particular, we can choose $a = 4k^2$ and $b = 5k^2 + 1$. See the examples for $k = 1$ in matrix 1, and $k = 2$ for matrix 3 below.

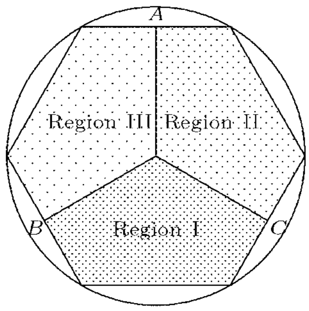
The case $n = 3k+2$. Let $C(3p+3, 3q+3) = -b$ for all $p, q = 0, 1, 2, \ldots, k-1$, and all other entries $C(i,j) = a$. Clearly, the sum of all integers in any 3×3 squares equals $8a - b$, and the sum of all integers in the $n \times n$ squares equals $[(3k + 2)^2 - k^2]a - k^2b = (8k^2 + 12k + 4)a - k^2b$. Hence we need integers a and b such that $8a < b < 8a + \frac{12k+4}{k^2}a$. Here we choose $a = k^2$ and $b = 8k^2 + 1$. See the examples for $k = 1$ in matrix 2, and $k = 2$ in matrix 4 below.

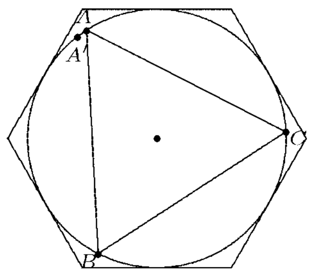
$$(1) \begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & -6 & -6 & 4 \\ 4 & -6 & -6 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \quad (2) \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -9 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$(3) \begin{pmatrix} 16 & 16 & 16 & 16 & 16 & 16 & 16 \\ 16 & -21 & -21 & 16 & -21 & -21 & 16 \\ 16 & -21 & -21 & 16 & -21 & -21 & 16 \\ 16 & 16 & 16 & 16 & 16 & 16 & 16 \\ 16 & -21 & -21 & 16 & -21 & -21 & 16 \\ 16 & -21 & -21 & 16 & -21 & -21 & 16 \\ 16 & 16 & 16 & 16 & 16 & 16 & 16 \end{pmatrix} \quad (4) \begin{pmatrix} 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & -33 & 4 & 4 & -33 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & -33 & 4 & 4 & -33 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}$$

3. Let S denote a regular hexagon and its interior with side length 1. Find the smallest real number r satisfying: one may colour the points in S using three different colours such that the distance between any two points with the same colour is smaller than r .

Solution Refer to the following colouring scheme. We divide the regular hexagon into three regions (figure on the left). The common edge of any two regions could follow the colour of either side. A, B and C are coloured the same as Region I, II, III respectively. In this case, $r = \frac{3}{2}$. (Here $AB = BC = CA = \frac{3}{2}$). We claim that $r < \frac{3}{2}$ is impossible. Suppose otherwise.





Let A be an arbitrary point on the circle inscribed inside the regular hexagon (figure on the right). Consider the equilateral triangle ABC. Refer to the above diagram on the right. We have $AB = BC = CA = \frac{3}{2}$. Therefore A, B and C must be coloured differently. Since $r < \frac{3}{2}$, there is a point A_b on the inscribed circle along the minor arc AB such that $A_bB = r$. Similarly, there is a point A_c on the inscribed circle along the minor arc AC such that $A_cC = r$. Then any point A' on the open arc A_bA_c containing A has $A'B$, $A'C > r$. Thus A' is coloured the same as A. We continue this argument with A'. It follows that the entire circle must be coloured the same as A, which is clearly absurd. In conclusion, $r_{min} = \frac{3}{2}$.

4. Let n be a positive integer. Determine all polynomials P with real coefficients such that

$$(x^n + 1)P(x) = P(x^2)$$

for all real numbers x.

Solution Note that the zero polynomial and $x^n - 1$ works, since $(x^n+1)(x^n-1) = x^{2n} - 1$. Note also that the only polynomial of degree zero that works is the zero polynomial. Suppose $P(x)$ satisfies the condition given and the degree N of $P(x)$ is positive. Then

$$\deg((x^n + 1)P(x)) = n + N, \quad \deg(P(x^2)) = 2N,$$

and thus $N = n$. Hence there exists $a \in \mathbb{R}$, $a \neq 0$ and U a polynomial with real coefficients such that

$$P(x) = ax^n + U(x), \quad \deg(U) < n.$$

Let $V = U + a$, such that we have

$$P(x) = a(x^n - 1) + V(x), \quad \deg(V) < n.$$

Then we have

$$(x^n + 1)P(x) = P(x^2) \Leftrightarrow (x^n + 1)(a(x^n - 1) + V(x)) = a(x^{2n} - 1) + V(x^2).$$

This implies

$$(x^n + 1)V(x) = V(x^2).$$

Hence V is also a solution. But this implies $V = 0$ since $\deg(V) < n$. Thus we have

$$P(x) = a(x^n - 1).$$

It is easy to check that all polynomials of this form satisfy the polynomial equation.

5. Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

Solution The answer is $\lfloor \frac{2n-1}{5} \rfloor$.

Consider x such pairs in $\{1, 2, \ldots, n\}$. The sum S of the $2x$ numbers satisfies

$$1 + 2 \cdots + 2x \leq S \leq n + (n-1) + \cdots + (n - x + 1).$$

Thus $x \leq \frac{2n-1}{5}$.

We show a construction with exactly $\lfloor \frac{2n-1}{5} \rfloor$ pairs. For $n = 5k+3, 5k+4, 5k+5$, the following gives $2k+1$ pairs, which is the required number.

Pairs	1 $4k+1$	2 $4k-1$	$\cdots$	k $2k+3$	$k+1$ $4k+2$	$k+2$ $4k$	$\cdots$	$2k+1$ $2k+2$
Sums	$4k+2$	$4k+1$	$\cdots$	$3k+3$	$5k+3$	$5k+2$	$\cdots$	$4k+3$

For $n = 5k+2$, the table is the same but with the pair $\{k+1, 4k+2\}$ removed. For $n = 5k+1$, remove the last pair and subtract 2 from each number in the second row.

6. A set A of integers is said to be *admissible* if for any $x, y \in A$ (not necessarily distinct), $x^2 + kxy + y^2 \in A$ for every integer k.

Determine all pairs m, n of nonzero integers such that the only admissible set containing both m, n is the set of all integers.

Solution First observe that if $\gcd(m,n) = d > 1$, then the set $d\mathbb{Z} \neq \mathbb{Z}$ consisting of all multiples of d is admissible and contains both m and n.

Now suppose that $\gcd(m,n) = 1$. Let A be an admissible set containing m, n. We have the following two observations.

(i) For all $x \in A$, by letting $y = x$ in the definition of admissible sets, we have $kx^2 \in A$, for all $k \in \mathbb{Z}$.

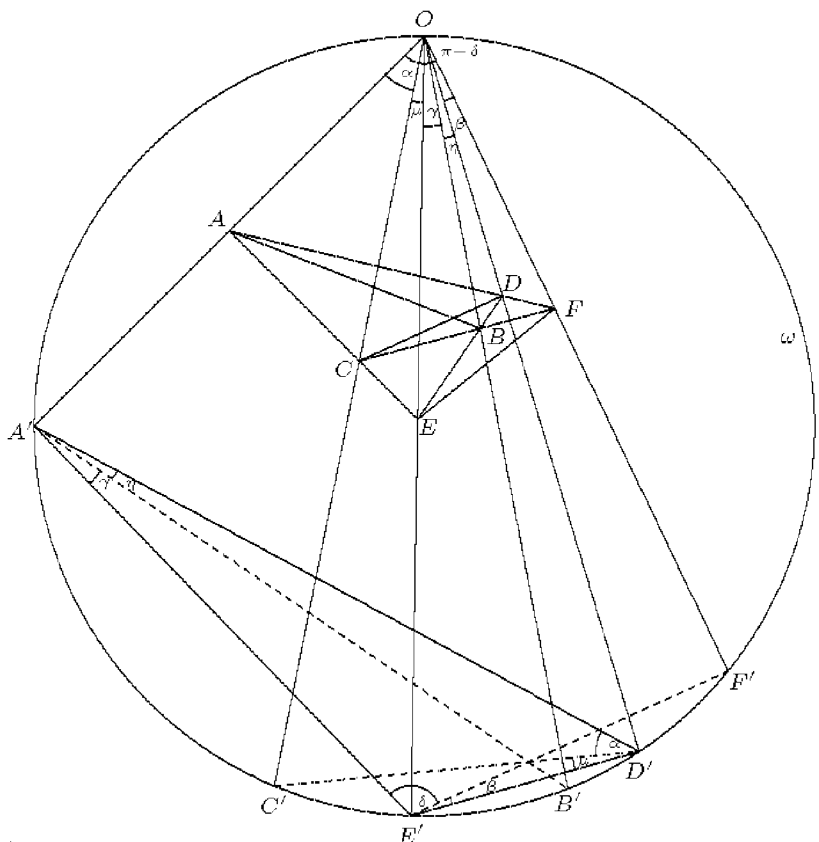
(ii) For all $x, y \in A$, $(x+y)^2 = x^2 + 2xy + y^2 \in A$.

Since $\gcd(m,n) = 1$, $\gcd(m^2, n^2) = 1$. Hence, there are integers a, b such that $am^2 + bn^2 = 1$. From (i), we have $am^2, bn^2 \in A$ and from (ii) we have $1 = (am^2 + bn^2) \in A$. Using (i) again, we have $k \times 1^2 = k \in A$ for all $k \in \mathbb{Z}$. Thus $A = \mathbb{Z}$.

7. Let $ACBD$ be a quadrilateral with AC intersecting DB at E and CB intersecting AD at F. Let O be a point on a circle ω. The rays OA, OC, OB, OD, OE and OF intersect ω at A', C', B', D', E' and F' respectively. Prove that $A'B'$, $C'D'$ and $E'F'$ are concurrent.

Solution Mark the angles $\alpha, \beta, \gamma, \mu, \eta$ and δ as shown in the figure. By the converse of Ceva's theorem, it suffices to show that in the triangle $A'E'D'$, we have

$$\frac{\sin\alpha}{\sin\mu} \cdot \frac{\sin\beta}{\sin\delta} \cdot \frac{\sin\gamma}{\sin\eta} = 1.$$



Using sine rule on the six triangles, OAC, OCE, OEB, OBD, ODF and OFA, we have

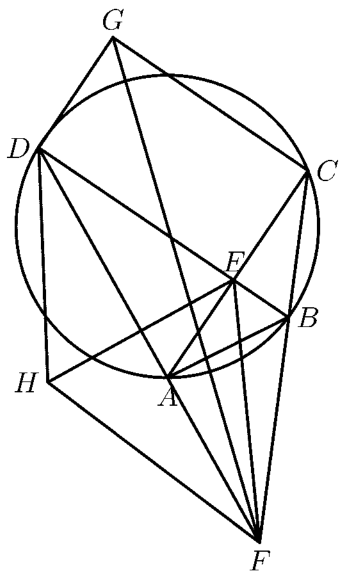
$$\begin{aligned}
\frac{\sin\alpha}{AC} &= \frac{\sin\angle ACO}{OA}\\
\frac{\sin\gamma}{EB} &= \frac{\sin\angle OBE}{OE}\\
\frac{\sin\beta}{DF} &= \frac{\sin\angle OFD}{OD}\\
\frac{\sin\mu}{CE} &= \frac{\sin(180^\circ-\angle ACO)}{OE}\\
\frac{\sin\eta}{BD} &= \frac{\sin(180^\circ-\angle OCE)}{OD}\\
\frac{\sin(180^\circ-\delta)}{AF} &= \frac{\sin(\angle OFD)}{OA}
\end{aligned}$$

Suitably multiplying these, we obtain

$$\frac{\sin\alpha}{\sin\mu}\cdot\frac{\sin\beta}{\sin\delta}\cdot\frac{\sin\gamma}{\sin\eta}=\frac{AC}{CE}\cdot\frac{EB}{BD}\cdot\frac{DF}{FA}.$$

By Menelaus' theorem applied to the triangle AED with transversal CCF, this product is equal to 1. Therefore, $A'B'$, $C'D'$ and $E'F'$ are concurrent.

8. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.



Solution We first show that $\triangle FDG \simeq \triangle FBE$. Since $ABCD$ is cyclic, $\triangle EAB \simeq \triangle EDC$ and $\triangle FAB \simeq \triangle FCD$. The parallelogram yields $GD = EC$ and

$\angle CDG = \angle DCE$. Also $\angle DCE = \angle DCA = \angle DBA$. Therefore

$$\angle FDG = \angle FDC + \angle CDG = \angle FBA + \angle ABD = \angle FBE,$$
$$\frac{GD}{EB} = \frac{CE}{EB} = \frac{CD}{AB} = \frac{FD}{FB}.$$

Since H is the reflection of E with respect to FD, we conclude that

$$\angle FHD = \angle FED = 180^\circ - \angle FEB = 180^\circ - \angle FGD.$$

This proves that D, H, F, G are concyclic.

9. Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

Solution The answer is all the prime numbers.

First we check that all prime numbers satisfy the condition. We'll show that if p is a prime, then $n \mid \binom{n}{p-2n}$ whenever $1 \leq n \leq \frac{p}{2}$. This is certainly true when $p = 2$. Now assume that p is an odd prime and $1 \leq n \leq p/2$. We have

$$(p-2n)\binom{n}{p-2n} = n\binom{n-1}{p-2n-1}.$$

Since $p \geq 2n$ and p is odd, all factors are nonzero. If $d = \gcd(p-2n, n)$, then $d \mid p$, but $d \leq n < p$ and hence $d = 1$. It follows that $\gcd(p-2n, n) = 1$. Thus $n \mid \binom{n}{p-2n}$.

Next we show that any composite number m does not satisfy the property.

Case (i): $m = 2k$ for some $k > 1$. Take $n = k$. Then $\frac{m}{3} \leq n \leq \frac{m}{2}$ but $\binom{m}{m-2n} = \binom{k}{0} = 1$ is not divisible by n.

Case (ii): m is odd. Then there exists an odd prime p and an integer $k \geq 1$ such that $m = p(2k+1)$. Pick $n = pk$. Then $\frac{m}{3} \leq n \leq \frac{m}{2}$. However,

$$\frac{1}{n}\binom{n}{m-2n} = \frac{1}{pk}\binom{pk}{p} = \frac{(pk-1)(pk-2)\cdots(pk-(p-1))}{p!}$$

is not an integer because p divides the denominator but not the numerator.

10. Players A and B play a game with N coins and 2013 boxes arranged around a circle. Player A starts the game by distributing the coins so that there is at least 1 coin in each box. They then take turns to make moves in the order $B, A, B, A \cdots$ as follows:

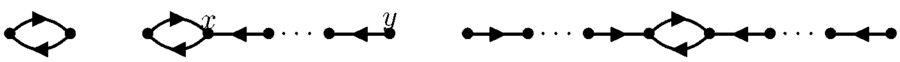
- B takes 1 coin from every box and puts it into an adjacent box.
- A takes several coins that were **not** involved in the previous move by B, with no two from the same box, and puts each of these coins into an adjacent box.

Find the minimum value of N so that A can ensure that after each of her moves all the boxes are nonempty.

Solution The answer is $N = 4024$. First let $N = 4024$. A *regular distribution* is one in which two of the boxes contains 1 coin each and the remaining boxes contains 2 coins each. A starts the game with a regular distribution. We call a box *red* if it contains more than 1 coin and white if it contains only 1 coin. After B has made his move, A cannot move any coin in a white box (since any coin in it has been moved by B) and can move one of the coins in a red box (since B can move only one of the two coins in it). For each red box, if B moves one of the coins to one of the neighbours, A will move one coin to the other neighbour. This will ensure that every red box has at most 2 coins and each of its neighbours is nonempty. Since there are only two white boxes, each has a red neighbour. Thus they are both nonempty and contain at most 2 coins each. A red box can become empty if it is adjacent to two white boxes and B has not put any coin into it. When this happens, A can ensure that this red box is nonempty by not moving the coin in it. Thus A can move the coins so that the distribution is regular.

An alternative solution:

The initial distribution is again regular. We consider B's move as a directed graph with the boxes as vertices. There is a directed edge from a to b if B moves a coin from a to b. Thus every vertex has an out edge to either its left or right neighbour. The graph formed can be a directed cycle. In this case A can respond by not moving any coin or if a move must be made, she can move a coin from a red box to a neighbouring white box. Otherwise, each connected component is a directed 2-cycle with 0, 1 or 2 tails as shown below.



The number of coins increases by 1 at x and decreases by 1 at y. Thus x will hold 3 coins if it is red and y will be empty if it is white. Since there are only two white boxes, they are either (i) on different tails or (ii) on the same tail. In case (i), for each vertex x that is red, A moves a coin from each box starting with x until it reaches y or until it reaches a white box whichever comes first. The resulting distribution will be regular. In case (ii), proceed as in (i). Then no box contains 3 coins. Only one box can be empty, the white box that is in position y. Then A can move a coin from a neighbouring red box from a neighbouring component, thereby obtaining a regular distribution.

Now consider the case where $N \leq 4023$. We'll show that B wins. We'll provide 2 solutions.

First solution:

Label the boxes 1, 2, ..., 2013 clockwise. Each coin x in box i is assigned a value $s(x)$ which is the shortest distance from box i to box 1 along the circumference. Let $S = \sum s(x)$. B will move coins counterclockwise from boxes 1 to 1000 and clockwise from boxes 1001 to 2013. This will decrease the value of S by 2011. Since there are at least 3 white boxes, A can move at most 2010 coins and thus can increase the value of S by at most 2010. Thus the value of S will decrease after a pair of moves of B followed by A. Thus A will not be able to keep all the boxes nonempty.

Second solution:

An interval I of ℓ consecutive boxes is said to be *sparse* if $2 \le \ell \le 2012$ and $C(I) \le 2\ell - 3$, where $C(I)$ is the total number of coins in the ℓ boxes.

Note that a sparse interval exists after the initial distribution by A since there is always a box containing ≥ 2 coins and removing that box leaves a sparse interval with 2012 boxes. We want to prove that for any sparse interval I of length ≥ 3, B can make his move so that no matter what A does, after her move there is a shorter sparse interval I'. Since a sparse interval with 2 boxes contains a total of 1 coin, one of the boxes must be empty and B wins.

Let boxes $1, 2, 3, \ldots, \ell$, $\ell \ge 3$, in a clockwise order, form a sparse interval I. (We always label the boxes in a sparse interval this way.) If box 1 contains more than 1 coin, it's clear that boxes $2, 3, \ldots, \ell$ also form a sparse interval. Thus we can assume that boxes 1 and ℓ contain 1 coin each. If $\ell = 3$, then each box in the sparse interval contains one coin each. If B moves clockwise from box 2 and 3 and counterclockwise from box 1, then box 2 will be empty and will remain empty after A's move. Thus B wins.

Thus we assume that $\ell \ge 4$. Now B moves coins from boxes ℓ and $\ell - 1$ clockwise and moves coins from boxes 1 and 2 counterclockwise. (The other moves are arbitrary). Then the interval from box 2 to box $\ell - 1$ loses a total of 2 coins and A will not be able to move any coins into this interval as boxes 1 and ℓ are white. Thus this shorter interval is sparse and we are done.

The selection and training of the Singapore team to the International Mathematical Olympiad is the responsibility of Singapore International Mathematical Olympiad Committee (SIMO). The national team is selected through the Singapore Mathematical Olympiad (Open section). These students undergo rigorous training from October to April. The final six members of the national team are selected based on the results of the National Team Selection Tests. The 2013 version of the tests is published in this issue.