Some Observations on Oblong Numbers

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An *oblong* number $O_n = n(n+1), n = 1,2,3,\ldots$ is a product of consecutive positive integers, necessarily even numbers. They begin 2, 6, 12, 20, 30,... . They are also known as *pronic* numbers. The Pythagorean, Theon of Smyrna, who flourished circa 100CE, knew that an oblong number is the sum of two equal triangular numbers, namely, $n(n+1) = \frac{n(n+1)}{2} + \frac{n(n+1)}{2}$. For me, Theon's observation, in so far as it involves triangularity, raised the question of how oblong numbers might be related to primitive Pythagorean triples. In fact, we can find all the oblong numbers that are the "even leg" of a primitive Pythagorean triple.

Some examples will guide us. The first few such numbers, along with (a,b,c), the corresponding primitive Pythagorean triple (PPT) are

O	(a,b,c)
$12 = 3 \cdot 4$	$(5,12,13)$
$20 = 4 \cdot 5$	$(21,20,29)$
$56 = 7 \cdot 8$	$(33,56,65)$
$72 = 8 \cdot 9$	$(65,72,97)$

These and other examples suggested that we consider the odd factor of O and classify it according to its having the form $4k+1$ or $4k-1$. We also use the well-known fact that a PPT is generated by two unequal numbers m and n that are relatively prime and of opposite parity. In this case the even leg of the PPT is given by $2mn$.

If O has odd factor of the form $4k+1$ then the other consecutive factor is either $4k$ or $4k+2$. The case $4k+1$ and $4k$ produces $O = (4k+1)(4k)$ which is the even leg of a PPT with generators $m = 4k+1$ and $n = 2k$. The case $4k+1$ and $4k+2$, yields $O = (4k+1)(4k+2)$ which, if it were the even leg of a PPT, would satisfy $(4k+1)(4k+2) = 2mn$ and so $(4k+1)(2k+1) = mn$. This is a contradiction because the left side is odd while the right side is even.

If O has odd factor of the form $4k-1$ then the other consecutive factor is either $4k$ or $4k-2$. The case $4k-1$ and $4k$ produces $O = (4k-1)(4k)$ which is the even leg of a PPT with generators $m = 4k-1$ and $n = 2k$. The case $4k-1$ and $4k-2$, yields $O = (4k-1)(4k-2)$ which, if it were the even leg of a PPT, would satisfy $(4k-1)(4k-2) = 2mn$ and so $(4k-1)(2k-1) = mn$. This is a contradiction as before. This proves that the oblong numbers that are the even legs of a PPT are those fitting the descriptions $O = (4k+1)(4k), k \geq 1$ and $O = (4k-1)(4k), k \geq 1$. The former produces the even legs 20, 72, 156, ... while the latter gives 12, 56, 132, ...

Next we considered some representation questions. We asked, are there infinitely many oblong numbers that are the sum of two oblong numbers or three oblong numbers?

We observed, among other such data, these instances:

$$
\begin{aligned}
O_6 &= 42 = O_3 + O_5 = 12 + 30 \\
O_{15} &= 240 = O_5 + O_{14} = 30 + 210 \\
O_{28} &= 812 = O_7 + O_{27} = 56 + 756 \\
O_{45} &= 2070 = O_9 + O_{44} = 90 + 1980 \\
O_{66} &= 4422 = O_{11} + O_{65} = 132 + 4290 \\
O_{91} &= 8372 = O_{13} + O_{90} = 182 + 8190
\end{aligned}
$$

and this suggested that $O_{2a^2-a} = O_{2a-1} + O_{2a^2-a-1}, \forall a \geq 2$.

The right hand side is

$$
\begin{aligned}
&(2a-1)(2a) + (2a^2-a-1)(2a^2-a) = (2a-1)(2a) + (2a^2-a-1)(a)(2a-1) \\
&= (2a-1)(2a+2a^3-a^2-a) = (2a-1)(2a^3-a^2+a) = (2a^2-a)(2a^2-a+1) = O_{2a^2-a}
\end{aligned}
$$

as desired.

We note that the subscripts above, 6, 15, 28, 45, ... are every other triangular number $T_n = \dfrac{n(n+1)}{2}, n = 3,5,7,9,...,2a-1,...$ and so we may write our discovery as $O_{T_n} = O_n + O_{T_n - 1}$.

As to oblong numbers that are the sum of three oblong numbers, we observed these examples:

$$
\begin{aligned}
O_1 + O_2 + O_3 &= O_4 \\
O_1 + O_3 + O_6 &= O_7 \\
O_1 + O_4 + O_{10} &= O_{11} \\
O_1 + O_5 + O_{15} &= O_{16} \\
O_1 + O_6 + O_{21} &= O_{22} \\
O_1 + O_7 + O_{28} &= O_{29}
\end{aligned}
$$

This suggested that $O_1 + O_n + O_{T_n} = O_{T_n+1}, \forall n \geq 2$. The left hand side is

$$
\begin{aligned}
&2+n(n+1)+\frac{n(n+1)}{2}\left(\frac{n(n+1)}{2}+1\right)\\
&=2+\frac{2n(n+1)}{2}+\frac{n(n+1)}{2}\left(\frac{n(n+1)}{2}+1\right)\\
&=2+\frac{n(n+1)}{2}\left(2+\frac{n(n+1)}{2}+1\right)\\
&=\left(\frac{n(n+1)}{2}\right)^2+3\left(\frac{n(n+1)}{2}\right)+2\\
&=\left(\frac{n(n+1)}{2}+1\right)\left(\frac{n(n+1)}{2}+2\right)\\
&=O_{T_n+1}
\end{aligned}
$$

as desired.

Finally we thought to ask whether there are infinitely many squares that are the sum of two or three oblong numbers. Some instances of the former are

$$
\begin{aligned}
O_5+O_2&=36=6^2\\
O_{11}+O_3&=144=12^2\\
O_{19}+O_4&=400=20^2\\
O_{29}+O_5&=900=30^2\\
O_{41}+O_6&=1764=42^2
\end{aligned}
$$

We will prove that this list extends indefinitely by mathematical induction. Assume that $O_n + O_x = (n+1)^2$ for some n and x. Thus $n^2+n+x^2+x=(n+1)^2$. We will deduce that $O_{n+2(x+1)} + O_{x+1} = (n+2x+3)^2$. The left hand side of the last equation equals

$$
\begin{aligned}
&(n+2x+2)(n+2x+3)+(x+1)(x+2)\\
&=n^2+4nx+5x^2+5n+13x+8\\
&=n^2+n+x^2+x+4x^2+4nx+4n+12x+8\\
&=(n+1)^2+4x^2+4nx+4n+12x+8\\
&=4x^2+4nx+12x+n^2+6n+9\\
&=(n+2x+3)^2
\end{aligned}
$$

as we wanted.

As to squares the sum of three oblong numbers, we observed that

$$O_1 + O_1 + O_3 = 4^2$$
$$O_2 + O_1 + O_7 = 8^2$$
$$O_3 + O_1 + O_{13} = 14^2$$
$$O_4 + O_1 + O_{21} = 22^2$$
$$O_5 + O_1 + O_{31} = 32^2$$

This suggested that $O_n + O_1 + O_{3n+(n-1)^2}$ is always a perfect square. This expression is

$$\begin{aligned} &n(n+1) + 2 + (n^2 + n + 1)(n^2 + n + 2) \\ &= n^4 + 2n^3 + 5n^2 + 4n + 4 \\ &= (n^2 + n + 2)^2 \end{aligned}$$

as we expected.

- Dedicated to my longtime friend and colleague Robert F. Sutherland

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