A Journey in Geometry:
from a Paradox to Theorems

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Geometry originated as the knowledge of length, angles and area. Indeed, in ancient Greek, the word *geometry* is constructed via earth (*geo*) and measurement (*metron*). It is natural and almost inevitable that one draws diagrams when studying geometry. A clear diagram not only describes the problem concisely, but may also give clues and insights. For example, a beginner learns in the geometry class that angles opposite to the equal sides of an isosceles triangle are equal. The conclusion (and its inverse) is clear from a diagram (figure 1).

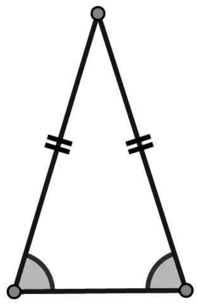

Figure 1

One may even find in *Elements* that Euclid, the great Grecian mathematician, developed a number of arguments referring to diagrams. However, this empirical (and perhaps intuitive) approach must be applied very carefully. Otherwise, mathematical rigour may be compromised. In the beginning of the twentieth century, the following paradox was introduced to show that *intuition* might be absurd in the study of geometry: one may prove that any triangle is isosceles!

"Proof "

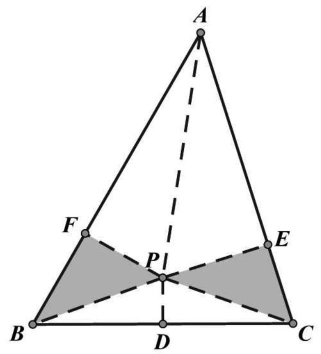

Figure 2

Refer to Figure 2 above. In ΔABC, D is the midpoint of BC. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at P. Draw $PE \perp AC$ at E and $PF \perp AB$ at F.

It is easy to see that $\Delta APE \cong \Delta APF$ (AAS).

Hence, we have $PE = PF$ and $AE = AF$. (1)

As $\Delta BPD \cong \Delta CPD$ (SAS), we have $PB = PC$.

Hence, $\Delta PBF \cong \Delta PCE$ (RHS), which implies $BF = CE$. (2)

It follows from (1) and (2) that $AB = AF + BF = AE + CE = AC$.

One may write down a similar argument if P is outside ΔABC. Refer to Figure 3 below.

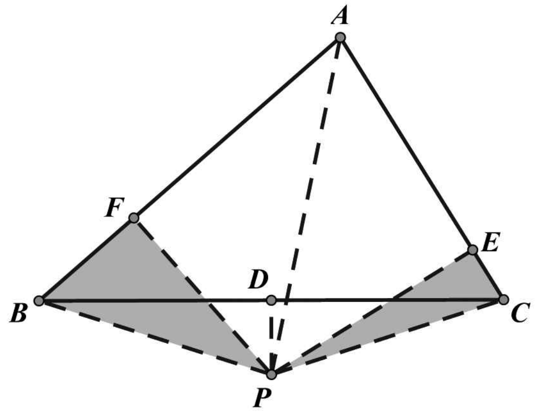

Figure 3

Challenge

Of course, the conclusion is absurd, if not catastrophic. However, it seems the argument follows step by step. Can you detect where the mistake is? (Think about it. Do not refer to the answer below immediately.)

Indeed, we shall draw a more accurate diagram (Figure 4) as follows:

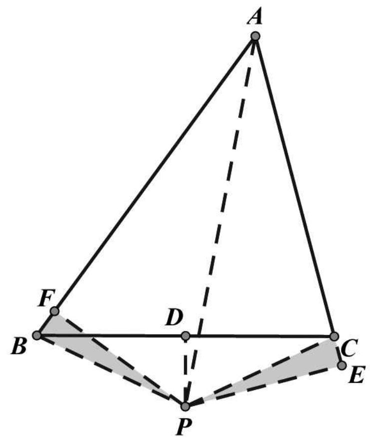

Figure 4

The previous argument still holds except for the last line, i.e., we still have $\Delta PBF \cong \Delta PCE$, $AE = AF$ and $BE = CF$. However, we do NOT have $AB = AC$ because $AB = AF + BF$ but $AC = AE - CE$.

The crucial part is that F lies on the line segment AB, but E lies on the extension of the line segment AC!

To think one step further

Our illustration above only gives one counter-example of the "theorem" that any triangle is isosceles. Notice that our previous proof still holds if both feet of perpendicular lie on the sides or both on the extensions of the sides. To what extent is this possible? Are there any other unexpected "isosceles" triangles?

Let us draw the *circumcircle* of ΔABC. Refer to Figure 5 on the right.

Let O denote the center of the circumcircle. By definition, $OB = OC$. Since D is the midpoint of BC, we have $\Delta OBD \cong \Delta OCD$.

Hence, $\angle BOD \cong \angle COD$.

Let OD extended intersects the circumcircle at P'. Connect AP'.

Recall that the angle at the center of the circle is twice as the angle at the circumference.

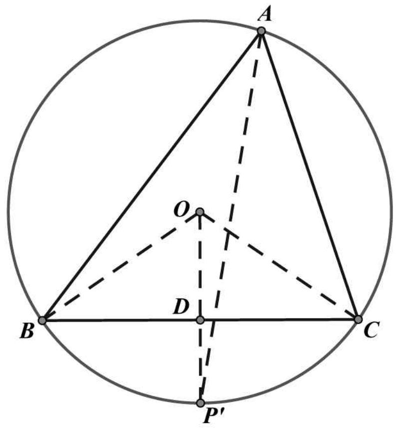

Figure 5

It follows that $\angle BAP' = \frac{1}{2}\angle BOP' = \frac{1}{2}\angle COP' = \angle CAP'$, i.e., AP' is the angle bisector of $\angle A$. Indeed, P and P' coincide.

We have shown that P must lie on the circumcircle of ΔABC. Refer to Figure 6 below.

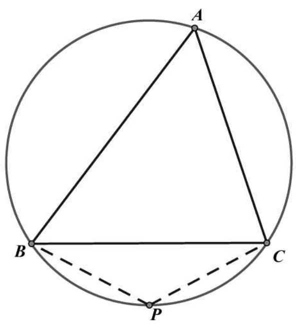

Figure 6

Recall that $\angle ABP + \angle ACP = 180°$. Hence,

(i) Either $\angle ABP = \angle ACP = 90°$, in which case $AB = AC$ and AP is the diameter of the circumcircle; or

(ii) Exactly one of $\angle ABP, \angle ACP$ is acute and the other is obtuse. In this case, exactly one of the feet of perpendicular E, F is on the side of the triangle and the other is on the extension. Refer to Figure 7.

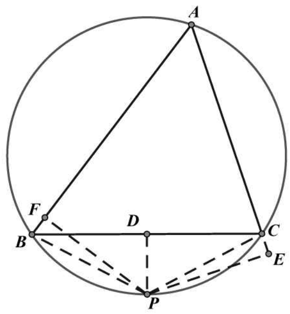

Figure 7

In conclusion, the diagram in the original proof can **never** be valid, i.e., if $AB \neq BC$, then exactly one of E, F lies on the sides of ΔABC. This solves the paradox completely.

Observation

One may observe that D, E, F are collinear (i.e., lying on the same line) in the previous diagram. Indeed, this is always true. To prove this statement, we shall recall the elementary circle geometry.

(i) If A, B, C, D are *concyclic* (i.e., lying on the same circle), then $\angle A = \angle D$. Refer to Figure 8 below.

(ii) The converse is also true, i.e., if $\angle A = \angle D$, then A, B, C, D are concyclic.

(iii) If A, B, C, D are *concyclic* (i.e., lying on the same circle), then $\angle A + \angle C = 180°$. Refer to Figure 9 below.

(iv) The converse is also true, i.e., if $\angle A + \angle C = 180°$, then A, B, C, D are concyclic.

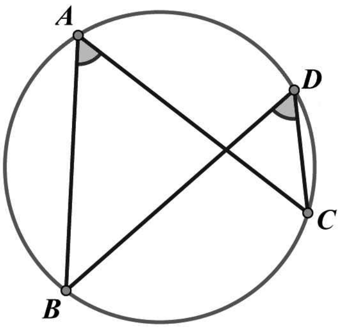

Figure 8

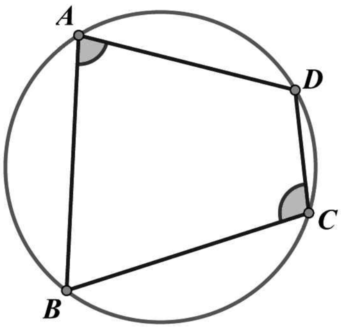

Figure 9

We will apply the properties above and show that D, E, F are always collinear.

Proof:

Refer to Figure 10 below.

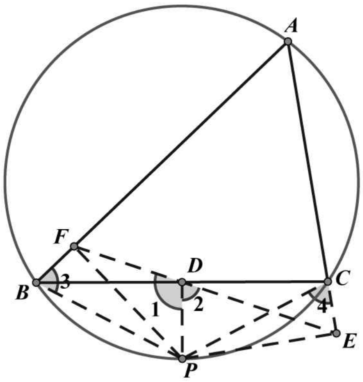

Figure 10

We are to show that $\angle 1 + \angle 2 = 180°$.

It is given that $\angle BFP = \angle BDP = 90°$. Hence, B, F, D, P are concyclic by (ii).

We must have $\angle 1 + \angle 3 = 180°$ by (iii).

Similarly, $\angle CDP = \angle CEP = 90°$ implies C, D, P, E are concyclic by (iv).

We must have $\angle 2 = \angle 4$.

Since A, B, P, C are concyclic, $\angle 3 = \angle 4$ by (iii).

Since $\angle 1 + \angle 3 = 180°$ and $\angle 2 = \angle 3 = \angle 4$, we conclude that $\angle 1 + \angle 2 = 180°$.

Notice that in the proof above, we never use the fact that PD is the perpendicular bisector of BC . Indeed, the conclusion holds for any point on the circumcircle of the triangle. This is an important result in elementary geometry called Simson's Theorem.

Simson's Theorem

Let P be a point on the circumcircle of ΔABC, then the feet of perpendicular from P to AB, BC, CA are collinear.

In particular, this line is called the Simson's Line of ΔABC, as illustrated in Figure 11 below.

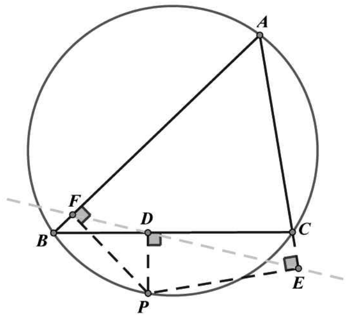

Figure 11

Exercise

Show that the inverse of Simson's Theorem holds, i.e., if the feet of perpendicular from a point P to the sides of ΔABC are collinear, then P lies on the circumcircle of ΔABC.

Hint: You may use most of the facts in the proof of Simson's Theorem. The only difference is that you are given $\angle 1 + \angle 2 = 180°$ and you are to show that $\angle 3 = \angle 4$.

The following example shows a simple application of Simson's Theorem.

Example

A quadrilateral $ABCD$ is inscribed inside a circle and $AD \perp CD$. Draw $BE \perp AC$ at E and $BF \perp AD$ at F. Show that line EF passes through the midpoint of line segment BD.

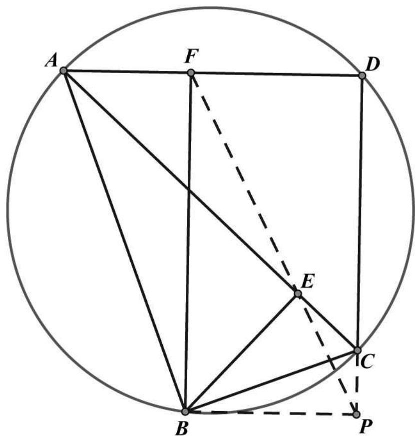

Figure 12

Proof:

Draw $BP \perp PD$ at P. Since $AD \perp PD$ and $BF \perp AD$, we have $AD // BP$ and $BF // PD$, i.e., $BPDF$ is a parallelogram.

Apply Simson's Theorem to ΔACD. We see that F, E, P are collinear.

The conclusion follows as the diagonals of a parallelogram bisect each other.

Note that Simson's Theorem also states that the **locus** of the point whose feet of perpendicular to the sides of a triangle (i.e., three straight lines) are collinear is exactly the circumcircle of the triangle. What if we have four straight lines?

Miquel Point

Say we have four lines $\ell_1, \ell_2, \ell_3, \ell_4$. Consider the locus of the point whose feet of perpendicular to three lines are collinear. The locus with respect to ℓ_1, ℓ_2, ℓ_3 is a circle, and that with respect to ℓ_2, ℓ_3, ℓ_4 is another circle. Hence, the intersection of the circles gives a unique point whose feet of perpendicular to the four lines are collinear. Figure 13 gives an illustration.

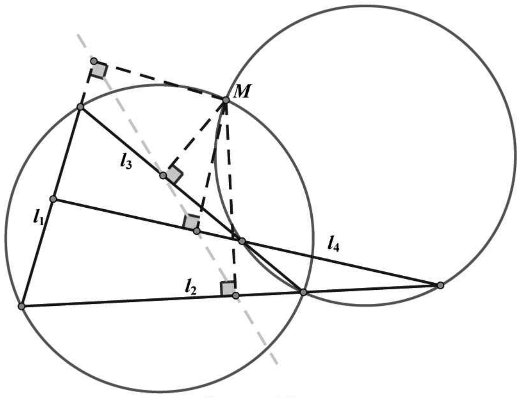

Figure 13

Call this point M in the diagram above.

Among the four lines $\ell_1, \ell_2, \ell_3, \ell_4$, any three would enclose a triangle, i.e., there are altogether four triangles. One may easily show, using the inverse of Simson's Theorem, that the circumcircles of all these triangles passing through M. The point M is called the Miquel Point.

In general, four straight lines would enclose a shape called a complete quadrilateral. Refer to Figure 14 below.

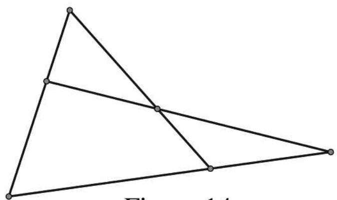

Figure 14

Each complete quadrilateral has a unique Miquel Point. Refer to Figure 15 below. This is the point whose feet of perpendicular to all four lines are collinear; the line is called the Simson's Line of the complete quadrilateral.

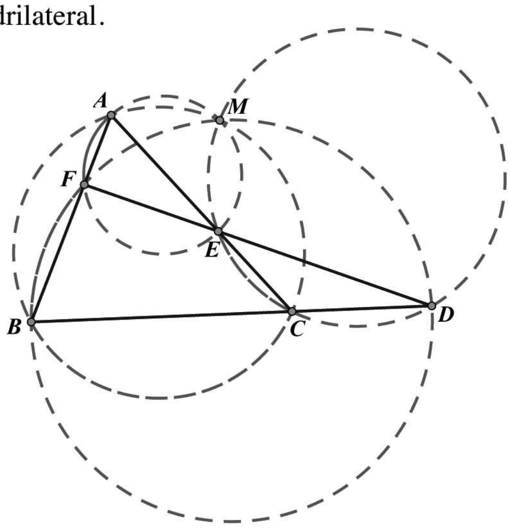

Figure 15

Miquel Point is an interesting property of complete quadrilaterals. Indeed, complete quadrilaterals give a very rich structure in Euclidean geometry and there are many other elegant results, making the entire picture full of mathematical beauty. We give two examples.

Refer to Figure 16 below.

(1) The midpoints of AD, BE, CF are collinear; the line is perpendicular to the Simson's Line.

(2) Let H_1, H_2, H_3, H_4 denote the orthocenter (where heights of the triangle meet) of $\Delta AEF, \Delta ABC, \Delta BDF, \Delta CDE$ respectively. We have H_1, H_2, H_3, H_4 collinear and the line is parallel to the Simson's Line.

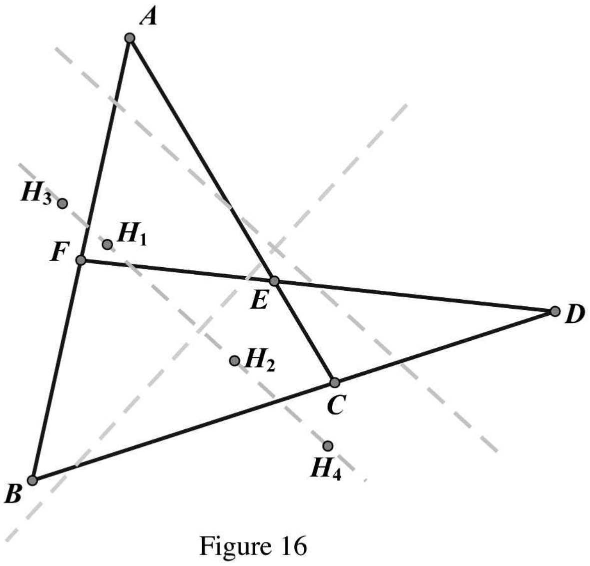

Figure 16

We have reached the end of this journey. In the previous few pages, we started from a paradox and studied it in details. Hence, by repeated trials of observation, proof and generalization, we see several important concepts and results. This is an expressway! For example, Miquel Point is not taught in most pre-university education, except for perhaps the very few top students nationwide preparing for the International Mathematical Olympiad. Nevertheless, we shall see that this concept is not difficult to perceive.

Hopefully the readers, especially those without much experience in mathematics, would find this journey enriching and exciting.

Enjoy Mathematics!

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