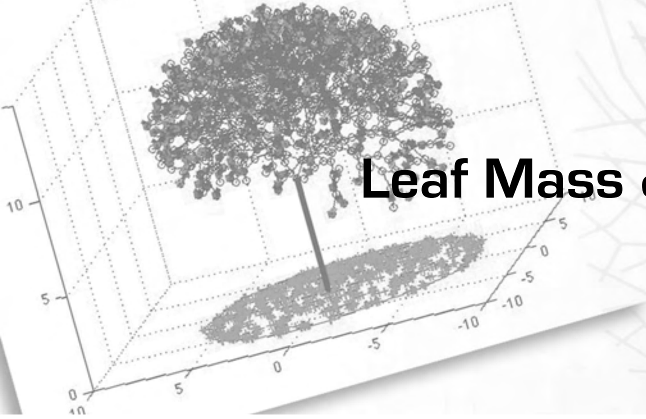

# Geometrical Tree: Leaf Mass & Leaf-Tree Relationship (Part 1)

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Winner in the Outstanding Winner category at the
International Mathematics Contest in Modeling 2012

## 1. Introduction

In this paper we present a mathematical model for investigating the relationship between tree profiles and leaf shapes and estimating the total leaf mass of a tree. Specifically, we created a tree profile using 3-dimensional coordinates and linear transformation. We simulate the sunlight irradiance in the tree crown using the brightness function of the sky and Monsi and Saeki equation [8]. We then incorporate the sunlight irradiance into different tree profiles to find a leaf shape that maximizes sunlight exposure. The chosen shapes match with real leaf shapes, suggesting a close relationship between leaf shape and tree profile: leaf shape maximizes the overall sunlight exposure under a given tree profile. We then incorporate the tree profile with the photosynthesis model [12], and use indexes like leaf mass per area (LMA) and leaf area index (LAI) to estimate the total leaf mass of a tree.

### 1.1. Outline of Our Approach.

We first introduce two models which will be useful in the latter applications:

- **Vector Tree Model**: The first part of our paper will be devoted to presenting the theoretical framework of this model. Our objective is to create a spatial structure of the tree crown (branching structure and leaf distribution). We use vectors to simulate leaf shape, distribution of leaves on branches, and the tree profile/branching structure. Linear transformation of vectors is used to represent the relationship between daughter branch and parent branch.
- **Sunlight Model**: The second part of the paper will introduce the sunlight model. Our objective is to simulate solar irradiance across a year. Brightness function over the celestial sphere is used to describe solar irradiance from different directions, Monsi and Saeki equation is used to calculate the light attenuation in a tree crown due to overlapping of leaf shadows.

The latter sections present two applications of the models:

- **Investigating the Relationship between Leaf Shape and Tree Profile**
  We use the vector tree model to construct different tree profiles. Combining the tree profile with the sunlight model, we are able to calculate the sunlight exposure rate of the leaves. We then adopt different leaf shapes

for the same tree profile and find the one that maximizes the sunlight exposure. Comparing this chosen leaf shape with the real leaf shape of the tree, if there is a match, we may conclude that the leaf shape is associated with the tree profile in that it maximizes the sunlight exposure.

- **Estimating Total Leaf Mass of a Tree**
  Given a tree, we use the vector tree model to simulate its profile, and use the sunlight model to determine the light irradiance at each leaf. Then, we derive the relationship between light irradiance and the leaf mass per area (LMA) according to a photosynthesis model. Thus, we can derive the LMA for each leaf and calculate the weighted average LMA for the entire tree. We then use the sunlight model again to find the shadow of the tree crown and calculate the total leaf area of the tree using leaf area index (LAI). Finally, the total leaf mass is calculated by multiplying LMA with the total leaf area.

### 1.2. General Assumptions.

- Trees are assumed to be paracladial, i.e. if any branch is cut off, it has the same structural characteristics, apart from size, as the parent from which it is cut.
- The majority of leaves will grow on the last generation of branches.
- Sun light is assumed to propagate in a straight line. Diffraction and refraction of lights are ignored.
- Photosynthesis rate is assumed to be affected only by the rate of light irradiance. Other factors such as $CO_2$ concentration in the air are assumed to be homogeneous at any part of the tree crown.
- Photosynthesis due to light sources other than the sun is neglected. (e.g. moonlight at night and artificial light)
- The environmental destructive effects are minimized and thus negligible, such as natural disasters and herbivory.
- Nutrient supplies are sufficient.

## 2. Vector Tree Model

### 2.1. Leaf classification.

The leaf is a major part of the plant-body plan. How to classify different leaves? Traditional plant taxonomy focuses on leaf functions and leaf shape. Recent research also suggests that venation is a strong indicator in leaf classification.

The general leaf shape is an important factor for the plant to receive sunlight. Traditional parameters in describing leaf shape include presence/absence of leaf petiole, flatness, leaf index (a ratio of leaf length to leaf width), margin type, and overall size [12].

Since we are going to construct a computerized simulation of a tree, it is important for us to classify leaves quantitatively such that they are easy for representation and simulation. Hence we develop a model to classify the leaves into 4 basic shape categories.

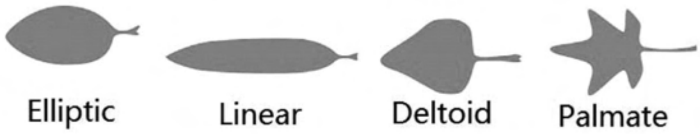



Figure 2.1

To classify a given leaf to one of the four basic shape categories, we focus on 3 factors of the leaf: shape convexity, leaf index (ratio of leaf length to leaf width), and the position of the longest width on a leaf.

To determine the shape convexity, we first approximate a leaf using a polygon. We first fit the leaf into a grid with each cell size 0.5cm×0.5cm. Next we select the outmost grids that have been covered more than half by the leaf. Then we plot and connect the centers of these grids. Hence we have a simulating polygon of the leaf. Two illustrations are shown as below.

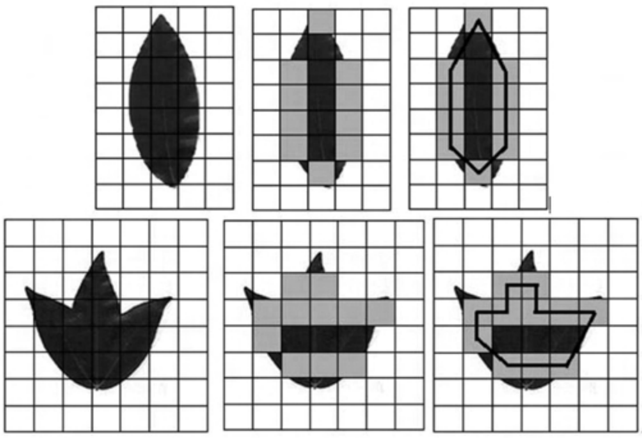


Figure 2.2

We are now able to determine the convexity of the polygon of leaf convex polygons are such that all diagonals lie entirely inside the polygon; concave polygons are such that some diagonals will lie outside the polygon. Hence we can determine the convexity of the leaf.

If a leaf is concave, we immediately classify it as Palmate. Convex leaves are left for further determination.

To classify other leaves, the second factor we look at is leaf index, which is the ratio of leaf length to leaf width. According to empirical data [12] [7], we classify leaves with leaf ratio of 3.5 and above as Linear. Those with leaf ratio below 3.5 will be either Elliptic or Deltoid.

To classify Elliptic leaf and Deltoid leaf, we look at the position of longest-width on a leaf. In order to have a more quantitative view, we are interested in the ratio of distance between leaf tip and longest-width-position (D) to the leaf length (L).

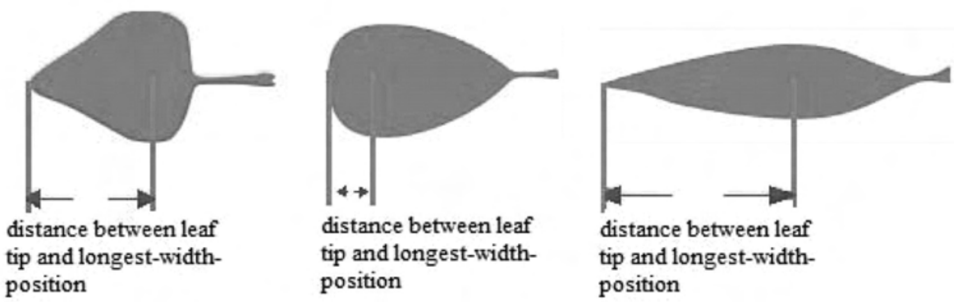



Figure 2.3

If the ratio (D/L) lies within [1/4, 3/4], we classify the leaf as Elliptic. Leaves whose ratio is less than 1/4 or greater than 3/4 are Deltoid.

To summarize the decision rule of classifying a leaf, we provide a decision flow chart.

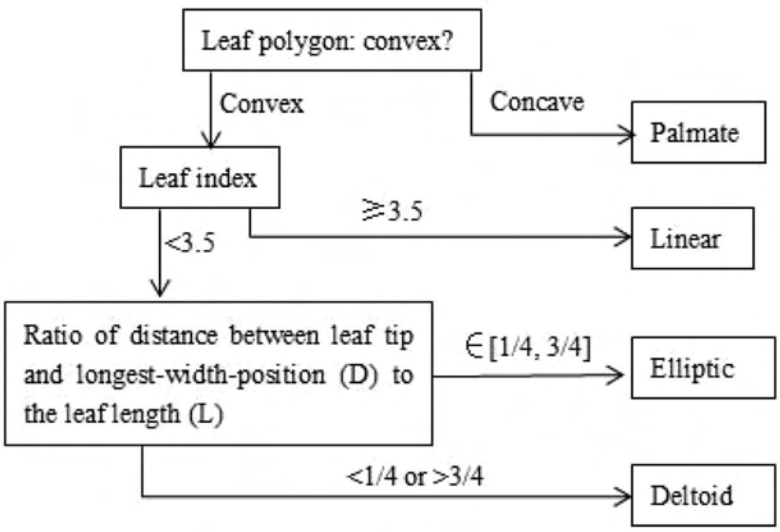



Figure 2.4

## 2.2. Branching Structure.

2.2.1. *Biological Backgrounds and Existing Models.* The existing geometrical models simulating branching structures of trees are essentially empirical based on a rule that specifies the relative angular direction and length of a daughter branch to its parent branch [7]. In order to obtain a more realistic model, studies considering whorls and bifurcations as rule in branching simulation have also been carried out [4] [5].

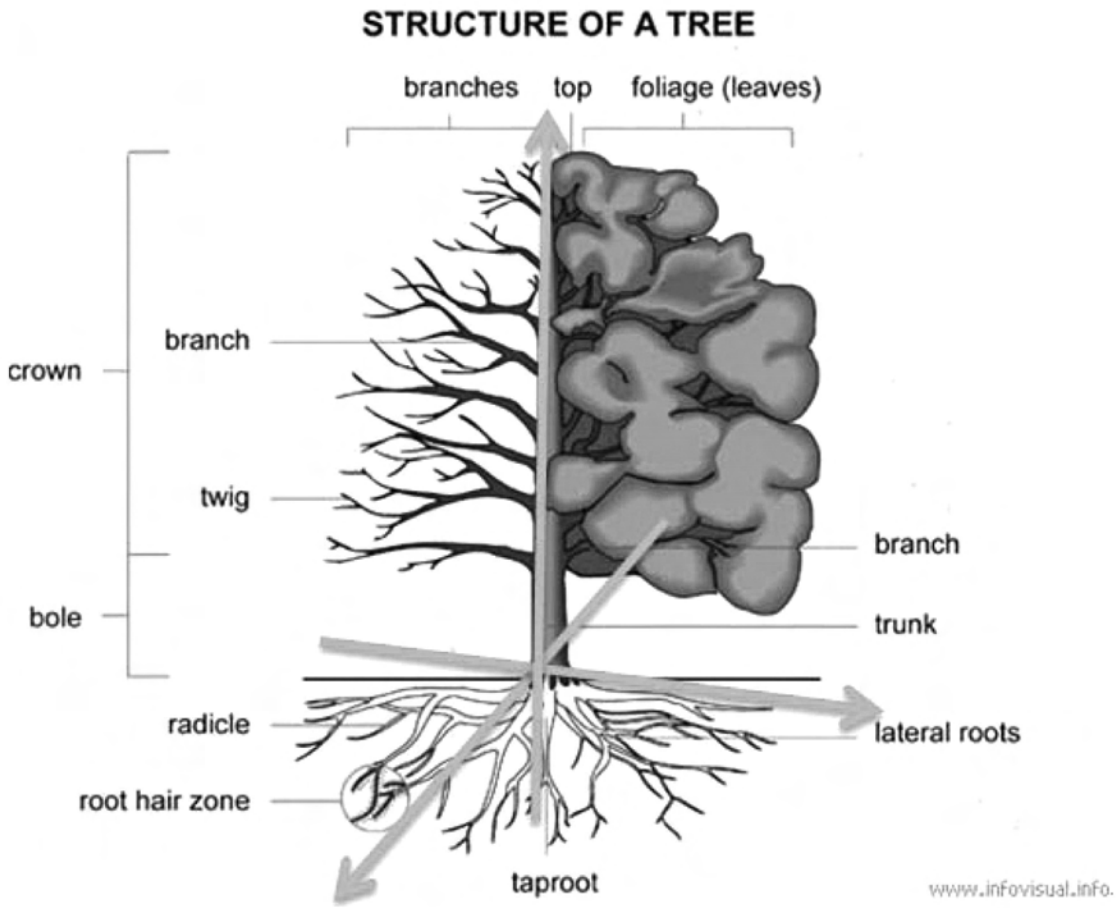



Figure 2.5

2.2.2. *Model Description.* In this model, our objective is to represent each branching point and the position of each leaf using their position vector (x,y,z) in a 3D coordinate shown above with z-axis set to be the main stem of the tree.

**Branch Formation**

Given the length and location of a parent branch, we want to find the direction and length of the child branch. At this point, the child branches generated will subject to the following:

- Number of child branches at the branching point $N$ is given by $N = k + \sigma$, where
  - $k$: A species-wise constant coefficient denoting the the number of child branches for the species.
  - $\sigma$: Uncertainty compensator which incorporates the genetic uncertainties and environmental uncertainties. $\sigma$ is a random variable with integer values.

- At each branching point, the length and direction of the $N$ child branches are determined by multiplying the direction vector of the parent branch with transformation matrices $A_1, A_2, \cdots, A_N$ respectively.

$A_i$ is defined as the following: $A_i(\theta_i, \phi_i) = \lambda_i * \begin{pmatrix} cos\phi_i & sin\phi_i & 0 \\ -cos\theta_i sin\phi_i & cos\theta_i cos\phi_i & sin\theta_i \\ sin\theta_i sin\phi_i & -sin\theta_i cos\phi_i & cos\theta_i \end{pmatrix}$

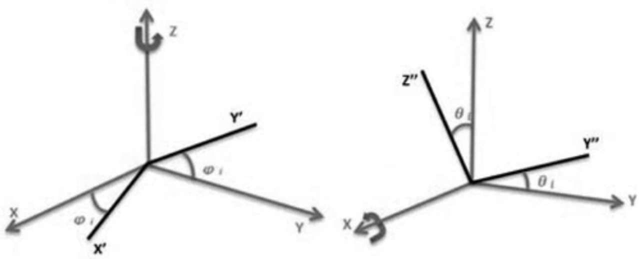



Figure 2.6

where $\lambda_i$ is the length change, $\phi_i$ is the rotated angle with respect to the parent branch about $O_z$ and $\theta_i$ is the rotated angle with respect to the parent branch about local $x$ axis $O_x$ with respect to the parent branch. Note that at each branching point, the coordinate system adopted for this transformation will be the local coordinate system with respect to the local parent branch, and the parent branch will lie on the z-axis.

- Now, we are able to determine the coordinate of each branch. We first number the branches according to the generating sequence. Let $r_N$ denotes the vector representing the branch of the $N^{th}$ generated branch. If $r_t$ is the parent branch of $r_k$, then the equation:

$$r_k = A_j r_t \tag{1}$$

is to be satisfied, where $A_j$ is the transformation matrix that governs the rule between $t^{th}$ branch and $k^{th}$ branch (abbreviated as branch t and branch k in later context). Let $R_N$ denote the coordinate of $N^{th}$ generated branch. Then $R_1 = r_1$ and $R_k = R_t + r_k$. In particular, in the situation where only bifurcations are concerned [7],

$$\begin{cases} r_{2k} = A_1 r_k \\ r_{2k+1} = A_2 r_k \end{cases} \tag{2}$$

and

$$\begin{cases} R_{2k} = r_{2k} + R_k \\ R_{2k+1} = r_{2k+1} + R_k \end{cases} \tag{3}$$

**Leaf Formation**

Now, we are ready to construct leaves on the branches. Define set $A = \{r_p \in Tree | \forall r_k \in Tree$, branch p is not the parent of branch k$\}$ and the growing of leaves is subject to the following:

- Basic phyllotaxis patterns: [6]
  - Distichous Phyllotaxis; in distichous phyllotaxis, leaves or other botanical elements grow one by one, each at 180 degrees from the previous one.
  - Whorled Phyllotaxis; In whorled phyllotaxis, two or more elements grow at the same node on the stem.;
  - Spiral Phyllotaxis; In spiral phyllotaxis, botanical elements grow one by one, each at a constant divergence angle d from the previous one.
  - Multijugate Phyllotaxis; In multijugate phyllotaxis, elements in a whorl (group of elements at a node) are spread evenly around the stem and each whorl is at a constant divergence angle d from the previous one.
- The spacing of leaves on the branch is determined by the function $d(I)$ which takes in the intensity of sunlight irradiance and outputs the spacing of the leaves.
- The probability that a leaf grows at a certain spot is determined by the function $p(I)$ which takes in the intensity of sunlight irradiance and outputs the probability that a leaf is likely to grow on a certain spot of the branch.

In our model, for each $r_p \in A$, we grow the leaves on branch p according to the archived Basic phyllotaxis patterns for different species. The spacing between two different potential leaf-growing spots is $d(I)$ and the probability of a leaf growing at a certain spot is $p(I)$.

2.2.3. *Different tree profiles generated by our model.* Using the model introduced above, we are able to generate a wide range of tree profiles by varying the coefficients in the model. Three examples are shown in Figure 2.7. If we zoom in to the trees, we will see the leaves represented as shown in Figure 2.8

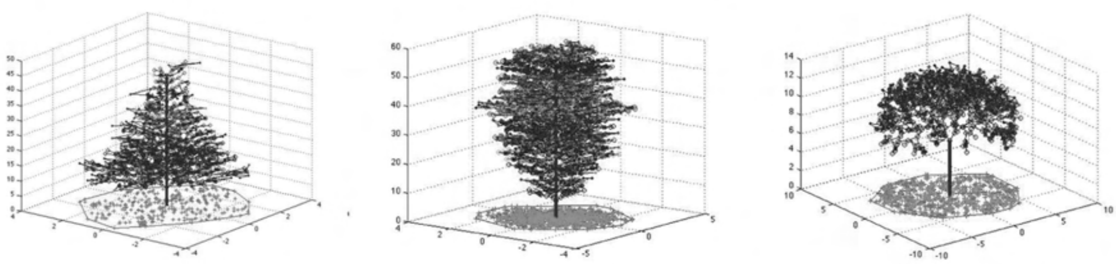


Figure 2.7

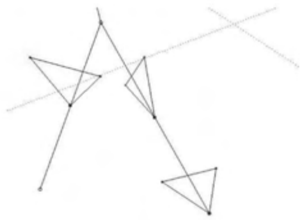


Figure 2.8

## 3. Sunlight Model

Photosynthesis is important for the growth of leaves, and it is driven by solar energy. Thus, in this section, we will consider the light irradiance in a tree crown. By modeling the sun radiation using brightness function, and using Monsi-Saeki equation [8] to describe the light attenuation within a tree crown, we will be able to calculate light irradiance at any point within the tree crown.

### 3.1. **Orbit of the sun & Brightness function for the sky.**

3.1.1. *Orbit of the sun.* Since the sun moves in a spiral during the course of a year (in the view of an observer standing on the earth), we will first model its orbit using celestial sphere. At different latitude, the sun will trace out a different trajectory in a year. Three examples are shown below.

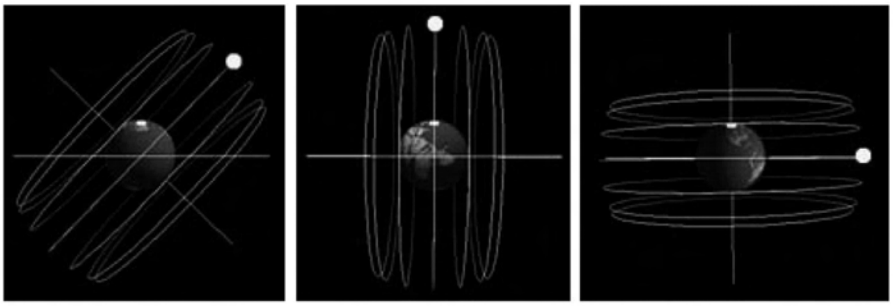


Figure 3.1 Orbit of the sun in the North Temperate Zone, on the Equator,and at the Arctic Pole
Source: http://www.math.nus.edu.sg/aslaksen/applets

3.1.2. *Brightness function for the sky.* The brightness function [7] for the sky is defined in terms of spherical polar coordinates $(r, \theta, \phi)$, although it does not involve the $r$ coordinate. When defining the position of the sun, $\theta$ is known as the zenith angle (when $\theta = 0$, the sun is directly overhead) and $\phi$ is termed the azimuth angle.

The brightness function of the sky as proposed in [7] is defined as $B(\theta, \phi)$, with units $Wm^{-2}srad^{-1}$. The steradian (srad) is the SI unit for solid angle: a solid

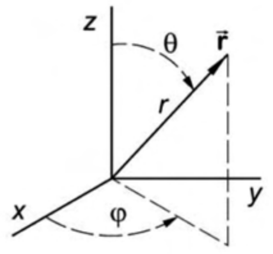



Figure 3.2

angle is the area of the surface of a portion of sphere divided by the square of the radius of the sphere. Thus the total solid angle of a sphere is $4\pi/srad$.

To illustrate the method of deriving $B(\theta,\phi)$, we use trees on the equator as an example. Brightness function for the sky at other latitude and longitude can be derived in a similar way with slight modification of the algorithm.

On the equator, the orbit of the sun is shown in the second sub-figure in Figure 3.1. The orbit will shift from the leftmost circle to rightmost circle from summer solstice to winter solstice (strictly speaking it is not a circle but a spiral).

We denote the rate of solar radiation that reaches the earth as E with unit of $Wm^{-2}s^{-1}$. E is known from previous research of solar energy.

When the sun is at a certain position $(\theta,\phi)$ on the celestial sphere, the rate of solar energy that is absorbed by the earth surface $(E_a)$ depends on the angle between the ground and the light beam $(\gamma)$.

$$E_a \approx E \times sin\gamma, (\gamma = \frac{\pi}{2} - \phi) \tag{4}$$

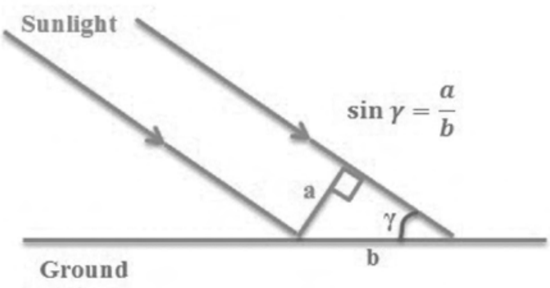



Figure 3.3

Thus, since we know the orbit of the sun in the course of a year, the brightness function can be derived by integration of $E_a$ over time (t). This should be done by computer simulation and discretization of the spiral trajectory is used for ease of implementation.

3.2. **Sunlight irradiation inside the tree crown.** In this section, we adopt the method in [7] to model the sunlight irradiation inside the tree crown. In plant models, light attenuation in a canopy is generally described by the equation (known as Monsi and Saeki equation [8]):

$$I(l) = l_0 e^{-ks} \tag{5}$$

where $I_0$ and $I(l)(W(m^2ground)^{-1})$ are the irradiances above and within the tree crown respectively at path-length s, and $k(m^2ground(m^2leaf)^{-1})$ is known as the extinction coefficient which is assumed to be constant.

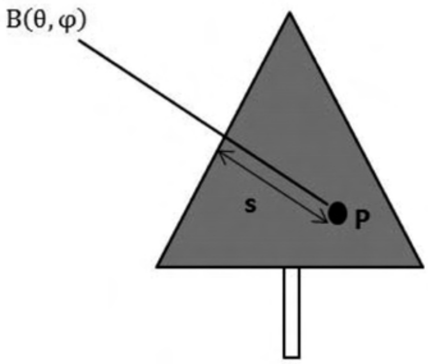



Figure 3.4

For any point P within the tree crown, the path-length s varies with the angles $\theta$ and $\phi$. Monsi-Saeki equation can be used to describe the attenuation along the path-length s that the radiation passes to reach P. And according to our definition of brightness function, $I_0 = B(\theta, \phi)$. Thus, the total irradiation at a point P within the plant $I_p$ is given by the integral

$$I_p = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} B(\theta, \phi) e^{-ks} cos\theta sin\phi \; d\theta d\phi \tag{6}$$

In practice, this integral is evaluated numerically using computer programs.

## 4. SHAPE & TREE PROFILE/BRANCHING STRUCTURE

In this section, we use the model developed in chapter 2 to investigate the relationship between leaf shape and tree profile/branching structure. We raise a hypothesis that for any tree profile, the leaf shape will maximize its exposure to sunlight.

To test this hypothesis, we calculate the sunlight exposure rate for each type of leaf shape under a given branching structure. Then, we will choose the leaf shape which maximizes sunlight exposure. If the leaf shape chosen coincides with the real life shape, our hypothesis is true.

4.1. **Sunlight exposure rate.** To define the sunlight exposure rate of leaves, we first ignore the motion of the sun in the sky and assume that the sun is always directly overhead of the tree. Then, the sunlight exposure rate (SE) is defined as

$$SE = \frac{\sum A_i}{TA} \tag{7}$$

Where $A_i$ is the area of shadow casted by an individual leave with index i (denoted by pink dots on the graph below), and $TA$ is the total area within the borderline of the shadow casted by the tree crown (denoted by the red circle on the graph below). $SE$ measures the proportion of sunlight on the tree crown that is blocked by leaves.

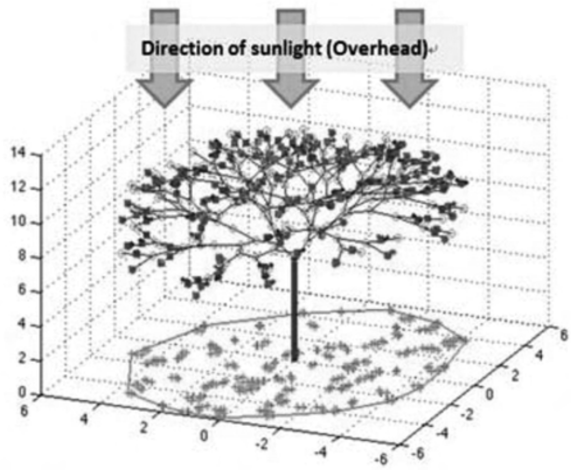



Figure 4.1

Now, we consider the motion of the sun in the sky. We have introduced the concept of brightness function for the sky $B(\theta, \phi)$ in section 2.2 which models the motion of sun in the course of a year by assigning each point $(\theta, \phi)$ on the celestial sphere a solar irradiance rate.

We define the sunlight exposure rate for each direction $(\theta, \phi)$ in a similar way to the overhead sunlight.

$$SE(\theta, \phi) = \frac{\sum A_i(\theta, \phi)}{TA(\theta, \phi)} \tag{8}$$

Then, the overall sunlight exposure rate for the tree is an integral of $SE(\theta, \phi)$ over the entire celestial sphere with weight $B(\theta, \phi)$ for each $(\theta, \phi)$:

$$SE_{total} = \frac{\int_0^{\frac{\pi}{2}} \int_0^{2\pi} B(\theta, \phi) SE(\theta, \phi) cos\theta sin\phi \; d\theta d\phi}{N} \tag{9}$$

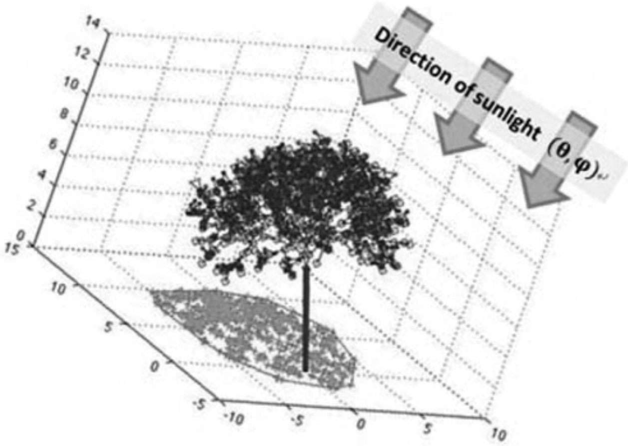



Figure 4.2

where $N = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} B(\theta, \phi) cos\theta sin\phi \ d\theta d\phi$ is a normalization factor. The range of $SE_{total}$ is [0, 1].

4.2. **Matching leaf shape with tree profile.** Now, we are ready to investigate the relationship between leaf shape and branching structures.

As illustrated in section 2, we classify leaf shapes into four categories shown in Figure 4.3. We choose the three types of tree profiles shown in Figure 4.4 to study.

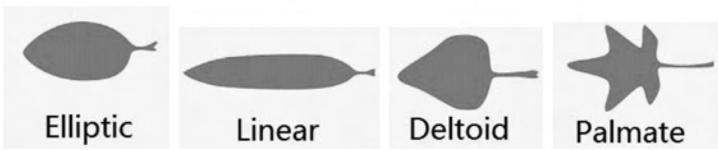



Figure 4.3

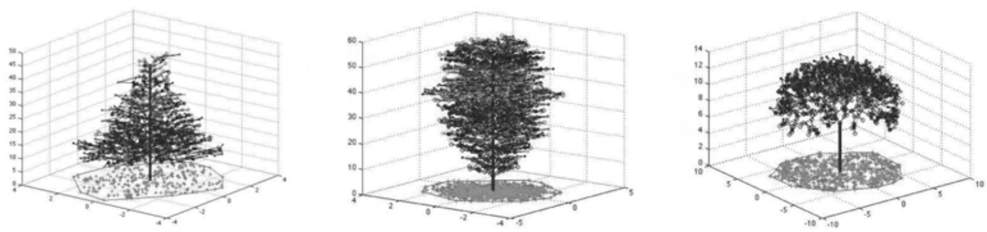


Figure 4.4: Profile 1: pine tree; Profile 2: poplar tree; Profile 3: Japanese banana

For each tree profile, we calculate the $SE_{total}$ for the four types of leaves and chose the one with the highest $SE_{total}$. We set d (the distance between adjacent leaves on the same branch) and S (the size of a leaf) based on data in manual of leaf architecture [2]. To derive the brightness function of the sky, we assume that the tree is located at a place with latitude: $45^0 N$. The results are shown in Table 4.1 and a detailed analysis of a specific case is shown in Figure 4.5.

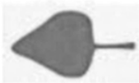
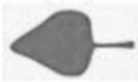
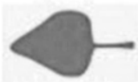
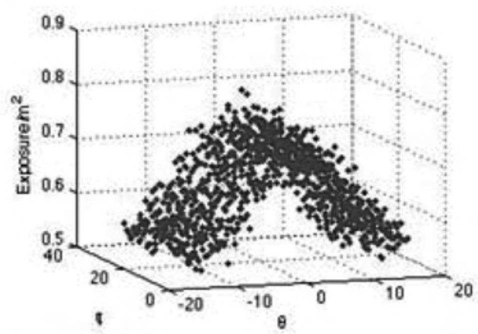
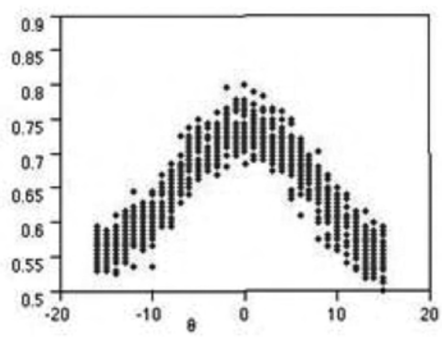
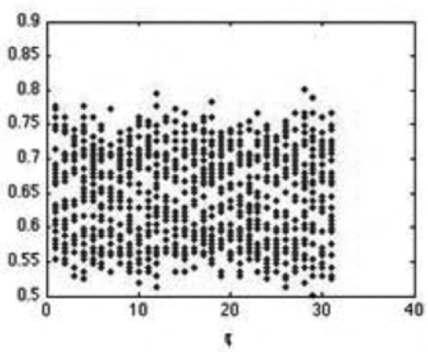
| Tree Profile: | Triangular: pine tree | Elliptic: poplar tree | Semi-spherical: Japanese banana |
|---|---|---|---|
| Leaf distribution on branches: |  |  |  |
|  | 0.53 | 0.62 | 0.82 |
|  | 0.67 | 0.49 | 0.83 |
|  | 0.49 | 0.63 | 0.82 |
|  | 0.47 | 0.62 | 0.85 |
| Shape chosen |  |  |  |
| Real leaf shape |  |  |  |

Table 4.1 $SE_{total}$ for different combinations of tree profile and leaf type




This figure is the detailed implementation of poplar tree with Deltoid leaves (numerical discretion is adopted). From the figure we could see that different angles of sun will influence the exposure area of the leaves on the tree. In our implementation, when θ and φ are small, the light irradiance on the tree is largest. We computed the weighted sum according to the method described in the previous sections.

Figure 4.5

Since the results are generally expected, our hypothesis stands, i.e. there is correlation between leaf shapes and tree structures: leaf shapes tend to maximize sunlight exposure of this tree. This provides insights into why leaves have different shapes. A larger pool of tree profiles should be studied using this model before we can draw a solid conclusion on the exact relationship. However, that is beyond the mathematical context of this topic, we leave it for readers who are interested to investigate further.

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*Part 2 of this article will appear in Volume 39 No. 2 of this magazine.*
*See next article about the authors.*

# Model Victory

Sarah Loke

Students Xu Wenji, Lu Jingyi and Zhang Jing did the Department of Mathematics (National University of Singapore) proud, scoring victory for the first time in the 2012 Mathematics Contest in Modeling (MCM). The trio clinched the Ben Fusaro Award, one of the highest awards in the contest, outperforming more than 3,500 teams that the MCM attracted from institutions around the world. They took part in MCM, which is primarily an online contest, held from 9 to 13 February 2012, on the advice of Professor Bao Weizhu, Department of Mathematics.

**Overcoming challenges**

As part of the contest, Wenji, Zhang Jing and Jingyi chose to tackle the continuous modelling problem that sought to find answers to the question, "How much do the leaves on a tree weigh". Wenji, a Year 3 student majoring in Quantitative Finance and also the leader of the team, said of their participation: ìTaking part in MCM is like doing a scientific research in a condensed version. The experience will definitely benefit us in our future careers."

Over the five-day of the contest, the trio brainstormed, consulted numerous reference books and past literatures and went through various sub-models as a team. One big challenge they had to overcome was to simulate the branching structures of trees. Zhang Jing, a Year 3 Mathematics major was able to use his talents in coding a programme for simulation algorithm and simulated the branching structures of trees. Jing is an expert of Matlab, a high-level technical computing language and interactive environment for algorithm development, data analysis and visualisation. He said: "Coding the programme for simulating the branching structures of trees using Matlab was definitely not easy; it was a most challenging task for me. I consulted various reference books, and combining that with what I had learnt previously, I managed to complete the simulation for my team."

**Good news**

When they received news of how well they did in the MCM, Jingyi, Year 3 and Quantitative Finance major, summed up her experience. She said: "I was interested in modeling, although I had little idea of it at first. Now, I know so much more about modeling and how it can be applied to solve real-life problems. From the contest, I discovered my analytical side, which I did not know I have."

They received a certificate for their win. In addition, their solution paper will be featured in The Undergraduate Mathematics and its Applications Journal, more commonly known as The UMAP Journal, along with commentaries from the author and judges.

On the secret of their success, Wenji said: "The key to our winning the award is that the three of us have strengths that complement each other. I am proud of my team and am honoured to have worked with each of them."

So, what would the answers offered by the winning team have served? An accurate model to measure the weight of the leaves of a tree will aid botanists in monitoring the health of trees, consistently and conveniently, by surveying the biomass of the leaves. Ecologists will also be able to use such models to observe environmental changes and ecological phenomena. These applications will greatly benefit botany or environment-related studies.

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