IMO 2012 Problems

Language: **English**

53rd International Mathematical Olympiad

MAR DEL PLATA - ARGENTINA

Day: **1**

Tuesday, July 10, 2012

Problem 1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC.

Prove that M is the midpoint of ST.

(The *excircle* of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 2. Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n \; > \; n^n.$$

Problem 3. The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k + 1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \geq 2^k$, then B can guarantee a win.
2. For all sufficiently large k, there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Language: **English**

Day: 2

Wednesday, July 11, 2012

Problem 4. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c=0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 \;=\; 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

Problem 5. Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

Show that $MK = ML$.

Problem 6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} \;=\; \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} \;=\; 1.$$

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

53rd International Mathematical Olympiad
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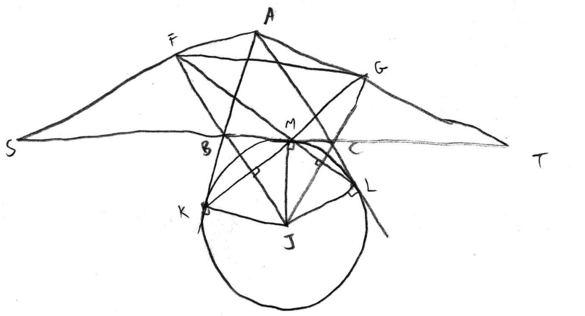
Name: Lim Jeck
Contestant code: SGP 1
Problem: 1 Page: 1



Note that $BMJK$ and $MCLJ$ are kites ($BM = BK$, $MC = CL$, $JK = JM = JL$)

$\therefore$ $MK \perp BJ$ and $ML \perp CJ$

$\therefore$ M is the orthocenter of $\Delta FGJ \Rightarrow JM \perp FG$

But $JM \perp BC \Rightarrow BC // FG$.

So now it suffices to show AM bisects FG by homothety.

$\angle FJK = \angle FJM = \angle FGM \Rightarrow FGJK$ is cyclic

similarly $FGJL$ is cyclic.

Also note that $\angle AKJ = \angle ALJ = 90°$ so $AKJL$ is cyclic.

$\therefore$ $AFKJLG$ is cyclic.

So $\angle AFG = \angle AKG = \angle BKM = \angle BMK = \angle FGM$

So $AF // GM$.

similarly $AG // FM$ so $AFMG$ is a parallelogram.

$\therefore$ AM bisects $FG \Rightarrow M$ is midpoint of ST.

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MAR DEL PLATA - ARGENTINA

Name: Lim Jeck
Contestant code: SGP 1
Problem: 2 Page: 1

Note that $1 + a_k = \underbrace{\frac{1}{k-1} + \frac{1}{k-1} + \ldots + \frac{1}{k-1}}_{k-1 \text{ of them}} + a_k \overset{AM-GM}{\geq} k \sqrt[k]{\frac{a_k}{(k-1)^{k-1}}}$ $\cancel{k \geq 2}$. $2 \leq k \leq n$.

So $(1+a_k)^k \geq \frac{k^k}{(k-1)^{k-1}} a_k$ with equality iff $a_k = \frac{1}{k-1}$.

$$\therefore \prod_{i=2}^{n} (1+a_i)^i \geq \prod_{i=2}^{n} \frac{i^i}{(i-1)^{i-1}} a_i$$

$$= \frac{2^2}{1^1} \cdot \frac{3^3}{2^2} \cdot \frac{4^4}{3^3} \cdots \frac{n^n}{(n-1)^{n-1}} \prod_{i=2}^{n} a_i$$

$= n^n$. with equality iff $a_i = \frac{1}{i-1}$.

~~If equality holds,~~ Equality does not hold because $\prod a_i = 1 \times \frac{1}{2} \times \frac{1}{3} \times \ldots \times \frac{1}{n-1} \neq 1$ for $n \geq 3$.

So $\prod_{i=2}^{n} (1+a_i)^i > n^n$ which is what we want.

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MAR DEL PLATA - ARGENTINA

Name: Lim Jeck
Contestant code: SGP 1
Problem: 3 Page: 1

3.1 We consider THE following GAME:

~~A chooses x~~

let ~~k~~ k be a positive integer known to A & B.

A think ~~of~~ a no. x with $1 \le x \le$ ~~...~~ 2^k.

B asks questions ~~by~~ the same way as the liar's guessing game

(choose a set S, ask A if $x \in S$ blah blah blah)

~~But~~ now B can only ask k questions, and among them, A must ~~...~~ answer truthfully at least once.

After the question and answer period, B must specify an ~~no.~~ integer y with $1 \le y \le 2^k$.

If $y \ne x$ B wins and B loses otherwise.

~~...~~

We claim that B can always guarantee a win.

We show this by induction.

$k=1$: B asks if $x \in \{1\}$.

If A replies yes, it must be true so B choose $y=2$.

" " no, it must be true so B choose $y=1$.

So B will win.

Now suppose true until some k, and we want to show for $k+1$.

~~Consider~~ B asks if $x \in \{1, 2, \dots, 2^k\}$.

Suppose A replies yes.

~~...~~ Then B ~~plays as if for k~~ pretends A is lying and asks the next k questions according to our induction hypothesis where $x \in \{2^k+1, 2^k+2, \dots, 2^{k+1}\}$.

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Name: Lim Jeck
Contestant code: SGP 1
Problem: 3 Page: 2

(we can imagine a different game where $k=k$, $2^k+1 \le x \le 2^k$ and B must choose ~~...~~ $2^k+1 \le y \le 2^{k+1}$

it is the same game as long as x can be 2^k different possible values).

So now B has a no. y with $2^k+1 \le y \le 2^{k+1}$, and $x \ne y$ confirmed, when A lies at first.

$\therefore$ If $x \notin \{1, 2, \dots, 2^k\}$, then $x \ne y$

If $x \in \{1, 2, \dots, 2^k\}$, then clearly $x \ne y$.

$\therefore$ $x \ne y$ is confirmed in all cases, and B wins.

Similarly if A replies no to $x \in \{1, 2, \dots, 2^k\}$, it is the same as

A replying yes to $x \in \{2^k+1, \dots, 2^{k+1}\}$. Same as $x \in \{1, 2, \dots, 2^k\}$ essentially

$\therefore$ B always has a winning strategy

So by induction our claim is true for all k, that B can win.

Back to our question, we want to make B win by eliminating the no. of possible values of x from N to at most 2^k. If $N \le 2^k$ done. Suppose $N \ge 2^k+1$.

If we can reduce 2^k+1 to 2^k, then we can reduce N to 2^k by reducing

N to $N-1$ to $N-2$ to ... to 2^k.

So we just have to show when $N = 2^k+1$.

B asks A if ~~$x \in \{1\}$~~ $x \in \{2^k+1\}$ (at least $k+1$) many times. If A keeps answering no, then since one of the no's is true, $x \ne 2^k+1$ and so we reduced 2^k+1 possibilities to 2^k.

Suppose A answered yes at least once, B ~~...~~ immediately stop asking A that.

Now we play the previous game where B asks k questions, with $1 \le x \le 2^k$.

then $\exists y$ ~~...~~ with $1 \le y \le 2^k$ and $x \ne y$, given that $x \in \{2^k+1\}$ is a lie.

If it is a lie, $x \ne y$. If it is the truth, $x \ne y$ clearly.

So $x \ne y$ confirmed so we reduced 2^k+1 possibilities to 2^k.

So we are done.

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Name:

Contestant code:

Problem: 3 Page: 3

3.2 Choose $N = n+1$. Let ~~d = 1.99993141592631000922631001110101101100162 → 1.99~~ d be a real no. s.t. $1.99 < d < 2$.

when A says $x \in S$ or $x \notin S$, he claims that

$x \neq y \ \forall y \notin S$ or $x \neq y \ \forall y \in S$.

~~Before~~ Right after the m^{th} question, let $u_{a,m}$ be the largest no. p satisfying: $1 \le a \le N$.

for the last p questions, A claims that $x \neq a$.

Then $u_{a,0} = 0 \ \forall a$ with $1 \le a \le N$.

Let $v_{a,m} = d^{u_{a,m}}$, and $S_m = \sum_{a=1}^{N} v_{a,m}$.

Then $S_0 = N$.

Now A's strategy is simply to answer yes or no, whichever leads to the lowest possible S_m score.

Note that $u_{a,m+1} = \begin{cases} u_{a,m}+1 & \text{if A claims } x \neq a \text{ on the } m+1^{th} \text{ turn.} \\ 0 & \text{otherwise.} \end{cases}$

$v_{a,m+1} = \begin{cases} d v_{a,m} & \text{if} \ldots \\ 1 & \ldots \end{cases}$

Suppose at the m^{th} turn A says yes and S_m will have a value S_m'.

" " no and S_m will have a value S_m''.

Then $S_m' + S_m'' = dS_{m-1} + N$.

So the ~~lowest possible~~ $S_m \le \frac{d}{2} S_{m-1} + \frac{N}{2}$.

$\Rightarrow$ $S_m - \frac{N}{2-d} \le \frac{d}{2}\left(S_{m-1} - \frac{N}{2-d}\right)$.

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Name:

Contestant code:

Problem: 3 Page: 4

Note that $S_0 - \frac{N}{2-d} < 0$ so $S_m - \frac{N}{2-d} < 0 \ \forall m$.

$\Rightarrow S_m < \frac{N}{2-d} \ \forall m$

Suppose there is a no. a where A claims $x \neq a$ in $k+1$ consecutive turns.

Then there is a time m where $S_m \ge d^{k+1}$.

So $d^{k+1} < \frac{N}{2-d}$.

We can choose $1.99^k < N < 1.999^k$ for sufficiently large k.

Then $d^{k+1} < \frac{1}{2-d} \times 1.999^k \Rightarrow \left(\frac{d}{1.999}\right)^k < \frac{1}{d(2-d)}$.

But when we choose $d = 1.99990702603 1062$, the above is not true for sufficiently large k.

$\therefore \ \forall$ sufficiently large k, A claims that $x \neq a$ in at most k consecutive turns.

So x can be any value from 1 to N and A can still be truthful once among any $k+1$ consecutive answers. So B cannot guarantee a win.

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Name: Lim Jeck
Contestant code: SGP 1
Problem: 4 Page: 1

Set $a=b=c=0$, we get $f(0)^2=0 \Rightarrow f(0)=0$.

Set $c=0$, $b=-a$, we get $f(a)^2+f(-a)^2-2f(a)f(-a)=0$

$$\Rightarrow (f(a)-f(-a))^2=0$$

$$\Rightarrow f(a)=f(-a).$$

Set $c=-a-b$, then since $f(a+b)=f(-a-b)$,

$$f(a)^2+f(b)^2+f(a+b)^2=2f(a)f(b)+2f(b)f(c)+2f(c)f(a)$$

$$\Leftrightarrow (f(a+b)-f(a)-f(b))^2=4f(a)f(b).$$

This means that $\forall a,b\in\mathbb{Z}$, $f(a)f(b)\geq 0$. So f is always non-negative or non-positive.

WLOG assume $f(x)\geq 0$ $\forall x$, since if $f(x)\leq 0$ $\forall x$, replace $f(x)$ by $-f(x)$.

$\therefore$ $f(a+b)-f(a)-f(b)=\pm 2\sqrt{f(a)f(b)}$

$\Leftrightarrow$ $f(a+b)=\left(\sqrt{f(a)}\pm\sqrt{f(b)}\right)^2$

$\Leftrightarrow$ $\sqrt{f(a+b)}=\pm\sqrt{f(a)}\pm\sqrt{f(b)}$ (both ± may not be the same sign).

Let $g:\mathbb{Z}\to\mathbb{R}$ where $g(x)=\sqrt{f(x)}$ $\forall x$.

Now the original condition is equivalent to $g(a+b)=\pm g(a)\pm g(b)$ $\forall a,b$ and $g(a)^2\in\mathbb{Z}$ and $g(a)=g(-a)$ and $g(a)\geq 0$.

So we only need to check these 4 conditions in order to satisfy the original condition.

But just to be safe lets just throw in $g(0)=0$ to have 5 conditions.

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Name:
Contestant code:
Problem: 4 Page: 2

If $g(z)=0$ for some $z\neq 0$.

Then $g(x+z)=\pm g(x)\pm g(z)=\pm g(x)=g(x)$ $\because g\geq 0$. So g has period z.

~~Setting $x=z, 2z, 3z, \ldots$ gives $g(z)=g(2z)=g(3z)=\ldots=0$.~~

~~$\therefore g(nz)=0$ $\forall n\in\mathbb{Z}$ (using $g(a)=g(-a)$).~~

Let $g(1)=k$. If $k=0$ then $g(x)=0$ $\forall x$ which works.

Otherwise $k>0$, then $k^2\in\mathbb{Z}$.

Then $g(x+1)=\pm g(x)\pm k$. ~~so we can easily induct to show~~

Suppose $g(1)=k, g(2)=2k, \ldots, g(\alpha)=\alpha k$ for some $\alpha\geq 1$, and $g(\alpha+1)=\alpha k-k$.

It is not hard to see α exists from $g(x+1)=\pm g(x)\pm k$, or $g(x)=|xk|$ $\forall x$.

Easy to check $g(x)=|x|k$ works, ~~for $k^2\in\mathbb{Z}$~~.

If $\alpha\geq 3$, ~~$g(\alpha+2)=\pm g(\alpha+1)\pm k=\alpha k-2k$ or αk~~

$g(\alpha+1)=\alpha k-k=\pm 2k\pm(\alpha-1)k$ ~~$=\pm g(\alpha)\pm 2k=\alpha k-2k$ or $\alpha k+2k$~~

impossible. ~~$=\pm g(\alpha-1)\pm 3k=\pm g(\alpha-1)\pm 3k=\alpha k-4k$ or $\alpha k+2k$~~

~~so $g(\alpha+2)=\alpha k-2k$ does not exist~~.

$\therefore$ $\alpha\leq 2$

Case 1: $\alpha=1$

$g(1)=k$, $g(2)=0$, so $g(x)=g(x+2)$ is periodic.

So $g(x)=\begin{cases}k & \text{when } x \text{ is odd}\\ 0 & \text{even}.\end{cases}$ easy to chek it works.

Case 2: $\alpha=2$

$g(1)=k$, $g(2)=2k$, $g(3)=k$, $g(4)=\pm k\pm k=\pm 2k\pm 2k \Rightarrow g(4)=0$.

So $g(x)=\begin{cases}0 & x\equiv 0\\ k & x\equiv 1,3\\ 2k & x\equiv 2\end{cases}$ (mod 4).

To check it works, ~~consider~~ we only need to check $g(a+b)=\pm g(a)\pm g(b)$ ~~only~~ for $a,b\in\{0,1,2,3\}$, which is easy.

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Name:

Contestant code:

Problem: 4 Page: 3

~~Case 3: a = 2~~

~~$g(1)=k,\ g(2)=2k,\ g(3)=3k,\ g(4)=$~~

~~$g(4) = 4k+3k = 17k+2k$~~

So all possible functions f are:

$f(x) = nx^2$ ⟵ easily checked

$f(x) = 0$ ⟵ easily checked

$f(x) = \begin{cases} n & x \equiv 1 \\ 0 & x \equiv 0 \end{cases} \pmod 2$ ⟵ just check $a,b,c \in \{0,1\} \pmod 2$

$f(x) = \begin{cases} 0 & x \equiv 0 \\ n & x \equiv 1,3 \\ 4n & x \equiv 2 \end{cases} \pmod 4$ ⟵ just check $a,b,c \in \{0,1,2,3\} \pmod 4$

where $n \in \mathbb{Z}$.

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

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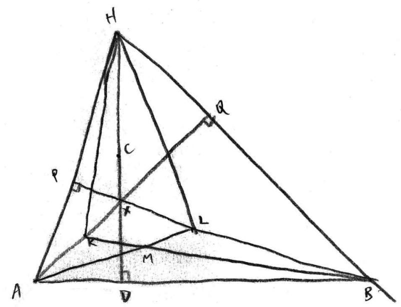
Contestant code: SGP 1

Problem: ~~4~~ 5 Page: 1



Let H be the orthocenter of ΔAXB.

Then X is orthocenter of ΔHAB.

$AL^2 = AC^2 = AD \cdot AB$

So $\angle ALD = \angle ABL$. (circle through D, L, B tangents AL).

Let AX, BX meet BH, AH at Q, P respectively.

Then $\angle ALD = \angle ABL = \angle AHD$

so HADL is cyclic $\Rightarrow \angle HLA = \angle HDA = 90°$

$\therefore$ $HL^2 = HP \cdot HA = HX \cdot HD$ $\because$ ~~APDX~~ APXD is cyclic.

Similarly $\angle HKB = 90°$ and $HK^2 = HX \cdot HD = HL^2$.

$\therefore HK = HL$.

$\therefore \Delta HKM \equiv \Delta HLM \Rightarrow MK = ML$ done.

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Name: Lim Jeck
Contestant code: SGP 1
Problem: 6 Page: 1

Consider $N = \max\{a_1, a_2, \dots, a_n\}$ then

$$3^{N-a_1} + 2\times 3^{N-a_2} + \dots + n\times 3^{N-a_n} = 3^N \equiv 1 \pmod 2.$$

$$\therefore \frac{n(n+1)}{2} \equiv 1 \pmod 2 \Rightarrow n \equiv 1,2 \pmod 4.$$

Consider small values of n. easy to check these are solutions:

n	$\{a_n\}$ in that order
1	0
2	1, 1
5	2, 2, 2, 3, 3
6	~~2, 3, 3, 3, 4, 4, 4, 4~~ 2, 2, 3, 3, 3, 3
9	2, 3, 3, 3, 3, 4, 4, 4, 4
10	2, 3, 3, 3, 4, 4, 4, 4, 4, 4.

(trust me)

We claim that all n of the form $1,2 \pmod 4$ works. We show this by extending the construction for n to $n+8$.

Note that $\frac{1}{2^\alpha} = \frac{1}{2^{\alpha+2}} + \frac{1}{2^{\alpha+2}} + \frac{1}{2^{\alpha+3}} + \frac{1}{2^{\alpha+4}} \times 6$.

Case 1: $n \equiv 0 \pmod 4$

Consider $\frac{k}{3^\alpha} = \frac{k}{3^{\alpha+2}} + \cancel{\frac{n+1}{3^{\alpha+4}}} + \cancel{\frac{n+2}{3^{\alpha+4}}} + \cancel{\frac{n+3}{3^{\alpha+3}}} + \frac{n+1+n+3+n+5+n+6+n+7+n+8}{3^{\alpha+4}} + \frac{n+2}{3^{\alpha+3}} + \frac{n+4}{3^{\alpha+2}}$

$\Leftrightarrow 4k = n+4$.

So k exists (in integer) and we just extended n to $n+8$.

(because if $(b_1, b_2, \dots, b_n)$ is a solution for n, $(b_1, b_2, \dots, b_{k-1}, b_k+2, b_{k+1}, \dots, b_n, b_k+4, b_k+3, b_k+4, b_k+2, b_k+4, b_k+4, b_k+4, b_k+4)$

53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Name:
Contestant code:
Problem: 6 Page: 2

is a solution for $n+8$)

Case 2: $n \equiv 1 \pmod 4$

Consider $\frac{k}{3^\alpha} = \frac{k}{3^{\alpha+2}} + \frac{n+3+n+4+n+5+n+6+n+7+n+8}{3^{\alpha+4}} + \frac{n+1}{3^{\alpha+3}} + \frac{n+2}{3^{\alpha+2}}$

$\Leftrightarrow 4k = n+3$ so k exists, and similarly we extended n to $n+8$.

Case 3: $n \equiv 2 \pmod 4$

$\frac{k}{3^\alpha} = \frac{k}{3^{\alpha+2}} + \frac{n+1+n+2+n+3+n+5+n+6+n+7}{3^{\alpha+4}} + \frac{n+4}{3^{\alpha+3}} + \frac{n+8}{3^{\alpha+2}}$

$\Leftrightarrow 4k = n+6$ so k exists, and similarly done.

Case 4: $n \equiv 3 \pmod 4$.

I just realised we don't need case 4 & 1, since n can only be $1,2 \pmod 4$.

$\therefore$ all possible n is ~~0~~ $1,2 \pmod 4$.

Using this method, we can extend $n=1$ to $n=9$

$0 \rightarrow 2,3,2,4,4,4,4,4,4$

extend $n=2$ to 10

$1,1 \rightarrow 1,3,5,5,5,4,5,5,5,3$.

extend $n=10$ to 18

$1,3,5,5,5,4,5,5,5,3 \rightarrow 1,3,5,7,5,4,5,5,5,3,9,9,9,8,9,9,9,7$