Problem Posing and Constellations in Mathematics:

Using a Biography of a Recent Mathematician for Teaching Mathematics

K. Immaculate Namukasa
with A. Katherine Minello, J.Sarah Dent, and Anthony Staibano

ABSTRACT

There are conceptual, socio-cultural, and motivational aspects to using the history of mathematics and science in teaching and learning. By using biographies - the lives and works of great mathematicians and scientists - teachers can integrate history of mathematics into their teaching. This paper is about mathematician Paul Erdős (1913-1996, Hungary), whose major work was in the field of number theory. Featured are Erdős's emphasis on conjecturing and problem solving, his mathematical accomplishments at a young age, his collaborative work, and his fascinating character. The paper demonstrates how to use a biography of a recent mathematician to promote key learning processes such as problem posing and collaboration.

Key Words: Problem solving, history of mathematics, biographies, number theory

> *Why are numbers beautiful? It's like asking someone why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you, I know numbers are beautiful. If they aren't beautiful, nothing is.*
>
> Paul Erdős

This paper is about the life and works of mathematician Paul Erdős (1913-1996, Hungary), whose major mathematics work was in the field of number theory. It outlines specific lessons for school mathematics from Erdős's life and work.

Since the 1980s, educators have encouraged teachers to use the history of mathematics and science in their teaching (see Bidwell, 1993; Liu, 2003). Gulikers and Blom (2001) have categorized the roles of history of mathematics and science in teaching and learning into conceptual roles, socio-cultural roles, and motivational roles. The use of biographies - the life and mathematical works of great mathematicians and scientists - is one way of integrating the history of mathematics into teaching. It demonstrates social, cultural, and personal aspects of mathematical activity (Fauvel, 1991; Ernest, 1998) and motivates students who are interested in stories. But most importantly, teachers may use biographies of recent and ancient mathematicians to illustrate the mathematical and scientific processes encouraged by most reform-based curriculum documents such as the NCTM Standards 2000.

Erdős's Life – An Introduction

Paul Erdős was born into a Hungarian-Jewish family on March 26, 1913 in Budapest, Hungary. Erdős was not even a year old when World War I broke out. His father was captured by the Russian army and sent to Siberia, where he spent six years in captivity. During that time Erdős's mother worked out of town, leaving her son behind (O'Connor & Robertson, 2000). The tragic death of Paul's two older sisters from scarlet fever resulted in Erdős being home-schooled by his parents, both high school mathematics teachers. His parents felt a need to protect their son from exposure to infectious diseases.

By age four, Erdős was able to convert the ages of his peers to seconds and the distances between planets to distances traveled by a train (Hoffman, 1998). When his parents' friend asked, "What is 100 less 250?" Erdős replied, "150 below zero." At ten Erdős was already interested in prime numbers. At fourteen he asked another teenager who loved mathematics, "Could you give me a four digit number?" "2532," said the teenager. "The square of it is 6411024," Erdős replied. "How many proofs of the Pythagorean theorem do you know?" Erdős asked. "One," said the teenager. "I know 37," Erdős responded (Chung & Graham, 1998, p. 119-120). By the time Erdős was 17 his father had already introduced him to the work of mathematician George Cantor (1845-1915, Russia/Germany), the inventor of set theory.

Figure 1. *Erdős at age 14 (left) and age 17 (right).*
Photograph courtesy of the J. Bolyai Mathematical Society.

As a Jewish youth living in Hungary, Erdős would not have attended the University of Budapest had he not won the national examination in 1930. He studied mathematics and in 1934 received a PhD in number theory. He traveled throughout his working life, taking up temporary university positions and collaborating with mathematicians in other countries, first in the United Kingdom, then in the United States, and later in Israel, France, the Netherlands, Canada and many other countries (Bollobas, 1998).

Erdős's life was both fruitful and peculiar. As a nomadic mathematician, he kept a few possessions in a suitcase, rarely stayed in one place for long, and often went from a university, to a conference, and then to a mathematician's home. "Another roof, another proof" was one of Erdős's maxims (Babai & Spencer, 1998). In the later decades of his life, he would show up unannounced and exclaim, "My mind is open." He would then stay long enough to collaborate on a proof. Erdős published around 1500 papers with 511 collaborators before he died of a heart attack in Warsaw, Poland at a graph theory conference in 1996 (Hoffman, 1998). The number of

Erdős's publications is second only to that of Leonhard Euler (1707-1783, Switzerland), who had more pages but fewer actual papers (Babai & Spencer, 1998; Hoffman 1998). About his numerous publications Erdős would encourage, "Weigh them, do not count them." Euler and Erdős were unique among mathematicians in that they produced most of their work in the last decades of their lives (Bollabas, 1998). In later years Erdős lived with his mother, traveling from conference to conference with her. Erdős suffered from cataracts, which he had difficulty finding time to have removed. Time spent in hospital, he explained to his friends, would delay progress in mathematics.

Figure 2. *Erdős deep in thought.*
Photo by Wlodzimierz Kuperberg, 1985

Erdős had an eccentric vocabulary. Children were *epsilons* (mathematicians often use epsilon, ε, to represent small positive quantities), and giving a math lecture was *preaching*. Erdős spoke of *The Book*, where he believed God keeps all elegant proofs. He said the same God often hid socks and passports.

Perfect Numbers – Collaborating with Students

Erdős worked with many students at many levels. As a teenager, he solved problems published in a secondary school journal in Hungary. Two other successful problem-solvers, Paul Turan and Tibor Gallai, became Erdős' university classmates and lifelong collaborators. Erdős published a mathematics article with a 14 year old, Lajos (Schechter, 1998).

Figure 3. *Erdős's university classmates Tibor Gallai at 18 (left) and Paul Turan at 17 (right).*
Photograph courtesy of the J. Bolyai Mathematical Society.

Some mathematicians in different parts of the world including India recount having interacted with Erdős while they were still students. At the urging of his mentor, a university professor, Hawaiian high school student David Williamson wrote Erdős a letter inquiring about the originality of a number theory proof he had just completed on perfect numbers. Perfect numbers are numbers such as 6, 28, 496, 8128, and 3355033 that are formed by adding their factors, say, 6 = 1 + 2 + 3 or 28 = 1 + 2 + 4 + 7 + 14. The last number in the sum equals the sum of the rest of the numbers in the sum. Perfect numbers can also be formed by adding consecutive numbers starting from one; that is to say, 28 = 1 + 2 + 3 + 4 + 5 + 6 + 7 and as such is also a triangular number. Perfect numbers have many other interesting properties and have engaged mathematicians since ancient times. Williamson had proved that an odd perfect number (if there is one, for none has ever been found for numbers less than 10^{300}) must have exactly one prime factor that leaves a remainder 1 when divided by 4 (Weisstein, 2008). Put differently, an odd perfect number is congruent to 1 *modulo* 4. Special numbers such as prime and triangular numbers and special number sequence are a topic in middle grades number and algebra concepts. Zazkis & Campbell (2006) maintain that by including elementary number theory in the school curriculum students get to explore algebraic formalizations of integer arithmetic. Activities on splitting numbers illuminate the equivalent meaning (as opposed to the "find the answer" meaning) of the equal sign.

Euclid (c. 350 BC, Ancient Greece) was the first to notice that the first four even perfect numbers 6, 28, 496, 8128 can be generated by the formula $2^{n-1}(2^n - 1)$ where $(2^n - 1)$ is prime. For instance, when $n = 2$, then $2^{n-1}(2^n - 1)$ is 2×3, which results in the first perfect number, 6. Later prime numbers of the form $2^n - 1$ were labeled the Mersenne primes after another mathematician. So n = 2, 3, 5, 7 … give Mersenne numbers $2^n - 1$ = 3, 7, 31, 127, respectively. Ibn al-Haytham (c. 965-1040, Persia-Egypt) conjectured that every even perfect number is of the form $2^{n-1}(2^n - 1)$. Thus, Mersenne numbers 3, 7, 127 have corresponding perfect numbers 2×3, $2^2{\times}7$, $2^4 \times 31$, $2^6 \times 127$. Ibn al-Haytham's conjecture remained unproved until Euler proved it 700 years later. Erdős promptly responded that Williamson's proof had been done earlier by Leonhard Euler. In the same letter Erdős shared a related conjecture, which Euler had also proved, and encouraged Williamson to solve one of Erdős's problems that involved perfect numbers (Babai & Spencer, 1998; Hoffman, 1998). Perfect numbers and Mersenne number sequences (3, 7, 15, 31, 127…) could be explored as special historical sequences when middle school students explore special number sequences such as Fibonacci, exponential and prime number sequences and at high school lesson when they explore exponential functions.

The Prime Number Theorem – a Beautiful Proof

Erdős developed an extraordinary interest in prime numbers after his father taught him that there are infinitely many primes and that these primes have arbitrarily large gaps - there are no patterns in the sequence of primes: 2, 3, 5, 7, 11, 13, 17…. Erdős eventually explored an elementary proof for the Prime Number Theorem (PNT). At age 18 Erdős had provided an elementary proof

- an elementary proof uses concepts that are intuitive as regards the theorem being proved - to another number theorem about primes that is usually mistaken for the PNT.

Joseph Bertrand (1822-1900, France) in 1849 conjectured and verified that between any two numbers n and $2n$, where n is a positive integer, there is a prime number. For example, between 2 and 4 there is the prime number 3, between 3 and 6 is 5 and so on. Pafnuty L. Chebyshev (1821-1894, Russia) in 1850 proved Bertrand-Chebyshev's theorem (Hoffman, 1998; O'Connor & Robertson, 2000), but it was Erdős who first provided an elementary proof. Erdős is famous for his elementary conjectures and proofs.

The PNT is a famous number theory that centers on the distribution of prime numbers. The prime counting function $\pi(x)$ denotes the number of primes less than or equal to a real number that is greater than 1. For instance $\pi(2) = 1$, there being only one prime number, 2, that is less than or equal to 2, $\pi(3) = 2$ two prime numbers are less than or equal to 3, $\pi(4) = 2$, $\pi(5) = 3$ and so on. (The use of the symbol π here is not related to the circle measure.) Just recently established is $\pi(10^{22})$, a number with 22 digits (Weisstein, 2008). The function $\pi(x)$ generates the sequence (1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, 6…), another special, historical sequence. Mathematicians have for centuries engaged in studying the function $\pi(x)$. Many people think that writing arithmetic tables and inventing algorithms is an ancient mathematics practice that was in days of early mathematics such as with Babylonian and Egyptian multiplication tables. Quite the contrary, modern number theorists wrote tables and invented algorithms to calculate values for $\pi(x)$. For small ranges, say up to 100, the graph of $\pi(x)$ shown in Figure 4 is as odd as the graph of prime numbers - it yields neither a line nor a smooth curve. It is not even a regular step function, as seen in Figure 4. It is at best an irregular, zig-zag curve.

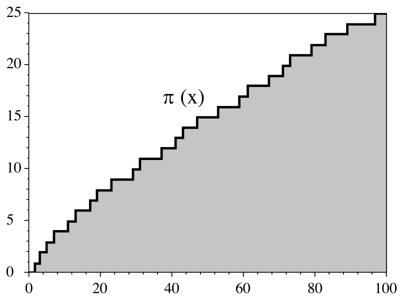

Figure 4. *π(x) for up to* 100

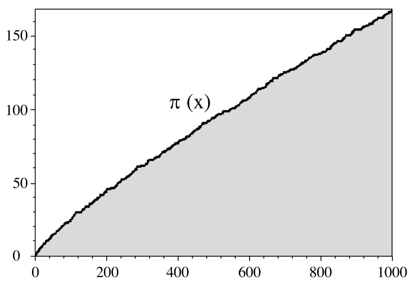

Figure 5. *π(x) for up to* 1000

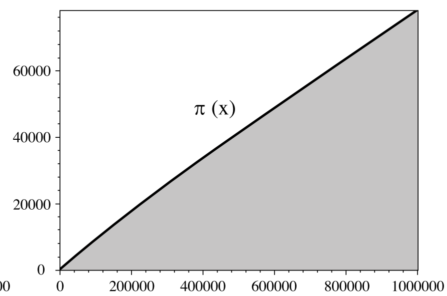

Figure 6. π(x) *for up to* 1000000

But, surprisingly, it shows order at a larger scale. The graph of $\pi(x)$, as shown in Figures 5 and 6, appears to have an interesting shape as the x scale gets larger, say to 1000 or even 1,000,000. To many mathematicians it resembles the graph of a logarithm. It always lies above the graph of $x/\log x$ (or $n/\ln n$ as shown in Figure 7) and though it smoothens toward a graph similar in shape to $x/\log x$, it never crosses that graph as π gets sufficiently large. Discrete functions, logarithms, limits, exponential, logarithmic and composite graphs, and graphing using technology are concepts explored in high schools curricular of many countries. Middle school students who explore Erastosthenes' sieve method of constructing a table of prime numbers would be familiar with bigger prime numbers and as such are likely to find the activity of generating and graphing the function $\pi(x)$, especially its regular nature when one zooms out, to be of interest.

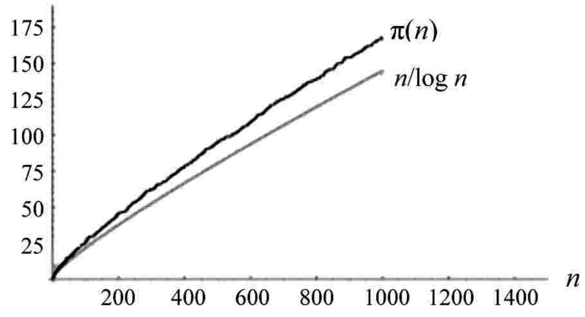

Figure 7. $\pi(n)$ *imposed on logarithm* (log *or* ln) *graphs*

The PNT states that for sufficiently large numbers x, $\pi(x)$, the upper graph in Figure 7, is approximately equal to $x/\log x$, the lower graph in Figure 7. Put differently, $\pi(x)$ is asymptotic to $x/\log x$ as x tends to ∞. Carl Frederic Gauss (1777-1855, German), at age 15, and Adrien Marie Legendre (1752-1833, France) made the PNT conjecture independently of one another. It was Jacques S. Hadarmard (1865-1963, France) and de la Valée Poussin (1866-1962, Belgium), following Riemann's famous analytic hypothesis, who independently solved the PNT conjecture a century later. Students who have heard about Gauss in the context of summing number sequences, and about Euler in the context of Euler formula that relates vertices, faces and edges of polygons would be motivated to hear about high school mathematics due to these mathematicians, in this case exponents and graphs of logarithmic functions.

Another century later, in 1949-1950, Erdős and Alte Selberg (1917-2007, Norway/United States) each provided a more elementary and beautiful proof of the PNT that used logarithmic properties and did not use modern complex analytic methods. Erdős (1949) and Selberg (1950) published their proofs separately, but not without furious disputes about intellectual property. Legend has it that Paul Turan (1910-1976, Hungary), a university classmate of Erdős, shared at a mathematics conference at which Erdős was in attendance the exciting work in progress of Selberg, not on the PNT theorem but on a distantly related conjecture. Erdős made and proved a related conjecture that built on Selberg's work, and quickly shared it with Selberg. Erdős's empathizers show letters in which Erdős, seeing the potential at collaborating on an elementary proof to the PNT theorem, persuaded Selberg to collaborate with him. Erdős and Selberg proceeded to accomplish and publish their proofs independently. The proof won Erdős the American Mathematical Society's Franklin Nelson Cole Prize in Number Theory and Selberg the more prestigious Fields medal (Schumer, 2004). Although Erdős won many other prizes, awards, and honorary degrees, legend has it that the Erdős-Selberg feud cost Erdős several prospects, including a permanent university job, in the mathematics community (Schechter, 1998).

Elementary Problems – Building Grander Theories

Erdős is remembered not only for elementary proofs but also for proofs to elementary problems. Consider the multiplication table used by students in the early elementary grades. A 10-by-10 multiplication table has 100 entries. More than half of these entries are repeats. Studying smaller cases of multiplication tables shows that a 1-by-1 multiplication table has 1 unique product out of the total of 1 product; a 2-by-2 multiplication table has 3 unique products out of the total of 4 products; a 3-by-3 table has 6 unique products; a 4-by-4 table has 9; 5-by-5 table has 14 (the pattern of multiples of 3 is broken at this step); a 6-by-6 table has 18, and a 10 by 10 has 42. Here is the question Erdős posed and later solved: Are there interesting patterns in the distribution of unique entries in multiplication tables?

Here is another elementary problem that Erdős formulated about Ancient Egyptian mathematics. Egyptians, in what are now recognized as limiting ways, conceived fractions, except for 2/3, as sums of unit fractions. For example, instead of 7/8 they wrote 1/2 + 1/ 4 + 1/8, and they wrote 4/10 as 1/3 + 1/15. They did not repeat a denominator in a presentation of a fraction (Chace, 1927). Leonardo Fibonacci (c.1175-1240, Pisa, Italy) proved that this is possible for all fractions. In 1932, Erdős proved that reciprocals of numbers with a common difference (i.e., reciprocals of numbers *a, a + d, a + 2d, a + 3d, …*) always sum to a non-integer number (Hoffman, 1998). The Egyptian way of representing fractions made it easier to compare sizes and to distribute food among people. It was used in Europe until the 12th century. For example, Europe was said to be "more than a third and eighth of the whole earth" (Hoffman, 1998). Erdős asked: How large do the denominators have to get to represent a given fraction? For a fraction such as 3/19, what is the largest possible denominator in its representation? This problem was solved three years after Erdős' death and earned the solver, a graduate student, a $750 check signed by Erdős, issued by Ronald Graham, a close collaborator of Erdős. When someone provided a proof to a conjecture, Erdős would not stop at congratulating the problem-solver. He used his sense of mathematical aesthetics and humor to judge the elegance of the solution. At times he would increase a cash prize. For one proof that he deemed ugly and easy, he lowered the prize to $50 from the promised $250 cash prize (Seife, 2002).

In addition to number theory, Erdős worked on Ramsey theory, graph theory, areas of classical analysis, complex functions, and probability theory (Babai, Pomerance, & Vertesi, 1998). Ross (1998), a close collaborator with Erdős, says that Erdős was well-read in other fields, including history, and well-informed in politics. He enjoyed classical music, although he referred to it as noise.

Here is another example of an Erdős problem, this time from geometry: For any five points on a flat surface, as long as they are not in a straight line, is it true that four of these points will always form a quadrilateral that is convex (i.e., that has no reflex interior angles)? (Chung & Graham, 1998; Hoffman, 1998). This conjecture may be paraphrased as: How many non-linear random points do you need to form a triangle? A quadrilateral? A heptagon? and so on. Put differently, if g(n) is the minimum number of random points needed to form an n-gon, is it true that g(n) exists for all n? What is g(3), the minimum number of random points needed to form a triangle? What is g(4)? If we know that g(3) is 3, g(4) is 5, and g(5) is 9, what is g(6)? Is there a pattern? A young female mathematician in Budapest, Esther Klein, who later married George Szekeres, posed this problem while the two were university classmates of Erdős. Erdős and Szekeres solved Klein's problem for g(4) and g(5) and later generalized the problem to state that whenever $1 + 2^{n-2}$ points are sprinkled on a plane, one can always draw a convex *n*-sided polygon, an *n*-gon (i.e., $g(n)$ is equal to or greater than $1 + 2^{n-2}$). g(6) = 19 - at least 19 random points are needed to form a hexagon. This proof, as well as many others by Erdős and his collaborators, formed a basis for Ramsey theory, which, generally speaking, is about the impossibility of disorder in sufficiently large random systems. There is at times a tendency to think that Ramsey theorem was invented by Erdős; Frank P. Ramsey (1903-1930, Britain) proved the theorem before Erdős became interested in Ramsey-type problems. Szekeres discovered Ramsey's proof after they solved Esther's problem. Erdős termed subsequent work on questions related to Esther's question, Ramsey Theory. Ramsey used the metaphor that shapes seen in star constellations are inevitable with a large enough random arrangement of stars. Erdős used the analogy of the party problem. Ramsey theory has many applications to computer science and to gaming. We will return to the grand Ramsey theory further on.

Many mathematicians are concerned with theory building. "One of the areas that set Erdős apart was his focus on problem-solving and problem-posing" - to conjecture and to prove, to pose and to solve mathematics problems (Gallian, 2002, p. 329). The many cheques Erdős issued as cash prizes to people who solved his problems are physical artifacts of his problem posing art. Other artifacts are the problems themselves, over a thousand (Seife, 2002), many of which have not yet been solved. Erdős is said to have considered the problems he posed and the conjectures he made as key to fundamental mathematical problems. In many cases his intuition was right - solutions to his problems became key in the development of grand theories (Bollobas, 1998). Some of his counterparts thought about Erdős what Erdős thought about God and his so-called book of beautiful proofs: That Erdős always knew these grander theories - such as Ramsey theory - but only revealed smaller conjectures one at a time. Historically, it is a common practice for a mathematician to challenge fellow mathematicians with difficult problems to which he or she already has solutions. Most of Erdős's problems were not in this vein. It is noted that Erdős shared his unsolved problems and conjectures to facilitate progress in the field of his mathematics projects.

Ramsey Theory

Erdős was influential in developing Ramsey theory, which is usually paraphrased as: What is the minimum number of people that must be invited to a party so that a given number of people will either be acquaintances or mutual strangers? To solve Ramsey's problems, Erdős is said to have developed the Probabilistic Method, which is used in combinatorics, a field of pure mathematics that deals with permutations and combinations.

Ramsey theory is closely related to an area of university mathematics known as graph theory. Graph theory was invented by Euler when he solved another historical problem, the Königsberg bridge problem.

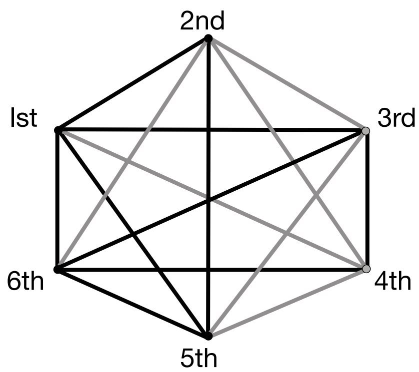

Figure 8. *Acquaintances and strangers, a dichromatic graph of* K_6 *at a party - one random case showing 4 dark triangles*

Consider the graph in Figure 8, called K_6. It is referred to as a complete graph on six vertices. In upper elementary school mathematics terms, a graph of order six is a hexagon with all its diagonals. It is a common geometrical representation that students may use to represent say, handshakes by each of six guests at a party, ordered and colored according to the handshakes made by each guest. In line with Ramsey theory, Figure 8 is shaded in only two colors: dark and light green. The problem becomes a Ramsey-type problem, when one color is used for acquaintances and the other for strangers. In the complete dichromatic (two-colored) graph of strangers and acquaintances there is likely to be monochromatic (one-colored) sub-graphs representing mutual strangers or acquaintances. Ramsey theorists have verified that no matter how you color the complete K_6 graph with two colors, you will always form complete monochromatic triangles, sub-graphs of order three, K_3. You might, in rare cases, get monochromatic squares, K_4, or even a monochromatic pentagon, K_5, in a K_6 graph shaded randomly with two colors. Here is a paraphrase of Ramsey conjecture: For a party of six guests there will always be at least either three strangers (an independent set), or three acquaintances (a clique). In technical terms, R(3 acquaintances, 3 strangers) = R(3, 3) = 6. The minimum number of guests necessary to get either three acquaintances or three strangers is six. In geometrical terms, the smallest *complete* dichromatic polygon in which monochromatic triangles can exist is a hexagon. This can be generalized to other polygons.

Various versions of Ramsey-type problems exist. Every two pairs of numbers (*m, n*) have a Ramsey number R(*n, m*) - this is a more generalized version of Ramsey's conjecture. Ramsey theorem states that for all numbers R(*n, m*) exists. R(1,1), the minimal guest list so as to get one friend or one stranger, is 1; R (1, *n*) is 1; R(2, *m*) is *m*; R(3, 3) is 6; R(3, 4) is 9; R(3, 5) is 14; R(3, 6) is 18, R(4, 4) is 18 and R(4, 5) is 25. Hoffman (1998) reports that in 1993 it took 110 computers running in sync to solve for R(4, 5). As was the case for generating tables and computational algorithms for $\pi(n)$, many current mathematicians specializing in Ramsey theory worked hard at generating tables and algorithms for $R(n, m)$ before they turned to generating lower and upper bounds of bigger $R(n, m)$. $R(n, n)$ is the Ramsey number for K_n. $R(5, 5)$ is bigger than $R(4, 5)$ and all that is known about $R(5, 5)$ is that it lies between 43 and 49. Erdős is quoted to have said that $R(5, 5)$ could be found if the need for it was aggravated enough to warrant collective effort and massive investment (Schechter, 1998). Erdős used what Chung and Graham (1998) referred to as a counting method to establish the lower bounds for any Ramsey number. It took decades to displace the lower bound that Erdős first suggested with a revised and more accurate one. Ramsey theory as is the case with Graph theory is filled with conjectures and theories by Erdős (Chung & Graham, 1998; Schechter, 1998). Chung & Graham (1998) outline over 200 Ramsey-type problems, conjectures and proofs by Erdős, 140 of which are joint works. For instance, Erdős conjectured on the gaps between consecutive Ramsey numbers: $R(n+1, n)$ and $R(n, n)$. Although Ramsey numbers and establishing lower and upper bounds are not part of today's school curriculum, tables, algorithms, gaps between consecutive numbers in sequences are. As well, there is a lesson to learn about the methods used by mathematicians including visual, probabilistic and counting methods.

The Erdős Number – Artifact of a Life's Work

Ramsey numbers and graphs are echoed in an artifact that was developed in honor of Erdős. As a humorous tribute to Erdős, his friend Casper Goffman created the *Erdős number*. The first Erdős number is 0, which represents Erdős himself. Erdős had 511 direct collaborators. Each of those people has an Erdős number of 1. Collaborators of these 511 original collaborators have Erdős number 2, and so on. The Erdős number has been extended to fractional quantities, so that if you co-authored fifty-seven papers with Erdős, as András Sarkőzy, you get the number 1/57. The Erdős number for anyone who has no co-authorship with Erdős or with his collaborators, then, is not 0; rather, it is undefined! A sweeping majority of today's mathematicians have an Erdős number below 5 (Buchanan, 2003). The Erdős number project reveals something unique about family, social, epidemic, and work networks. It has resonance in collaboration research, systems theory, and network theory.

Buchanan's (2003) work illustrates how small the world is when it comes to networks; there are small degrees of separation and a small number such as 5 may not be that small. An electronic search using a Collaboration Distance tool[1] on the American Mathematical Society website reveals that Mark Buchanan, one of the inventors of Small World Network Theory, has an Erdős number of 5. Consider a current mathematician, an acquaintance of one of the authors, who works at an African University in a country that Erdős most likely never visited. Does such a mathematician have a Erdős number that is bigger than Buchanan's? The collaboration distance calculator gives this mathematician an Erdős number of 3 - he co-authored with a mathematician who co-authored with a co-author of Erdős. The same African author has an Albert Einstein number of 5. Erdős and Albert Einstein had one collaborator in common, Ernst

[1] Available at http://www.ams.org/mathscinet/collaborationDistance.html

Gabor Strauss, which means that Einstein has an Erdős number of 2. Karl Friederich Gauss (1777-1855, German), who lived a hundred years before Erdős, has a Erdős number of 4.

Much has been written about Erdős, his trials as a transient worker in the post-World War II era, his mathematical works, his humorous moments, and what his contemporary Einstein, had to say about him. Many of his biographies contain elaborate bibliographies (see Babai, Pomerance, & Vertesi, 1998; Bollobas, 1998; Hoffman, 1998; Seife, 2002). Let us now turn to lessons for school mathematics drawn from Erdős's life and work.

Lessons for School Students – Erdős as Inspiration

Fauvel (1991) maintains that biographies of great mathematicians give mathematics a human face. To tell stories of great mathematicians in classrooms, Fauvel (1991), Liu (2003) and Bidwell (1993) encourage using drama activities, anecdotal and portrait displays, artifacts of mathematical works, maps and calendars of mathematicians' birth places and dates, as well as outlines of obstacles faced by mathematicians. Using this paper as a resource, teachers may develop drama activities around the life and mathematical works of Erdős, including his interest in formulating mathematics problems and his non-stop travel.

Erdős's biography shows that even theorems already proved and problems already solved may be revisited so that new, more elegant, and innovative proofs and solutions can be created. An abbreviated version of Erdős's story would be a great motivator for many students, especially gifted students and those fascinated by history. Students who have heard about George Polya and his four steps in problem solving will be interested to hear that Erdős was born in the same country - Hungary - only a few decades later. Students may also find themselves drawn to Erdős because of his unique personality and socio-cultural background.

Erdős had a unique perspective on life and on mathematics. His work illuminates the importance of mathematical processes. He is a model for problem posing and collective work. For Erdős, the process was often as important as the result. To end up with another question or a conjecture is as interesting as getting a solution to a problem. Erdős had this to say about problem solving: "A well chosen problem can isolate an essential difficulty in a particular area … An innocent looking problem … Might be like a 'marshmallow,' serving as a tasty tidbit supplying a few moments of fleeting enjoyment. Or it might be like an 'acorn,' requiring deep and subtle new insights from which a mighty oak can develop" (Chung & Graham, 1988, p. ix). Indeed many of Erdős' simple questions - such as, what happens when you randomly add edges to a graph - later spawned theories such as random graphs theory. Many of these theories later became tools in newer areas such as autocatalytic networks in complexity science

Working in groups, collaborating on projects, and communicating with others are increasingly valued learning skills in mathematics classrooms (e.g., NCTM, 2000). The story of Erdős's life, particularly the development of the Erdős numbers, demonstrates the importance of collaborative learning and collective work that transcends geographical and historical boundaries. That Erdős always consulted with colleagues when working on mathematics conjectures and proofs shows how he embodied mathematics as a social activity and as a collective project based on interaction and collaboration.

Biographies of mathematicians might also serve as sources of mathematics problems for the classroom (Ernest, 1998; Fauvel, 1991). Consider the problem of coloring a complete graph, such as K_4 (a square together with its two diagonals) and bigger polygons randomly with two colors to explore possible monochromatic polygons. Middle school students may explore the problem of possible monochromatic subgraphs. Also consider the problem of distribution of other sequences, such as sequence of even numbers (2, 4, 6, 8, 10, …) labeled as say E(x). E(x) is the even counting function used to denote the number of even numbers less than or equal to a real number that is greater than 1. It follows that E(2) is 1, there being only one even number, 2, that is less or equal to 2. E(3) is 1, E(4) is 2, E(5) is 2, and so on. E(n) = (0, 1, 2, 2, 3, 3, 4, 4, 5, 5 …). What is the nature of the function E(x) and how is it different from or similar to the distribution of prime numbers? Teachers may guide students to explore more distribution functions of special numbers such as odd numbers, triangular numbers, exponents of two, and exponents of three. As well interesting patterns emerge when not only patterns in sequences are studied but also when patterns in sequences of first differences and second differences are studied. For elementary school students this might be done as an elementary school focal point that is later connected to slope and differentiation at high school.

Erdős's mathematics work was interdisciplinary in nature - he connected geometry to number theory to probability theory. In a pre-service teacher education classroom that explored history of mathematics, including biographies of mathematicians, in the context of teaching and learning, two students - co-authors of this papers - independently chose to focus on Erdős's biography. This suggests that Erdős biography and mathematical works might be of interest to teachers and students. By using biographies, teachers may demonstrate links between school mathematics and university mathematics. Examples of links between university and school mathematics activities include the problems mathematicians pose about elementary mathematics artifacts, the processes that mathematicians engage in such as generating computation tables and computational algorithms that elementary school students also engage in. As well, biographies of mathematicians in general might be a source of illustrations for engaging in school mathematics in ways that facilitate making connections, non-routine problems solving and exploring big mathematical ideas. Generally, the use of biographies in teaching, that is more common in science teaching than in mathematics teaching, according to pre-service teachers, may serve as a motivation for students to become mathematicians and scientists at a time when fewer and fewer students are taking this route.

Erdős's story exemplifies a recent mathematician who participated in the evolution of old and new fields of mathematical study. Mathematics is not a dead subject; there is much more to discover and construct. New branches of mathematics are still sprouting, even from elementary and historical problems.

Whether it is through his emphasis on conjecturing and problem solving, his mathematical accomplishments at a young age, his highly collaborative work, or his fascinating character, Paul Erdős's biography has the potential to demonstrate key processes in mathematics.

Notes:

The authors wish to thank Khadheeja Ibrahim-Didi for her helpful discussion on ideas to do with small worlds and degrees of separation.

References

Babai, L., Pomerance, C., & Vertesi, P. (1998). The Mathematics of Paul Erdös. *Notices of the AMS, 45(1),* 19-31. Retrieved from http://www.ams.org/notices/199801/vertesi.pdf

Babai L., & Spencer, J. (1998). Paul Erdös. *Notices of the AMS, 45(1)*, 64-73.

Bidwell, J. K. (1993). Humanize your classroom with the history of mathematics. *The Mathematics Teacher*, 86(6), 461-464.

Bollobas, B. (1998). To prove and conjecture. Paul Erdös and his mathematics. *The American Mathematical Monthly*, 105(3), 209-237.

Buchanan, M. (2003). *Nexus: Small Worlds and the Groundbreaking Theory of Networks*. New York: W. W. Norton.

Chung, F., & Graham, R. (1998). *Erdös on graphs: his legacy of unsolved problems.* Wellesley, Mass: AK Peters.

Chace, B. A. (1927). *The Rhind mathematical papyrus*. Oberlin, OH: AMS.

Erdös, P. (1949). On a new method in elementary number theory which leads to an elementary proof of the Prime Number Theorem. *Proceedings of the National Academy of Sciences of the United States of America*, 35(7), 374-384.

Ernest, P. (1998). The history of mathematics in the classroom. *Mathematics in Schools*, 27 (4), 25.

Fauvel, J. (1991). Using History in Mathematics Education. *For the Learning of Mathematics*, 11(2), 3-6.

Gallian, J. (2002). *Contemporary Abstract Algebra (5th Ed.)*. New York: Houghton Mifflin Company.

Gulikers, I., & Blom, K. (2001). A historical angle, a survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics, 47*, 223-258.

Hoffman, P. (1998). *The man who loved only numbers : The story of Paul Erdös and the search for mathematical truth*. New York: Hyperion Books.

NCTM (2000). *Principles and Standards*. Reston, VA: NCTM.

O'Connor, J. J. & Robertson, E. F. (2000). *Paul Erdös. School of Mathematics and statistics*, University of St Andrews. Retrieved from http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Erdös.html

Liu, P. H. (2003). Do teachers need to incorporate the history of mathematics in their teaching? *The Mathematics Teacher,* 96(6), 416-421.

NCTM (2006). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics*. Reston, VA.: NCTM.

Ponte, J. P., & Chapman, O. (2008). Preservice mathematics teachers' knowledge and development. In L. D. Engish (Ed.), *Handbook of international research in mathematics education* (2nd, ed.), 223-261. New York: Routledge.

Ross, P. (1998). Reminiscences of Paul Erdös. *The College Mathematics Journal*, 29(1), 77-79.

Schechter, B. (1998). *My brain is open: The mathematical journey of Paul Erdös*. New York: Simon & Schuster.

Seife, C. (2002). Erdös's hard-to-win prizes still draw bounty hunters. *Science*, 29, 39-40.

Selberg, A. (1950). An Elementary Proof of the Prime-Number Theorem. *The Annals of Mathematics, Second Series*, 50(2), 305-313.

Schumer, P. D. (2004). *Mathematical Journeys*. Hoboken, NJ: John Wiley.

Weisstein, E. W. (2008). *Prime number theorem*. Retrieved from Wolfram MathWorld Website: http://mathworld.wolfram.com/PrimeNumberTheorem.html

Zazkis, R., & Campbell, S. R. (2006). *Number Theory in Mathematics Education: Perspectives and prospects*. Mahwah, NJ.: LEA.

The main author of this article, K. Immaculate Namukasa, is an Associate Professor at the Faculty of Education, The University of Western Ontario.

The co-authors A. Katherine Minello, J. Sarah Dent and Anthony Staibano were Teacher Candidates (Year 2008) at the Faculty of Education, The University of Western Ontario.